Research Article

Solution and Hyers-Ulam-Rassias Stability of Generalized Mixed Type Additive-Quadratic Functional Equations in Fuzzy Banach Spaces

M. Eshaghi Gordji,¹ H. Azadi Kenary,² H. Rezaei,²
 Y. W. Lee,³ and G. H. Kim⁴

¹ Department of Mathematics, Semnan University, Semnan 35131-19111, Iran

² Department of Mathematics, Yasouj University, Yasouj 75918-74831, Iran

³ Department of Computer Hacking and Information Security, Daejeon University,
 Youngwoondong Donggu, Daejeon 300-716, Republic of Korea

⁴ Department of Mathematics, Kangnam University, Yongin, Gyeonggi 446-702, Republic of Korea

Correspondence should be addressed to G. H. Kim, ghkim@kangnam.ac.kr

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By using fixed point methods and direct method, we establish the generalized Hyers-Ulam stability of the following additive-quadratic functional equation $f(x + ky) + f(x - ky) = f(x + y) + f(x - y) + (2(k+1)/k)f(ky) - 2(k+1)f(y)$ for fixed integers k with $k \neq 0, \pm 1$ in fuzzy Banach spaces.

1. Introduction and Preliminaries

The stability problem of functional equations was originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let $(G_1, \cdot)$ be a group and let $(G_2, *, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \rightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta, \quad (1.1)$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta, \quad (1.2)$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. In 1978, Rassias [3] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded. In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [5–17]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad (1.3)$$

is related to a symmetric biadditive function. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.3) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x, x)$ for all x (see [6, 18]). The biadditive function B is given by

$$B(x, y) = \frac{1}{4}(f(x+y) - f(x-y)). \quad (1.4)$$

A Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.3) was proved by Skof for functions $f : A \rightarrow B$, where A is normed space and B Banach space (see [19–22]). Borelli and Forti [23] generalized the stability result of quadratic functional equations as follows (cf. [24, 25]): let G be an Abelian group, and X a Banach space. Assume that a mapping $f : G \rightarrow X$ satisfies the functional inequality:

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varphi(x, y), \quad (1.5)$$

for all $x, y \in G$, and $\varphi : G \times G \rightarrow [0, \infty)$ is a function such that

$$\Phi(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < \infty, \quad (1.6)$$

for all $x, y \in G$. Then there exists a unique quadratic mapping $Q : G \rightarrow X$ with the property

$$\|f(x) - Q(x)\| \leq \Phi(x, x), \quad (1.7)$$

for all $x \in G$.

Now, we introduce the following functional equation for fixed integers k with $k \neq 0, \pm 1$:

$$f(x+ky) + f(x-ky) = f(x+y) + f(x-y) + \frac{2(k+1)}{k} f(ky) - 2(k+1)f(y), \quad (1.8)$$

with $f(0) = 0$ in a non-Archimedean space. It is easy to see that the function $f(x) = ax + bx^2$ is a solution of the functional equation (1.8), which explains why it is called additive-quadratic functional equation. For more detailed definitions of mixed type functional equations, we can refer to [26–47].

Definition 1.1 (see [48]). Let X be a real vector space. A function $N : X \times \mathbb{R} \rightarrow [0, 1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

$$(N1) \ N(x, t) = 0 \text{ for } t \leq 0;$$

$$(N2) \ x = 0 \text{ if and only if } N(x, t) = 1 \text{ for all } t > 0;$$

$$(N3) \ N(cx, t) = N(x, t/|c|) \text{ if } c \neq 0;$$

$$(N4) \ N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\};$$

$$(N5) \ N(x, \cdot) \text{ is a nondecreasing function of } \mathbb{R} \text{ and } \lim_{t \rightarrow \infty} N(x, t) = 1;$$

$$(N6) \ \text{for } x \neq 0, N(x, \cdot) \text{ is continuous on } \mathbb{R}.$$

The pair (X, N) is called a fuzzy normed vector space.

Example 1.2. Let $(X, \|\cdot\|)$ be a normed linear space and $\alpha, \beta > 0$. Then

$$N(x, t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta \|x\|}, & t > 0, \ x \in X, \\ 0, & t \leq 0, \ x \in X, \end{cases} \quad (1.9)$$

is a fuzzy norm on X .

Definition 1.3. Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{n \rightarrow \infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X and one denotes it by $N - \lim_{n \rightarrow \infty} x_n = x$.

Definition 1.4. Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, one has $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

Example 1.5. Let $N : \mathbb{R} \times \mathbb{R} \rightarrow [0, 1]$ be a fuzzy norm on $\mathbb{R}$ defined by

$$N(x, t) = \begin{cases} \frac{t}{t + |x|}, & t > 0, \\ 0, & t \leq 0. \end{cases} \quad (1.10)$$

The $(\mathbb{R}, N)$ is a fuzzy Banach space. Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$, $\delta > 0$, and $\epsilon = \delta/(1 + \delta)$. Then there exist $m \in \mathbb{N}$ such that for all $n \geq m$ and all $p > 0$, one has

$$\frac{1}{1 + |x_{n+p} - x_n|} \geq 1 - \epsilon. \quad (1.11)$$

So $|x_{n+p} - x_n| < \delta$ for all $n \geq m$ and all $p > 0$. Therefore $\{x_n\}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$. Let $x_n \rightarrow x_0 \in \mathbb{R}$ as $n \rightarrow \infty$. Then $\lim_{n \rightarrow \infty} N(x_n - x_0, t) = 1$ for all $t > 0$.

We say that a mapping $f : X \rightarrow Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \rightarrow Y$ is continuous at each $x \in X$, then $f : X \rightarrow Y$ is said to be continuous on X ([49]).

Definition 1.6. Let X be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:

- (1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.7. Let (X, d) be a complete generalized metric space and let $J : X \rightarrow X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty, \quad (1.12)$$

for all nonnegative integers n , or there exists a positive integer n_0 such that

- (1) $d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n$;
- (2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;
- (3) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;
- (4) $d(y, y^*) \leq 1/(1 - L)d(y, Jy)$ for all $y \in Y$.

We have the following theorem from [42], which investigates the solution of (1.8).

Theorem 1.8. A function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies (1.8) for all $x, y \in X$ if and only if there exist functions $A : X \rightarrow Y$ and $Q : X \times X \rightarrow Y$, such that $f(x) = A(x) + Q(x, x)$ for all $x \in X$, where the function Q is symmetric biadditive and A is additive.

2. A Fixed Point Method

Using the fixed point methods, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (1.8) in fuzzy Banach spaces. Throughout this paper, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

Theorem 2.1. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a mapping such that there exists an $\alpha < 1$ with

$$\varphi(x, y) \leq |k|\alpha\varphi\left(\frac{x}{k}, \frac{y}{k}\right), \quad (2.1)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be an odd function satisfying $f(0) = 0$ and

$$\begin{aligned} N\left(f(k(x+y)) + f(k(x-y)) - f(kx+y) - f(kx-y) - \frac{2(k+1)}{k}f(ky) + 2(k+1)f(y), t\right) \\ \geq \frac{t}{t + \varphi(x, y)}, \end{aligned} \quad (2.2)$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N - \lim_{n \rightarrow \infty} (f(k^n x)/k^n)$ exists for all $x \in X$ and defines a unique additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(|2k+2| - |2k+2|\alpha)t}{(|2k+2| - |2k+2|\alpha)t + \varphi(0, x)}, \quad (2.3)$$

for all $x \in X$ and $t > 0$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd function. Putting $x = 0$ in (2.2), we get

$$N\left(\frac{f(ky)}{k} - f(y), \frac{t}{|2k+2|}\right) \geq \frac{t}{t + \varphi(0, y)}, \quad (2.4)$$

for all $y \in X$ and all $t > 0$. Replacing y by x in (2.4), we have

$$N\left(\frac{f(kx)}{k} - f(x), \frac{t}{|2k+2|}\right) \geq \frac{t}{t + \varphi(0, x)}, \quad (2.5)$$

for all $x \in X$ and all $t > 0$. Consider the set $S := \{h : X \rightarrow Y; h(0) = 0\}$ and introduce the generalized metric on S :

$$d(g, h) = \inf_{\mu \in (0, +\infty)} \left\{ N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(0, x)}, \forall x \in X \right\}, \quad (2.6)$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [50]). We consider the mapping $J : (S, d) \rightarrow (S, d)$ as follows:

$$Jg(x) := \frac{1}{k}g(kx), \quad (2.7)$$

for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) = \beta$. Then

$$N(g(x) - h(x), \beta t) \geq \frac{t}{t + \varphi(0, x)}, \quad (2.8)$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), \alpha\beta t) &= N\left(\frac{1}{k}g(kx) - \frac{1}{k}h(kx), \alpha\beta t\right) \\ &= N(g(kx) - h(kx), |k|\alpha\beta t) \\ &\geq \frac{|k|\alpha t}{|k|\alpha t + \varphi(0, x)} \\ &\geq \frac{|k|\alpha t}{|k|\alpha t + |k|\alpha\varphi(0, x)} \\ &= \frac{t}{t + \varphi(0, x)}, \end{aligned} \quad (2.9)$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \beta$ implies that $d(Jg, Jh) \leq \alpha\beta$. This means that $d(Jg, Jh) \leq \alpha d(g, h)$ for all $g, h \in S$. It follows from (2.5) that

$$d(f, Jf) \leq \frac{1}{|2k + 2|}. \quad (2.10)$$

By Theorem 1.7, there exists a mapping $A : X \rightarrow Y$ satisfying the following.

(1) A is a fixed point of J , that is,

$$kA(x) = A(kx), \quad (2.11)$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $M = \{g \in S : d(h, g) < \infty\}$. This implies that A is a unique mapping satisfying (2.11) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(0, x)}, \quad (2.12)$$

for all $x \in X$.

(2) $d(J^n f, A) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality $\lim_{n \rightarrow \infty} (f(k^n x)/k^n) = A(x)$, for all $x \in X$.

(3) $d(f, A) \leq (1/(1 - \alpha))d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{1}{|2k + 2| - |2k + 2|\alpha}. \quad (2.13)$$

This implies that the inequality (2.3) holds.

It follows from (2.1) and (2.2) that

$$\begin{aligned}
 N & \left(\frac{f(k^n(x+ky))}{k^n} + \frac{f(k^n(x-ky))}{k^n} - \frac{f(k^n(x+y))}{k^n} - \frac{f(k^n(x-y))}{k^n} \right. \\
 & \quad \left. - \frac{2(k+1)}{k} \frac{f(k^{n+1}y)}{k^n} + 2(k+1) \frac{f(k^n y)}{k^n}, \frac{t}{k^n} \right) \\
 & \geq \frac{t}{t + \varphi(k^n x, k^n y)},
 \end{aligned} \tag{2.14}$$

for all $x, y \in X$, all $t > 0$, and all $n \in \mathbb{N}$. So

$$\begin{aligned}
 N & \left(\frac{f(k^n(x+ky))}{k^n} + \frac{f(k^n(x-ky))}{k^n} - \frac{f(k^n(x+y))}{k^n} - \frac{f(k^n(x-y))}{k^n} \right. \\
 & \quad \left. - \frac{2(k+1)}{k} \frac{f(k^{n+1}y)}{k^n} + 2(k+1) \frac{f(k^n y)}{k^n}, t \right) \\
 & \geq \frac{|k|^n t}{|k|^n t + |k|^n \alpha^n \varphi(x, y)},
 \end{aligned} \tag{2.15}$$

for all $x, y \in X$, all $t > 0$, and all $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} (|k|^n t / (|k|^n t + |k|^n \alpha^n \varphi(x, y))) = 1$ for all $x, y \in X$ and all $t > 0$, we obtain that

$$\begin{aligned}
 N & \left(A(k(x+y)) + A(k(x-y)) - A(kx+y) - A(kx-y) - \frac{2(k+1)}{k} A(ky) \right. \\
 & \quad \left. + 2(k+1)A(y), t \right) = 1,
 \end{aligned} \tag{2.16}$$

for all $x, y, z \in X$ and all $t > 0$. Hence the mapping $A : X \rightarrow Y$ is additive, as desired. $\square$

Corollary 2.2. Let $\theta \geq 0$ and let r be a real positive number with $r < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \rightarrow Y$ be an odd mapping satisfying

$$\begin{aligned}
 N & \left(f(k(x+y)) + f(k(x-y)) - f(kx+y) - f(kx-y) - \frac{2(k+1)}{k} f(ky) + 2(k+1)f(y), t \right) \\
 & \geq \frac{t}{t + \theta(\|x\|^r + \|y\|^r)},
 \end{aligned} \tag{2.17}$$

for all $x, y \in X$ and all $t > 0$. Then the limit $A(x) := N\text{-}\lim_{n \rightarrow \infty} (f(k^n x)/k^n)$ exists for each $x \in X$ and defines a unique additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{|2k+2|(|k| - |k|^r)t}{|2k+2|(|k| - |k|^r)t + |k|\theta\|x\|^r}, \tag{2.18}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.1 by taking $\varphi(x, y) := \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $\alpha = |k|^{r-1}$ and we get the desired result. $\square$

Theorem 2.3. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a mapping such that there exists an $\alpha < 1$ with

$$\varphi\left(\frac{x}{k}, \frac{y}{k}\right) \leq \frac{\alpha}{|k|} \varphi(x, y), \quad (2.19)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be an odd mapping satisfying $f(0) = 0$ and (2.2). Then the limit $A(x) := N - \lim_{n \rightarrow \infty} k^n f(x/k^n)$ exists for all $x \in X$ and defines a unique additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(|2k+2| - |2k+2|\alpha)t}{(|2k+2| - |2k+2|\alpha)t + \alpha\varphi(0, x)}, \quad (2.20)$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined as in the proof of Theorem 2.1.

Consider the mapping $J : S \rightarrow S$ by

$$Jg(x) := kg\left(\frac{x}{k}\right), \quad (2.21)$$

for all $g \in S$. Let $g, h \in S$ be given such that $d(g, h) = \beta$. Then

$$N(g(x) - h(x), \beta t) \geq \frac{t}{t + \varphi(0, x)}, \quad (2.22)$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), \alpha\beta t) &= N\left(kg\left(\frac{x}{k}\right) - kh\left(\frac{x}{k}\right), \alpha\beta t\right) \\ &= N\left(g\left(\frac{x}{k}\right) - h\left(\frac{x}{k}\right), \frac{\alpha\beta t}{|k|}\right) \\ &\geq \frac{(\alpha t/|k|)}{\alpha t/|k| + \varphi(0, x/k)} \geq \frac{t}{t + \varphi(0, x)}, \end{aligned} \quad (2.23)$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \beta$ implies that $d(Jg, Jh) \leq \alpha\beta$. This means that $d(Jg, Jh) \leq \alpha d(g, h)$ for all $g, h \in S$. It follows from (2.5) that

$$N\left(kf\left(\frac{x}{k}\right) - f(x), \frac{kt}{|2k+2|}\right) \geq \frac{t}{t + \varphi(0, x/k)} \geq \frac{t}{t + (\alpha/|k|)\varphi(0, x)}, \quad (2.24)$$

for all $x \in X$ and all $t > 0$. Therefore

$$N\left(kf\left(\frac{x}{k}\right) - f(x), \frac{\alpha t}{|2k+2|}\right) \geq \frac{t}{t + \varphi(0, x)}. \quad (2.25)$$

So $d(f, Jf) \leq \alpha$. By Theorem 1.7, there exists a mapping $A : X \rightarrow Y$ satisfying the following.

(1) A is a fixed point of J , that is,

$$A\left(\frac{x}{k}\right) = \frac{1}{k}A(x), \quad (2.26)$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that A is a unique mapping satisfying (2.26) such that there exists $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(0, x)}, \quad (2.27)$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n f, A) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality $N - \lim_{n \rightarrow \infty} k^n f(x/k^n) = A(x)$ for all $x \in X$.

(3) $d(f, A) \leq d(f, Jf)/(1 - L)$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \leq \frac{\alpha}{|2k+2| - |2k+2|\alpha}. \quad (2.28)$$

This implies that the inequality (2.20) holds.

The rest of proof is similar to the proof of Theorem 2.1. $\square$

Corollary 2.4. Let $\theta \geq 0$ and let r be a real number with $r > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \rightarrow Y$ be an odd mapping satisfying (2.17). Then $A(x) := N - \lim_{n \rightarrow \infty} k^n f(x/k^n)$ exists for each $x \in X$ and defines a unique additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{|2k+2|(|k|^r - |k|)t}{|2k+2|(|k|^r - |k|)t + |k|\theta\|x\|^r}, \quad (2.29)$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.3 by taking $\varphi(x, y) := \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $\alpha = |k|^{1-r}$ and we get the desired result. $\square$

Theorem 2.5. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $\alpha < 1$ with

$$\varphi(x, y) \leq k^2 \alpha \varphi\left(\frac{x}{k}, \frac{y}{k}\right), \quad (2.30)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be an even mapping with $f(0) = 0$ and satisfying (2.2). Then $Q(x) := N - \lim_{n \rightarrow \infty} (f(k^n x)/k^{2n})$ exists for all $x \in X$ and defines a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2|k| - 2|k|\alpha)t}{(2|k| - 2|k|\alpha)t + \varphi(0, x)}, \quad (2.31)$$

for all $x \in X$ and all $t > 0$.

Proof. Replacing x by kx in (2.2), we get

$$\begin{aligned} N\left(f(k(x+y)) + f(k(x-y)) - f(kx+y) - f(kx-y) - \frac{2(k+1)}{k}f(ky) + 2(k+1)f(y), t\right) \\ \geq \frac{t}{t + \varphi(kx, y)}, \end{aligned} \quad (2.32)$$

for all $x, y \in X$ and all $t > 0$. Putting $x = 0$ and replacing y by x in (2.32), we have

$$N\left(\frac{f(kx)}{k} - kf(x), \frac{t}{2}\right) \geq \frac{t}{t + \varphi(0, x)}, \quad (2.33)$$

for all $x \in X$ and all $t > 0$. By (2.33), (N3), and (N4), we get

$$N\left(\frac{f(kx)}{k^2} - f(x), \frac{t}{2|k|}\right) \geq \frac{t}{t + \varphi(0, x)}, \quad (2.34)$$

for all $x \in X$ and all $t > 0$. Consider the set $S^* := \{h : X \rightarrow Y; h(0) = 0\}$ and introduce the generalized metric on S^* :

$$d(g, h) = \inf_{\mu \in (0, +\infty)} \left\{ N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(0, x)}, \forall x \in X \right\}, \quad (2.35)$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (S^*, d) is complete (see [50]). Now we consider the linear mapping $J : (S^*, d) \rightarrow (S^*, d)$ such that

$$Jg(x) := \frac{1}{k^2}g(kx), \quad (2.36)$$

for all $x \in X$. Proceeding as in the proof of Theorem 2.1, we obtain that $d(g, h) = \beta$ implies that $d(Jg, Jh) \leq \alpha\beta$. This means that $d(Jg, Jh) \leq \alpha d(g, h)$ for all $g, h \in S$. It follows from

(2.34) that

$$d(f, Jf) \leq \frac{1}{2|k|}. \quad (2.37)$$

By Theorem 1.7, there exists a mapping $Q : X \rightarrow Y$ such that one has the following.

(1) Q is a fixed point of J , that is,

$$k^2 Q(x) = Q(kx), \quad (2.38)$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set $M = \{g \in S^* : d(h, g) < \infty\}$. This implies that Q is a unique mapping satisfying (2.38) such that there exists a $\mu \in (0, \infty)$ satisfying $N(f(x) - Q(x), \mu t) \geq t/(t + \varphi(0, x))$ for all $x \in X$.

(2) $d(J^n f, Q) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality $\lim_{n \rightarrow \infty} (f(k^n x)/k^{2n}) = Q(x)$ for all $x \in X$.

(3) $d(f, Q) \leq (1/(1 - \alpha))d(f, Jf)$, which implies the inequality $d(f, Q) \leq 1/(2|k| - 2|k|\alpha)$. This implies that the inequality (2.31) holds.

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

Corollary 2.6. Let $\theta \geq 0$ and let r be a real positive number with $r < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \rightarrow Y$ be an even mapping with $f(0) = 0$ and satisfying (2.17). Then the limit $Q(x) := N - \lim_{n \rightarrow \infty} (f(k^n x)/k^{2n})$ exists for each $x \in X$ and defines a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2k^2 - 2k^{2r})t}{(2k^2 - 2k^{2r})t + |k|\theta\|x\|^r}, \quad (2.39)$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.5 by taking $\varphi(x, y) := \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $\alpha = k^{2r-2}$ and we get the desired result. $\square$

Theorem 2.7. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $\alpha < 1$ with

$$\varphi\left(\frac{x}{k}, \frac{y}{k}\right) \leq \frac{\alpha}{k^2} \varphi(x, y), \quad (2.40)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be an even mapping with $f(0) = 0$ and satisfying (2.2). Then the limit $Q(x) := N - \lim_{n \rightarrow \infty} k^{2n} f(x/k^n)$ exists for all $x \in X$ and defines a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2|k| - 2|k|\alpha)t}{(2|k| - 2|k|\alpha)t + \alpha\varphi(0, x)}, \quad (2.41)$$

for all $x \in X$ and $t > 0$.

Proof. Let (S^*, d) be the generalized metric space defined as in the proof of Theorem 2.5. It follows from (2.34) that

$$N\left(k^2 f\left(\frac{x}{k}\right) - f(x), \frac{|k|t}{2}\right) \geq \frac{t}{t + \varphi(0, x/k)} \geq \frac{t}{t + (\alpha/k^2)\varphi(0, x)}, \quad (2.42)$$

for all $x \in X$ and $t > 0$. So

$$N\left(f(x) - k^2 f\left(\frac{x}{k}\right), \frac{\alpha t}{2|k|}\right) \geq \frac{t}{t + \varphi(0, x)}. \quad (2.43)$$

The rest of the proof is similar to the proofs of Theorems 2.1 and 2.3. $\square$

Corollary 2.8. Let $\theta \geq 0$ and let r be a real number with $r > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \rightarrow Y$ be an even mapping with $f(0) = 0$ and satisfying (2.17). Then $Q(x) := N - \lim_{n \rightarrow \infty} k^{2n} f(x/k^n)$ exists for each $x \in X$ and defines a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2|k|^{2r+1} - 2|k|^3) t}{(2|k|^{2r+1} - 2|k|^3)t + k^2 \theta \|x\|^r}, \quad (2.44)$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from Theorem 2.7 by taking $\varphi(x, y) := \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $\alpha = k^{2-2r}$ and we get the desired result. $\square$

3. Direct Method

In this section, using direct method, we prove the Hyers-Ulam stability of functional equation (1.8) in fuzzy Banach spaces. Throughout this section, we assume that X is a linear space, (Y, N) is a fuzzy Banach space, and (Z, N') is a fuzzy normed space. Moreover, we assume that $N(x, \cdot)$ is a left continuous function on $\mathbb{R}$.

Theorem 3.1. Assume that a mapping $f : X \rightarrow Y$ is an odd mapping with $f(0) = 0$ satisfying the inequality

$$\begin{aligned} & N\left(f(k(x+y)) + f(k(x-y)) - f(kx+y) - f(kx-y) - \frac{2(k+1)}{k}f(ky) + 2(k+1)f(y), t\right) \\ & \geq N'(\varphi(x, y), t), \end{aligned} \quad (3.1)$$

for all $x, y \in X$, $t > 0$, and $\varphi : X^2 \rightarrow Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < 1/|k|$ such that

$$N'\left(\varphi\left(\frac{x}{k}, \frac{y}{k}\right), t\right) \geq N'\left(\varphi(x, y), \frac{t}{|r|}\right), \quad (3.2)$$

for all $x, y \in X$ and all $t > 0$. Then there exists a unique additive mapping $A : X \rightarrow Y$ satisfying (1.8) and the inequality

$$N(f(x) - A(x), t) \geq N'\left(\varphi(0, x), \frac{|2k+2|(1-|kr|)t}{|r|}\right), \quad (3.3)$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (3.2) that

$$N'\left(\varphi\left(\frac{x}{k^j}, \frac{y}{k^j}\right), t\right) \geq N'\left(\varphi(x, y), \frac{t}{|r|^j}\right), \quad (3.4)$$

for all $x, y \in X$ and all $t > 0$. Putting $x = 0$ in (3.1) and then replacing y by x/k , we get

$$N\left(kf\left(\frac{x}{k}\right) - f(x), \frac{|k|t}{|2k+2|}\right) \geq N'\left(\varphi\left(0, \frac{x}{k}\right), t\right), \quad (3.5)$$

for all $x \in X$ and all $t > 0$. Replacing x by x/k^j in (3.5), we have

$$N\left(k^{j+1}f\left(\frac{x}{k^{j+1}}\right) - k^j f\left(\frac{x}{k^j}\right), \frac{|k|^{j+1}t}{|2k+2|}\right) \geq N'\left(\varphi\left(0, \frac{x}{k^{j+1}}\right), t\right) \geq N'\left(\varphi(0, x), \frac{t}{|r|^{j+1}}\right), \quad (3.6)$$

for all $x \in X$, all $t > 0$, and all integer $j \geq 0$. So

$$\begin{aligned} & N\left(f(x) - k^n f\left(\frac{x}{k^n}\right), \sum_{j=0}^{n-1} \frac{|k|^{j+1}|r|^{j+1}t}{|2k+2|}\right) \\ &= N\left(\sum_{j=0}^{n-1} k^{j+1}f\left(\frac{x}{k^{j+1}}\right) - k^j f\left(\frac{x}{k^j}\right), \sum_{j=0}^{n-1} \frac{|k|^{j+1}|r|^{j+1}t}{|2k+2|}\right) \\ &\geq \min_{0 \leq j \leq n-1} \left\{ N\left(k^{j+1}f\left(\frac{x}{k^{j+1}}\right) - k^j f\left(\frac{x}{k^j}\right), \frac{|k|^{j+1}|r|^{j+1}t}{|2k+2|}\right) \right\} \\ &\geq \min_{0 \leq j \leq n-1} \{N'(\varphi(0, x), t)\} \\ &= N'(\varphi(0, x), t), \end{aligned} \quad (3.7)$$

which yields

$$N\left(k^{n+p}f\left(\frac{x}{k^{n+p}}\right) - k^p f\left(\frac{x}{k^p}\right), \sum_{j=0}^{n-1} \frac{|k|^{j+p+1}|r|^{j+1}t}{|2k+2|}\right) \geq N'\left(\varphi\left(0, \frac{x}{k^p}\right), t\right) \geq N'\left(\varphi(0, x), \frac{t}{|r|^p}\right), \quad (3.8)$$

for all $x \in X, t > 0$, and all integers $n > 0, p \geq 0$. So

$$N\left(k^{n+p}f\left(\frac{x}{k^{n+p}}\right) - k^p f\left(\frac{x}{k^p}\right), \sum_{j=0}^{n-1} \frac{|k|^{j+p+1}|r|^{j+p+1}t}{|2k+2|}\right) \geq N'(\varphi(0, x), t), \quad (3.9)$$

for all $x \in X, t > 0$, and any integers $n > 0, p \geq 0$. Hence one can obtain

$$N\left(k^{n+p}f\left(\frac{x}{k^{n+p}}\right) - k^p f\left(\frac{x}{k^p}\right), t\right) \geq N'\left(\varphi(0, x), \frac{t}{\sum_{j=0}^{n-1} (|k|^{j+p+1}|r|^{j+p+1}/|2k+2|)}\right), \quad (3.10)$$

for all $x \in X, t > 0$, and any integers $n > 0, p \geq 0$. Since the series $\sum_{j=0}^{+\infty} k^j |r|^j$ is a convergent series, we see by taking the limit $p \rightarrow \infty$ in the last inequality that the sequence $\{k^n f(x/k^n)\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N) and so it converges in Y . Therefore a mapping $A : X \rightarrow Y$ defined by $A(x) := N - \lim_{n \rightarrow \infty} k^n f(x/k^n)$ is well defined for all $x \in X$. This means that

$$\lim_{n \rightarrow \infty} N\left(A(x) - k^n f\left(\frac{x}{k^n}\right), t\right) = 1, \quad (3.11)$$

for all $x \in X$ and all $t > 0$. In addition, it follows from (3.10) that

$$N\left(f(x) - k^n f\left(\frac{x}{k^n}\right), t\right) \geq N'\left(\varphi(0, x), \frac{t}{\sum_{j=0}^{n-1} (|k|^{j+1}|r|^{j+1}/|2k+2|)}\right), \quad (3.12)$$

for all $x \in X$ and all $t > 0$. So

$$\begin{aligned} N(f(x) - A(x), t) &\geq \min\left\{N\left(f(x) - k^n f\left(\frac{x}{k^n}\right), (1-\epsilon)t\right), N\left(A(x) - k^n f\left(\frac{x}{k^n}\right), \epsilon t\right)\right\} \\ &\geq N'\left(\varphi(0, x), \frac{\epsilon t}{\sum_{j=0}^{n-1} (|k|^{j+1}|r|^{j+1}/|2k+2|)}\right) \\ &\geq N'\left(\varphi(0, x), \frac{|2k+2|(1-|k||r|)\epsilon t}{|kr|}\right), \end{aligned} \quad (3.13)$$

for sufficiently large n and for all $x \in X, t > 0$, and ϵ with $0 < \epsilon < 1$. Since ϵ is arbitrary and N' is left continuous, we obtain

$$N(f(x) - A(x), t) \geq N'\left(\varphi(0, x), \frac{|2k+2|(1-|k||r|)t}{|kr|}\right), \quad (3.14)$$

for all $x \in X$ and $t > 0$. It follows from (3.1) that

$$\begin{aligned}
 & N\left(\frac{f(k^n(x+ky))}{k^n} + \frac{f(k^n(x-ky))}{k^n} - \frac{f(k^n(x+y))}{k^n} - \frac{f(k^n(x-y))}{k^n} \right. \\
 & \quad \left. - \frac{2(k+1)}{k} \frac{f(k^{n+1}y)}{k^n} + 2(k+1) \frac{f(k^n y)}{k^n}, t\right) \\
 & \geq N'\left(\varphi(k^n x, k^n y), \frac{t}{|k|^n}\right) \geq N'\left(\varphi(x, y), \frac{t}{|r|^n |k|^n}\right) \longrightarrow 1 \quad \text{as } n \longrightarrow +\infty,
 \end{aligned} \tag{3.15}$$

for all $x, y \in X$ and all $t > 0$. Therefore, we obtain in view of (3.11)

$$\begin{aligned}
 & N\left(A(k(x+y)) + A(k(x-y)) - A(kx+y) - A(kx-y) - \frac{2(k+1)}{k} A(ky) \right. \\
 & \quad \left. + 2(k+1)A(y), t\right) \\
 & \geq \min\left\{N\left(A(k(x+y)) + A(k(x-y)) - A(kx+y) - A(kx-y) - \frac{2(k+1)}{k} A(ky) \right. \right. \\
 & \quad \left. + 2(k+1)A(y) - \frac{f(k^n(x+ky))}{k^n} + \frac{f(k^n(x-ky))}{k^n} - \frac{f(k^n(x+y))}{k^n} \right. \\
 & \quad \left. - \frac{f(k^n(x-y))}{k^n} - \frac{2(k+1)}{k} \frac{f(k^{n+1}y)}{k^n} + 2(k+1) \frac{f(k^n y)}{k^n}, \frac{t}{2}\right), \\
 & \quad N\left(\frac{f(k^n(x+ky))}{k^n} + \frac{f(k^n(x-ky))}{k^n} - \frac{f(k^n(x+y))}{k^n} - \frac{f(k^n(x-y))}{k^n} \right. \\
 & \quad \left. - \frac{2(k+1)}{k} \frac{f(k^{n+1}y)}{k^n} + 2(k+1) \frac{f(k^n y)}{k^n}, \frac{t}{2}\right)\bigg\} \\
 & = N\left(\frac{f(k^n(x+ky))}{k^n} + \frac{f(k^n(x-ky))}{k^n} - \frac{f(k^n(x+y))}{k^n} - \frac{f(k^n(x-y))}{k^n} \right. \\
 & \quad \left. - \frac{2(k+1)}{k} \frac{f(k^{n+1}y)}{k^n} + 2(k+1) \frac{f(k^n y)}{k^n}, \frac{t}{2}\right) \quad (\text{for sufficiently large } n) \\
 & \geq N'\left(\varphi(x, y), \frac{t}{2|k|^n |r|^n}\right) \longrightarrow 1 \quad \text{as } n \longrightarrow +\infty,
 \end{aligned} \tag{3.16}$$

for all $x, y \in X$ and all $t > 0$, which implies that

$$A(k(x+y)) + A(k(x-y)) = A(kx+y) + A(kx-y) + \frac{2(k+1)}{k} A(ky) - 2(k+1)A(y). \tag{3.17}$$

Hence the mapping $A : X \rightarrow Y$ is additive, as desired.

To prove the uniqueness, let there be another mapping $L : X \rightarrow Y$ which satisfies the inequality (3.3). Since $L(k^n x) = k^n L(x)$ for all $x \in X$, we have

$$\begin{aligned}
 N(A(x) - L(x), t) &= N\left(k^n A\left(\frac{x}{k^n}\right) - k^n L\left(\frac{x}{k^n}\right), t\right) \\
 &\geq \min\left\{N\left(k^n A\left(\frac{x}{k^n}\right) - k^n f\left(\frac{x}{k^n}\right), \frac{t}{2}\right), N\left(k^n f\left(\frac{x}{k^n}\right) - k^n L\left(\frac{x}{k^n}\right), \frac{t}{2}\right)\right\} \\
 &\geq N'\left(\varphi\left(0, \frac{x}{k^n}\right), \frac{|2k+2|(1-|k||r|)t}{2|k|^{n+1}|r|}\right) \\
 &\geq N'\left(\varphi(0, x), \frac{|2k+2|(1-|k||r|)t}{2|k|^{n+1}|r|^{n+1}}\right) \rightarrow 1 \quad \text{as } n \rightarrow \infty,
 \end{aligned} \tag{3.18}$$

for all $t > 0$. Therefore $A(x) = L(x)$ for all $x \in X$. This completes the proof. $\square$

Corollary 3.2. *Let X be a normed space and let $(\mathbb{R}, N')$ be a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $p > 1$ such that an odd mapping $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the following inequality:*

$$\begin{aligned}
 N\left(f(k(x+y)) + f(k(x-y)) - f(kx+y) - f(kx-y) - \frac{2(k+1)}{k}f(ky) + 2(k+1)f(y), t\right) \\
 \geq N'(\theta(\|x\|^p + \|y\|^p), t),
 \end{aligned} \tag{3.19}$$

for all $x, y \in X$ and $t > 0$. Then there is a unique additive mapping $A : X \rightarrow Y$ satisfying (1.8) and the inequality

$$N(f(x) - A(x), t) \geq N'\left(\frac{\theta\|x\|^p}{|2k+2|}, \left(\frac{|k|^p - |k|}{|k|}\right)t\right). \tag{3.20}$$

Proof. Let $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ and $|r| = |k|^{-p}$. Applying Theorem 3.1, we get desired results. $\square$

Theorem 3.3. *Let $f : X \rightarrow Y$ be an odd mapping with $f(0) = 0$ satisfying the inequality (3.1) and let $\varphi : X^2 \rightarrow Z$ be a mapping for which there exists a constant $r \in \mathbb{R}$ satisfying $0 < |r| < |k|$ such that*

$$N'(\varphi(x, y), |r|t) \geq N'\left(\varphi\left(\frac{x}{k}, \frac{y}{k}\right), t\right), \tag{3.21}$$

for all $x, y \in X$ and all $t > 0$. Then there exists a unique additive mapping $A : X \rightarrow Y$ satisfying (1.8) and the following inequality:

$$N(f(x) - A(x), t) \geq N'\left(\varphi(0, x), \frac{|2k + 2|(|k| - |r|)t}{|k|}\right), \quad (3.22)$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (3.5) that

$$N\left(\frac{f(kx)}{k} - f(x), \frac{t}{|2k + 2|}\right) \geq N'(\varphi(0, x), t), \quad (3.23)$$

for all $x \in X$ and all $t > 0$. Replacing x by $k^n x$ in (3.41), we obtain

$$N\left(\frac{f(k^{n+1}x)}{k^{n+1}} - \frac{f(k^n x)}{k^n}, \frac{t}{|2k + 2|k^n}\right) \geq N'(\varphi(0, k^n x), t) \geq N'\left(\varphi(0, x), \frac{t}{|r|^n}\right). \quad (3.24)$$

So

$$N\left(\frac{f(k^{n+1}x)}{k^{n+1}} - \frac{f(k^n x)}{k^n}, \frac{|r|^n t}{|2k + 2||k|^n}\right) \geq N'(\varphi(0, x), t), \quad (3.25)$$

for all $x \in X$ and all $t > 0$. Proceeding as in the proof of Theorem 3.1, we obtain that

$$N\left(f(x) - \frac{f(k^n x)}{k^n}, \sum_{j=0}^{n-1} \frac{|r|^j t}{|2k + 2||k|^j}\right) \geq N'(\varphi(0, x), t), \quad (3.26)$$

for all $x \in X$, all $t > 0$, and any integer $n > 0$. So

$$N\left(f(x) - \frac{f(k^n x)}{k^n}, t\right) \geq N'\left(\varphi(0, x), \frac{t}{\sum_{j=0}^{n-1} (|r|^j / |2k + 2||k|^j)}\right). \quad (3.27)$$

The rest of the proof is similar to the proof of Theorem 3.1. $\square$

Corollary 3.4. Let X be a normed space and let $(\mathbb{R}, N')$ be a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $0 < p < 1$ such that an odd mapping $f : X \rightarrow Y$ with $f(0) = 0$ satisfies (3.19). Then there exists a unique additive mapping $A : X \rightarrow Y$ satisfying (1.8) and the inequality

$$N(f(x) - A(x), t) \geq N'\left(\varphi(0, x), \frac{|2k + 2|(|k| - |k|^p)t}{|k|}\right). \quad (3.28)$$

Proof. Let $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ and $|r| = |k|^p$. Applying Theorem 3.3, we get the desired results. $\square$

Theorem 3.5. *Let $f : X \rightarrow Y$ be an even mapping with $f(0) = 0$ satisfying the inequality (3.1) and let $\varphi : X^2 \rightarrow Z$ be a mapping for which there exists a constant $r \in \mathbb{R}$ such that $0 < |r| < 1/k^2$ and that*

$$N'\left(\varphi\left(\frac{x}{k}, \frac{y}{k}\right), t\right) \geq N'\left(\varphi(x, y), \frac{t}{|r|}\right), \quad (3.29)$$

for all $x, y \in X$ and all $t > 0$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ satisfying (1.8) and the inequality

$$N(f(x) - Q(x), t) \geq N'\left(\varphi(0, x), \frac{2(1 - |k^2 r|)t}{|kr|}\right), \quad (3.30)$$

for all $x \in X$ and all $t > 0$.

Proof. Replacing x by kx in (3.1), we get

$$\begin{aligned} N\left(f(k(x+y)) + f(k(x-y)) - f(kx+y) - f(kx-y) - \frac{2(k+1)}{k}f(ky) + 2(k+1)f(y), t\right) \\ \geq N'(\varphi(kx, y), t), \end{aligned} \quad (3.31)$$

for all $x, y \in X$ and all $t > 0$. Putting $x = 0$ and replacing y by x in (3.31), we have

$$N\left(\frac{f(kx)}{k^2} - f(x), \frac{t}{|2k|}\right) \geq N'(\varphi(0, x), t), \quad (3.32)$$

for all $x \in X$ and all $t > 0$. Replacing x by x/k in (3.32), we find

$$N\left(k^2 f\left(\frac{x}{k}\right) - f(x), \frac{|k|t}{2}\right) \geq N'\left(\varphi\left(0, \frac{x}{k}\right), t\right), \quad (3.33)$$

for all $x \in X$ and all $t > 0$. Also, replacing x by x/k^n in (3.33), we obtain

$$N\left(k^{2n+2} f\left(\frac{x}{k^n}\right) - k^{2n} f\left(\frac{x}{k^n}\right), \frac{|k|^{2n+1} t}{2}\right) \geq N'\left(\varphi\left(0, \frac{x}{k^{n+1}}\right), t\right) \geq N'\left(\varphi(0, x), \frac{t}{|r|^{n+1}}\right). \quad (3.34)$$

So

$$N\left(k^{2n+2} f\left(\frac{x}{k^n}\right) - k^{2n} f\left(\frac{x}{k^n}\right), \frac{|k|^{2n+1} |r|^{n+1} t}{2}\right) \geq N'(\varphi(0, x), t), \quad (3.35)$$

for all $x \in X$ and all $t > 0$. Proceeding as in the proof of Theorem 3.1, we obtain that

$$N\left(f(x) - k^{2n}f\left(\frac{x}{k^n}\right), \sum_{j=0}^{n-1} \frac{|k|^{2j+1}|r|^{j+1}t}{2}\right) \geq N'(\varphi(0, x), t), \quad (3.36)$$

for all $x \in X$, all $t > 0$, and any integer $n > 0$. So

$$N\left(f(x) - k^{2n}f\left(\frac{x}{k^n}\right), t\right) \geq N'\left(\varphi(0, x), \frac{t}{\sum_{j=0}^{n-1} (|k|^{2j+1}|r|^{j+1}t/2)}\right). \quad (3.37)$$

The rest of the proof is similar to the proof of Theorem 3.1. $\square$

Corollary 3.6. *Let X be a normed space and let $(\mathbb{R}, N')$ be a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $p > 1$ such that an even mapping $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality (3.19). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ satisfying (1.8) and the inequality*

$$N(f(x) - Q(x), t) \geq N'\left(\theta\|x\|^p, \frac{2(k^{2p} - k^2)t}{|k|}\right). \quad (3.38)$$

Proof. Let $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ and $|r| = |k|^{-2p}$. Applying Theorem 3.5, we get the desired results. $\square$

Theorem 3.7. *Assume that an even mapping $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality (3.1) and $\varphi : X^2 \rightarrow Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < k^2$ such that*

$$N'(\varphi(x, y), |r|t) \geq N'\left(\varphi\left(\frac{x}{k}, \frac{y}{k}\right), t\right), \quad (3.39)$$

for all $x, y \in X$ and all $t > 0$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ satisfying (1.8) and the following inequality

$$N(f(x) - Q(x), t) \geq N'\left(\varphi(0, x), \frac{2(k^2 - |r|)t}{|k|}\right), \quad (3.40)$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (3.32) that

$$N\left(\frac{f(kx)}{k^2} - f(x), \frac{t}{|2k|}\right) \geq N'(\varphi(0, x), t), \quad (3.41)$$

for all $x \in X$ and all $t > 0$. Replacing x by $k^n x$ in (3.41), we obtain

$$N\left(\frac{f(k^{n+1}x)}{k^{2n+2}} - \frac{f(k^n x)}{k^{2n}}, \frac{t}{2|k|^{2n+1}}\right) \geq N'(\varphi(0, k^n x), t) \geq N'\left(\varphi(0, x), \frac{t}{|r|^n}\right), \quad (3.42)$$

for all $x \in X$ and all $t > 0$. So

$$N\left(\frac{f(k^{n+1}x)}{k^{2n+2}} - \frac{f(k^n x)}{k^{2n}}, \frac{|r|^n t}{2|k|^{2n+1}}\right) \geq N'(\varphi(0, x), t), \quad (3.43)$$

for all $x \in X$ and all $t > 0$. So

$$N\left(f(x) - \frac{f(k^n x)}{k^{2n}}, t\right) \geq N'\left(\varphi(0, x), \frac{t}{\sum_{j=0}^{n-1} (|r|^j t / 2|k|^{2j+1})}\right). \quad (3.44)$$

The rest of the proof is similar to the proof of Theorem 3.1. $\square$

Corollary 3.8. *Let X be a normed space and let $(\mathbb{R}, N')$ be a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $0 < p < 1$ such that an even mapping $f : X \rightarrow Y$ with $f(0) = 0$ satisfies (3.19). Then there is a unique quadratic mapping $Q : X \rightarrow Y$ satisfying (1.8) and the inequality*

$$N(f(x) - Q(x), t) \geq N'\left(\varphi(0, x), \frac{2(k^2 - k^{2p})t}{|k|}\right), \quad (3.45)$$

for all $x \in X$, all $t > 0$.

Proof. Let $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ and $|r| = k^{2p}$. Applying Theorem 3.7, we get the desired results. $\square$

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References

- [1] S. M. Ulam, *Problems in Modern Mathematics*, John Wiley & Sons, Inc, New York, NY, USA, 1964.
- [2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
- [3] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
- [4] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

- [5] S. Abbaszadeh, "Intuitionistic fuzzy stability of a quadratic and quartic functional equation," *International Journal of Nonlinear Analysis and Applications*, vol. 1, no. 2, pp. 100–124, 2010.
- [6] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31, Cambridge University Press, Cambridge, UK, 1989.
- [7] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.
- [8] D. G. Bourgin, "Classes of transformations and bordering transformations," *Bulletin of the American Mathematical Society*, vol. 57, pp. 223–237, 1951.
- [9] M. B. Savadkouhi, M. E. Gordji, J. M. Rassias, and N. Ghobadipour, "Approximate ternary Jordan derivations on Banach ternary algebras," *Journal of Mathematical Physics*, vol. 50, no. 4, article 042303, p. 9, 2009.
- [10] P. Găvruta, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
- [11] P. Găvruta and L. Găvruta, "A new method for the generalized Hyers-Ulam-Rassias stability," *Journal of Mathematical Analysis and Applications*, vol. 1, no. 2, pp. 11–18, 2010.
- [12] D. H. Hyers, G. Isac, and T. M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and their Applications, 34, Birkhäuser Boston Inc., Boston, Mass, USA, 1998.
- [13] G. Isac and T. M. Rassias, "On the Hyers-Ulam stability of ψ -additive mappings," *Journal of Approximation Theory*, vol. 72, no. 2, pp. 131–137, 1993.
- [14] C. Park and M. E. Gordji, "Comment on "Approximate ternary Jordan derivations on Banach ternary algebras" [B. Savadkouhi Journal of Mathematical Physics vol. 50, article 042303, 2009]," *Journal of Mathematical Physics*, vol. 51, no. 4, 2010.
- [15] C. Park and A. Najati, "Generalized additive functional inequalities in Banach algebras," *International Journal of Nonlinear Analysis and Applications*, vol. 1, no. 2, pp. 54–62, 2010.
- [16] C. Park and T. M. Rassias, "Isomorphisms in unital C^* -algebras," *International Journal of Nonlinear Analysis and Applications*, vol. 1, no. 2, pp. 1–10, 2010.
- [17] T. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.
- [18] P. Kannappan, "Quadratic functional equation and inner product spaces," *Results in Mathematics*, vol. 27, no. 3–4, pp. 368–372, 1995.
- [19] F. Skof, "Local properties and approximation of operators," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, pp. 113–129, 1983.
- [20] P. W. Cholewa, "Remarks on the stability of functional equations," *Aequationes Mathematicae*, vol. 27, no. 1–2, pp. 76–86, 1984.
- [21] S. Czerwik, "On the stability of the quadratic mapping in normed spaces," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, pp. 59–64, 1992.
- [22] A. Grabiec, "The generalized Hyers-Ulam stability of a class of functional equations," *Publicationes Mathematicae Debrecen*, vol. 48, no. 3–4, pp. 217–235, 1996.
- [23] C. Borelli and G. L. Forti, "On a general Hyers-Ulam stability result," *International Journal of Mathematics and Mathematical Sciences*, vol. 18, no. 2, pp. 229–236, 1995.
- [24] C.-G. Park, "Generalized quadratic mappings in several variables," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 57, no. 5–6, pp. 713–722, 2004.
- [25] C.-G. Park, "On the stability of the quadratic mapping in Banach modules," *Journal of Mathematical Analysis and Applications*, vol. 276, no. 1, pp. 135–144, 2002.
- [26] A. Ebadian, A. Najati, and M. Eshaghi Gordji, "On approximate additive-quartic and quadratic-cubic functional equations in two variables on abelian groups," *Results in Mathematics*, vol. 58, no. 1–2, pp. 39–53, 2010.
- [27] M. Eshaghi Gordji, "Stability of a functional equation deriving from quartic and additive functions," *Bulletin of the Korean Mathematical Society*, vol. 47, no. 3, pp. 491–502, 2010.
- [28] M. Eshaghi Gordji, "Stability of an additive-quadratic functional equation of two variables in F-spaces," *Journal of Nonlinear Science and its Applications*, vol. 2, no. 4, pp. 251–259, 2009.
- [29] M. Eshaghi Gordji and M. B. Savadkouhi, "Stability of cubic and quartic functional equations in non-Archimedean spaces," *Acta Applicandae Mathematicae*, vol. 110, no. 3, pp. 1321–1329, 2010.
- [30] M. Eshaghi Gordji and M. B. Savadkouhi, "Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1198–1202, 2010.

- [31] M. Eshaghi Gordji and M. B. Savadkouhi, "Approximation of generalized homomorphisms in quasi-Banach algebras," *Mathematical Journal of the Ovidius University of Constantza*, vol. 17, no. 2, pp. 203–213, 2009.
- [32] M. Eshaghi Gordji and M. Bavand Savadkouhi, "On approximate cubic homomorphisms," *Advances in Difference Equations*, vol. 2009, Article ID 618463, 11 pages, 2009.
- [33] M. Eshaghi Gordji, S. Zolfaghari, J. M. Rassias, and M. B. Savadkouhi, "Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces," *Abstract and Applied Analysis*, vol. 2009, Article ID 417473, 14 pages, 2009.
- [34] M. Eshaghi Gordji and M. B. Savadkouhi, "Stability of mixed type cubic and quartic functional equations in random normed spaces," *Journal of Inequalities and Applications*, vol. 2009, Article ID 527462, 9 pages, 2009.
- [35] M. Eshaghi Gordji, A. Ebadian, and S. Zolfaghari, "Stability of a functional equation deriving from cubic and quartic functions," *Abstract and Applied Analysis*, vol. 2008, Article ID 801904, 17 pages, 2008.
- [36] M. Eshaghi Gordji, S. Kaboli Gharetapeh, J. M. Rassias, and S. Zolfaghari, "Solution and stability of a mixed type additive, quadratic, and cubic functional equation," *Advances in Difference Equations*, vol. 2009, Article ID 826130, 17 pages, 2009.
- [37] M. Eshaghi-Gordji, S. Kaboli-Gharetapeh, C. Park, and S. Zolfaghari, "Stability of an additive-cubic-quartic functional equation," *Advances in Difference Equations*, vol. 2009, Article ID 395693, 20 pages, 2009.
- [38] M. Eshaghi Gordji, T. Karimi, and S. Kaboli Gharetapeh, "Approximately n -Jordan homomorphisms on Banach algebras," *Journal of Inequalities and Applications*, vol. 2009, Article ID 870843, 8 pages, 2009.
- [39] M. Eshaghi Gordji and H. Khodaei, "Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 11, pp. 5629–5643, 2009.
- [40] M. Eshaghi Gordji, H. Khodaei, and R. Khodabakhsh, "General quartic-cubic-quadratic functional equation in non-Archimedean normed spaces," *"Politehnica" University of Bucharest Scientific Bulletin. Series A*, vol. 72, no. 3, pp. 69–84, 2010.
- [41] M. Eshaghi Gordji and A. Najati, "Approximately J^* -homomorphisms: a fixed point approach," *Journal of Geometry and Physics*, vol. 60, no. 5, pp. 809–814, 2010.
- [42] M. Eshaghi Gordji, S. Zolfaghari, S. Kaboli Gharetapeh, A. Ebadian, and C. Park, "Solution and stability of generalized mixed type additive and quadratic functional equation in non-Archimedean spaces," to appear in *Annali dell'Universita di Ferrara*.
- [43] R. Farokhzad and S. A. R. Hosseinioun, "Perturbations of Jordan higher derivations in Banach ternary algebras: an alternative fixed point approach," *International Journal of Nonlinear Analysis and Applications*, vol. 1, no. 1, pp. 42–53, 2010.
- [44] M. Eshaghi Gordji and H. Khodaei, "On the generalized Hyers-Ulam-Rassias stability of quadratic functional equations," *Abstract and Applied Analysis*, vol. 2009, Article ID 923476, 11 pages, 2009.
- [45] M. Eshaghi Gordji, S. Kaboli Gharetapeh, J. M. Rassias, and S. Zolfaghari, "Solution and stability of a mixed type additive, quadratic, and cubic functional equation," *Advances in Difference Equations*, vol. 2009, Article ID 826130, 17 pages, 2009.
- [46] M. Eshaghi Gordji, S. Kaboli Gharetapeh, E. Rashidi, T. Karimi, and M. Aghaei, "Ternary Jordan-derivations in C^* -ternary algebras," *Journal of Computational Analysis and Applications*, vol. 12, no. 2, pp. 463–470, 2010.
- [47] M. Eshaghi Gordji, J. M. Rassias, and N. Ghobadipour, "Generalized Hyers-Ulam stability of generalized (N, K) -derivations," *Abstract and Applied Analysis*, vol. 2009, Article ID 437931, 8 pages, 2009.
- [48] T. Bag and S. K. Samanta, "Finite dimensional fuzzy normed linear spaces," *Journal of Fuzzy Mathematics*, vol. 11, no. 3, pp. 687–705, 2003.
- [49] T. Bag and S. K. Samanta, "Fuzzy bounded linear operators," *Fuzzy Sets and Systems*, vol. 151, no. 3, pp. 513–547, 2005.
- [50] D. Miheţ and V. Radu, "On the stability of the additive Cauchy functional equation in random normed spaces," *Journal of Mathematical Analysis and Applications*, vol. 343, no. 1, pp. 567–572, 2008.