Research Article

Smooth Solutions of a Class of Iterative Functional Differential Equations

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By Faà di Bruno's formula, using the fixed-point theorems of Schauder and Banach, we study the existence and uniqueness of smooth solutions of an iterative functional differential equation $x'(t) = 1/(c_0x^{[0]}(t) + c_1x^{[1]}(t) + \cdots + c_mx^{[m]}(t))$.

1. Introduction

There has been a lot of monographs and research articles to discuss the kinds of solutions of functional differential equations since the publication of Jack Hale's paper [1]. Several papers discussed the iterative functional differential equations of the form

$$x'(t) = H\left(x^{[0]}(t), x^{[1]}(t), \dots, x^{[m]}(t)\right), \quad (1.1)$$

where $x^{[0]}(t) = t$, $x^{[1]}(t) = x(t)$, $x^{[k]}(t) = x(x^{[k-1]}(t))$, $k = 2, \dots, m$. More specifically, Eder [2] considered the functional differential equation

$$x'(t) = x^{[2]}(t) \quad (1.2)$$

and proved that every solution either vanishes identically or is strictly monotonic. Furthermore, Fečkan [3] and Wang [4] studied the equation

$$x'(t) = f\left(x^{[2]}(t)\right) \quad (1.3)$$

with different conditions. Staněk [5] considered the equation

$$x'(t) = x(t) + x^{[2]}(t) \quad (1.4)$$

and obtained every solution either vanishes identically or is strictly monotonic. Si and his coauthors [6, 7] studied the following equations:

$$\begin{aligned} x'(t) &= x^{[m]}(t), \\ x'(t) &= \frac{1}{x^{[m]}(t)}, \end{aligned} \quad (1.5)$$

$$x'(t) = \frac{1}{c_0 x^{[0]}(t) + c_1 x^{[1]}(t) + \dots + c_m x^{[m]}(t)} \quad (1.6)$$

and established sufficient conditions for the existence of analytic solutions. Especially in [8, 9], the smooth solutions of the following equations:

$$\begin{aligned} x'(t) &= \sum_{j=1}^m a_j x^{[j]}(t) + F(t), \\ x'(t) &= \sum_{j=1}^m a_j(t) x^{[j]}(t) + F(t), \end{aligned} \quad (1.7)$$

have been studied by the fixed-point theorems of Schauder and Banach.

A smooth function is taken to mean one that has a number of continuous derivatives and for which the highest continuous derivative is also Lipschitz. Let $x \in C^n$ if $x', \dots, x^{(n)}$ are continuous, $C^n(I, I)$ is the set in which $x \in C^n$ and maps a closed interval I into I . As in [9], we, using the same symbols, denote the norm

$$\|x\|_n = \sum_{k=0}^n \|x^{(k)}\|, \quad \|x\| = \max_{t \in I} |x(t)|, \quad (1.8)$$

then $C^n(I, R)$ with $\|\cdot\|_n$ is a Banach space, and $C^n(I, I)$ is a subset of $C^n(I, R)$. For given $M_i > 0$ ($i = 1, 2, \dots, n+1$), let

$$\begin{aligned} \Omega(M_1, \dots, M_{n+1}; I) &= \left\{ x \in C^n(I, I) : \left| x^{(i)}(t) \right| \leq M_i, i = 1, 2, \dots, n; \right. \\ &\quad \left. \left| x^{(n)}(t_1) - x^{(n)}(t_2) \right| \leq M_{n+1} |t_1 - t_2|, t, t_1, t_2 \in I \right\}. \end{aligned} \quad (1.9)$$

For convenience, we will make use of the notation

$$x_{ij}(t) = x^{(i)}(x^{[j]}(t)), \quad x_{*jk}(t) = \left(x^{[j]}(t) \right)^{(k)}, \quad (1.10)$$

where i, j , and k are nonnegative integers. Let I be a closed interval in R . By induction, we may prove that

$$x_{*jk}(t) = P_{jk}(x_{10}(t), \dots, x_{1,j-1}(t); \dots; x_{k0}(t), \dots, x_{k,j-1}(t)), \quad (1.11)$$

$$\beta_{jk} = P_{jk} \left(\overbrace{x'(\xi), \dots, x'(\xi)}^{j \text{ terms}}; \dots; \overbrace{x^{(k)}(\xi), \dots, x^{(k)}(\xi)}^{j \text{ terms}} \right), \quad (1.12)$$

$$H_{jk} = P_{jk} \left(\overbrace{1, \dots, 1}^{j \text{ terms}}; \overbrace{M_2, \dots, M_2}^{j \text{ terms}}; \dots; \overbrace{M_k, \dots, M_k}^{j \text{ terms}} \right), \quad (1.13)$$

where P_{jk} is a uniquely defined multivariate polynomial with nonnegative coefficients. The proof can be found in [8].

In order to seek a solution $x(t)$ of (1.6), in $C^n(I, I)$ such that ξ is a fixed point of the function $x(t)$, that is, $x(\xi) = \xi$, it is natural to seek an interval I of the form $[\xi - \delta, \xi + \delta]$ with $\delta > 0$.

Let us define

$$\begin{aligned} X(\xi; \xi_0, \dots, \xi_n; 1, M_2, \dots, M_{n+1}; I) \\ = \left\{ x \in \Omega(1, M_2, \dots, M_{n+1}; I) : x(\xi) = \xi_0 = \xi, x^{(i)}(\xi) = \xi_i, i = 1, 2, \dots, n \right\}. \end{aligned} \quad (1.14)$$

2. Smooth Solutions of (1.6)

In this section, we will prove the existence theorem of smooth solutions for (1.6). First of all, we have the inequalities in the following for all $x(t), y(t) \in X$:

$$\left| x^{[j]}(t_1) - x^{[j]}(t_2) \right| \leq |t_1 - t_2|, \quad t_1, t_2 \in I, \quad j = 0, 1, \dots, m, \quad (2.1)$$

$$\left\| x^{[j]} - x^{[l]} \right\| \leq j \|x - y\|, \quad j = 1, \dots, m, \quad (2.2)$$

$$\|x - y\| \leq \delta^n \|x^{(n)} - y^{(n)}\|, \quad (2.3)$$

and the proof can be found in [9].

Theorem 2.1. Let $I = [\xi - \delta, \xi + \delta]$, where ξ and δ satisfy

$$\xi \geq \frac{1}{|c_0| - \sum_{i=1}^m |c_i|}, \quad 0 < \delta \leq \xi - \frac{1}{|c_0| - \sum_{i=1}^m |c_i|}, \quad (2.4)$$

where $|c_0| > \sum_{i=0}^m |c_i|$, then (1.6) has a solution in

$$X(\xi; \xi_0, \dots, \xi_n; 1, M_2, \dots, M_{n+1}; I), \quad (2.5)$$

provided the following conditions hold:

(i)

$$\xi_1 = \xi^{-1} \left(\sum_{i=0}^m c_i \right)^{-1}, \quad (2.6)$$

$$\begin{aligned} \xi_k = & \sum \frac{(-1)^s (k-1)!s!}{s_1!s_2! \cdots s_{k-1}!} \left(\xi \sum_{i=0}^m c_i \right)^{-s-1} \left(\frac{\sum_{i=0}^m c_i \beta_{i1}}{1!} \right)^{s_1} \\ & \times \left(\frac{\sum_{i=0}^m c_i \beta_{i2}}{2!} \right)^{s_2} \cdots \left(\frac{\sum_{i=0}^m c_i \beta_{ik-1}}{(k-1)!} \right)^{s_{k-1}}, \end{aligned} \quad (2.7)$$

where $k = 2, \dots, n$, and the sum is over all nonnegative integer solutions of the Diophantine equation $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$,

(ii)

$$\begin{aligned} & \sum \frac{(k-1)!s!}{s_1!s_2! \cdots s_{k-1}!} (\xi - \delta)^{-s-1} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left(\frac{\sum_{i=0}^m |c_i| H_{i1}}{1!} \right)^{s_1} \\ & \times \left(\frac{\sum_{i=0}^m |c_i| H_{i2}}{2!} \right)^{s_2} \cdots \left(\frac{\sum_{i=0}^m |c_i| H_{ik-1}}{(k-1)!} \right)^{s_{k-1}} \leq M_k, \quad k = 2, \dots, n, \end{aligned} \quad (2.8)$$

where $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$,

(iii)

$$\begin{aligned} & \sum \frac{(n-1)!s!}{s_1!s_2! \cdots s_{n-1}!1!^{s_1}2!^{s_2} \cdots (n-1)!^{s_{n-1}}} \\ & \times \left[(s+1)(\xi - \delta)^{-s-2} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-s-2} \left(\sum_{i=0}^m |c_i| H_{i1} \right)^{s_1+1} \left(\sum_{i=0}^m |c_i| H_{i2} \right)^{s_2} \right. \\ & \quad \times \cdots \times \left(\sum_{i=0}^m |c_i| H_{in-1} \right)^{s_{n-1}} + s_1(\xi - \delta)^{-s-1} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left(\sum_{i=0}^m |c_i| H_{i1} \right)^{s_1-1} \\ & \quad \times \left(\sum_{i=0}^m |c_i| H_{i2} \right)^{s_2+1} \left(\sum_{i=0}^m |c_i| H_{i3} \right)^{s_3} \cdots \left(\sum_{i=0}^m |c_i| H_{in-1} \right)^{s_{n-1}} \\ & \quad + \cdots + s_{n-1}(\xi - \delta)^{-s-1} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left(\sum_{i=0}^m |c_i| H_{i1} \right)^{s_1} \cdots \left(\sum_{i=0}^m |c_i| H_{in-2} \right)^{s_{n-2}} \\ & \quad \left. \times \left(\sum_{i=0}^m |c_i| H_{in-1} \right)^{s_{n-1}-1} \left(\sum_{i=0}^m |c_i| H_{in} \right) \right] \\ & \leq M_{n+1}, \end{aligned} \quad (2.9)$$

where $s_1 + 2s_2 + \cdots + (n-1)s_{n-1} = n-1$ and $s = s_1 + s_2 + \cdots + s_{n-1}$,

Proof. Define an operator T from X into $C^n(I, I)$ by

$$(Tx)(t) = \xi + \int_{\xi}^t \frac{1}{c_0 x^{[0]}(s) + c_1 x^{[1]}(s) + \cdots + c_m x^{[m]}(s)} ds. \quad (2.10)$$

We will prove that for any $x \in X$, $Tx \in X$,

$$|(Tx)(t) - \xi| = \left| \int_{\xi}^t \frac{1}{\sum_{i=0}^m c_i x^{[i]}(s)} ds \right| \leq \left((\xi - \delta) \left(|c_0| - \sum_{i=1}^m |c_i| \right) \right)^{-1} |t - \xi| \leq \delta, \quad (2.11)$$

where the second inequality is from (2.4) and $x(I) \subseteq I$. Thus, $(Tx)(I) \subseteq I$.

It is easy to see that

$$(Tx)'(t) = \frac{1}{\sum_{i=0}^m c_i x^{[i]}(t)}, \quad (2.12)$$

and by Faà di Bruno's formula, for $k = 2, \dots, n$, we have

$$\begin{aligned} (Tx)^{(k)}(t) &= \left(\frac{1}{\sum_{i=0}^m c_i x^{[i]}(t)} \right)^{(k-1)} = \sum \frac{(-1)^s (k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \frac{1}{\left(\sum_{i=0}^m c_i x^{[i]}(t) \right)^{s+1}} \left(\frac{\left(\sum_{i=0}^m c_i x^{[i]}(t) \right)'}{1!} \right)^{s_1} \\ &\quad \times \left(\frac{\left(\sum_{i=0}^m c_i x^{[i]}(t) \right)''}{2!} \right)^{s_2} \cdots \left(\frac{\left(\sum_{i=0}^m c_i x^{[i]}(t) \right)^{(k-1)}}{(k-1)!} \right)^{s_{k-1}} \\ &= \sum \frac{(-1)^s (k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \frac{1}{\left(\sum_{i=0}^m c_i x^{[i]}(t) \right)^{s+1}} \left(\frac{\sum_{i=0}^m c_i x_{*i1}(t)}{1!} \right)^{s_1} \\ &\quad \times \left(\frac{\sum_{i=0}^m c_i x_{*i2}(t)}{2!} \right)^{s_2} \cdots \left(\frac{\sum_{i=0}^m c_i x_{*ik-1}(t)}{(k-1)!} \right)^{s_{k-1}}, \end{aligned} \quad (2.13)$$

where the sum is over all nonnegative integer solutions of the Diophantine equation $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$.

Furthermore, note $(Tx)(\xi) = \xi$, by (2.6) and (2.7),

$$\begin{aligned}
 (Tx)'(\xi) &= \frac{1}{\sum_{i=0}^m c_i x^{[i]}(\xi)} = \frac{1}{\xi \sum_{i=0}^m c_i} = \xi_1, \\
 (Tx)^{(k)}(\xi) &= \sum \frac{(-1)^s (k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \frac{1}{(\xi \sum_{i=0}^m c_i)^{s+1}} \left(\frac{\sum_{i=0}^m c_i x_{*i1}(\xi)}{1!} \right)^{s_1} \\
 &\quad \times \left(\frac{\sum_{i=0}^m c_i x_{*i2}(\xi)}{2!} \right)^{s_2} \cdots \left(\frac{\sum_{i=0}^m c_i x_{*ik-1}(\xi)}{(k-1)!} \right)^{s_{k-1}} \\
 &= \sum \frac{(-1)^s (k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \frac{1}{(\xi \sum_{i=0}^m c_i)^{s+1}} \left(\frac{\sum_{i=0}^m c_i \beta_{i1}}{1!} \right)^{s_1} \\
 &\quad \times \left(\frac{\sum_{i=0}^m c_i \beta_{i2}}{2!} \right)^{s_2} \cdots \left(\frac{\sum_{i=0}^m c_i \beta_{ik-1}}{(k-1)!} \right)^{s_{k-1}} \\
 &= \xi_k, \quad k = 2, \dots, n,
 \end{aligned} \tag{2.14}$$

where $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$. Thus, $(Tx)^{(k)}(\xi) = \xi_k$ for $k = 0, 1, \dots, n$,

$$|(Tx)'(t)| = \left| \frac{1}{\sum_{i=0}^m c_i x^{[i]}(t)} \right| \leq \left((\xi - \delta) \left(|c_0| - \sum_{i=1}^m |c_i| \right) \right)^{-1} \leq 1 = M_1, \tag{2.15}$$

By (2.8), we have

$$\begin{aligned}
 |(Tx)^{(k)}(t)| &\leq \sum \frac{(k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \frac{1}{|\sum_{i=0}^m c_i x^{[i]}(t)|^{s+1}} \left(\frac{\sum_{i=0}^m |c_i| |x_{*i1}(t)|}{1!} \right)^{s_1} \\
 &\quad \times \left(\frac{\sum_{i=0}^m |c_i| |x_{*i2}(t)|}{2!} \right)^{s_2} \cdots \left(\frac{\sum_{i=0}^m |c_i| |x_{*ik-1}(t)|}{(k-1)!} \right)^{s_{k-1}} \\
 &\leq \sum \frac{(k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} (\xi - \delta)^{-s-1} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left(\frac{\sum_{i=0}^m |c_i| |H_{i1}|}{1!} \right)^{s_1} \\
 &\quad \times \left(\frac{\sum_{i=0}^m |c_i| |H_{i2}|}{2!} \right)^{s_2} \cdots \left(\frac{\sum_{i=0}^m |c_i| |H_{ik-1}|}{(k-1)!} \right)^{s_{k-1}} \\
 &\leq M_k, \quad k = 2, \dots, n,
 \end{aligned} \tag{2.16}$$

where $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$.

Finally,

$$\begin{aligned}
& \left| (Tx)^{(n)}(t_1) - (Tx)^{(n)}(t_2) \right| \\
& \leq \sum \frac{(n-1)!s!}{s_1!s_2!\cdots s_{n-1}!1!^{s_1}2!^{s_2}\cdots (n-1)!^{s_{n-1}}} \\
& \quad \times \left| \left(\sum_{i=0}^m c_i x^{[i]}(t_1) \right)^{-s-1} \left(\sum_{i=0}^m c_i x_{*i1}(t_1) \right)^{s_1} \left(\sum_{i=0}^m c_i x_{*i2}(t_1) \right)^{s_2} \cdots \left(\sum_{i=0}^m c_i x_{*in-1}(t_1) \right)^{s_{n-1}} \right. \\
& \quad \left. - \left(\sum_{i=0}^m c_i x^{[i]}(t_2) \right)^{-s-1} \left(\sum_{i=0}^m c_i x_{*i1}(t_2) \right)^{s_1} \left(\sum_{i=0}^m c_i x_{*i2}(t_2) \right)^{s_2} \cdots \left(\sum_{i=0}^m c_i x_{*in-1}(t_2) \right)^{s_{n-1}} \right| \\
& \leq \sum \frac{(n-1)!s!}{s_1!s_2!\cdots s_{n-1}!1!^{s_1}2!^{s_2}\cdots (n-1)!^{s_{n-1}}} \\
& \quad \times \left[\left| \frac{1}{\left(\sum_{i=0}^m c_i x^{[i]}(t_1) \right)^{s+1}} - \frac{1}{\left(\sum_{i=0}^m c_i x^{[i]}(t_2) \right)^{s+1}} \right| \left(\sum_{i=0}^m |c_i| x_{*i1}(t_1) \right)^{s_1} \right. \\
& \quad \times \cdots \left(\sum_{i=0}^m |c_i| x_{*in-1}(t_1) \right)^{s_{n-1}} + \left| \left(\sum_{i=0}^m c_i x^{[i]}(t_2) \right)^{-s-1} \right| \\
& \quad \times \left| \left(\sum_{i=0}^m c_i x_{*i1}(t_1) \right)^{s_1} - \left(\sum_{i=0}^m c_i x_{*i1}(t_2) \right)^{s_1} \right| \left(\sum_{i=0}^m |c_i| x_{*i2}(t_1) \right)^{s_2} \cdots \\
& \quad \times \left(\sum_{i=0}^m |c_i| x_{*in-1}(t_1) \right)^{s_{n-1}} + \cdots + \left| \left(\sum_{i=0}^m c_i x^{[i]}(t_2) \right)^{-s-1} \right| \left(\sum_{i=0}^m |c_i| x_{*i1}(t_2) \right)^{s_1} \\
& \quad \times \left(\sum_{i=0}^m |c_i| x_{*i2}(t_2) \right)^{s_2} \cdots \left(\sum_{i=0}^m |c_i| x_{*in-2}(t_2) \right)^{s_{n-2}} \\
& \quad \times \left| \left(\sum_{i=0}^m c_i x_{*in-1}(t_1) \right)^{s_{n-1}} - \left(\sum_{i=0}^m c_i x_{*in-1}(t_2) \right)^{s_{n-1}} \right| \Bigg] \\
& \leq \sum \frac{(n-1)!s!}{s_1!s_2!\cdots s_{n-1}!1!^{s_1}2!^{s_2}\cdots (n-1)!^{s_{n-1}}} \\
& \quad \times \left[(s+1)(\xi-\delta)^{-s-2} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-s-2} \left(\sum_{i=0}^m |c_i| H_{i1} \right)^{s_1+1} \left(\sum_{i=0}^m |c_i| H_{i2} \right)^{s_2} \right. \\
& \quad \times \cdots \times \left(\sum_{i=0}^m |c_i| H_{in-1} \right)^{s_{n-1}} + s_1(\xi-\delta)^{-s-1} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left(\sum_{i=0}^m |c_i| H_{i1} \right)^{s_1-1} \\
& \quad \times \left(\sum_{i=0}^m |c_i| H_{i2} \right)^{s_2+1} \left(\sum_{i=0}^m |c_i| H_{i3} \right)^{s_3} \cdots \left(\sum_{i=0}^m |c_i| H_{in-1} \right)^{s_{n-1}} \\
& \quad + \cdots + s_{n-1}(\xi-\delta)^{-s-1} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left(\sum_{i=0}^m |c_i| H_{i1} \right)^{s_1} \cdots \left(\sum_{i=0}^m |c_i| H_{in-2} \right)^{s_{n-2}} \\
& \quad \times \left(\sum_{i=0}^m |c_i| H_{in-1} \right)^{s_{n-1}-1} \left(\sum_{i=0}^m |c_i| H_{in} \right) \Bigg] |t_1 - t_2|,
\end{aligned} \tag{2.17}$$

where $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$. By (2.9), we see that

$$\left| (Tx)^{(n)}(t_1) - (Tx)^{(n)}(t_2) \right| \leq M_{n+1}|t_1 - t_2|. \quad (2.18)$$

Now, we can say that T is an operator from X into itself.

Next, we will show that T is continuous. Let $x, y \in X$, then

$$\begin{aligned} \|Tx - Ty\|_n &= \|Tx - Ty\| + \left\| (Tx)' - (Ty)' \right\| + \sum_{k=2}^n \left\| (Tx)^{(k)} - (Ty)^{(k)} \right\| \\ &= \max_{t \in I} \left| \int_{\xi}^t \left(\frac{1}{\sum_{i=0}^m c_i x^{[i]}(s)} - \frac{1}{\sum_{i=0}^m c_i y^{[i]}(s)} \right) ds \right| \\ &\quad + \max_{t \in I} \left| \frac{1}{\sum_{i=0}^m c_i x^{[i]}(t)} - \frac{1}{\sum_{i=0}^m c_i y^{[i]}(t)} \right| \\ &\quad + \sum_{k=2}^n \max_{t \in I} \left| \sum \frac{(k-1)!s!}{s_1!s_2! \cdots s_{k-1}!1!^{s_1}2!^{s_2} \cdots (k-1)!^{s_{k-1}}} \right. \\ &\quad \times \left[\frac{1}{\left(\sum_{i=0}^m c_i x^{[i]}(t) \right)^{s+1}} \left(\sum_{i=0}^m c_i x_{*i1}(t) \right)^{s_1} \left(\sum_{i=0}^m c_i x_{*i2}(t) \right)^{s_2} \cdots \right. \\ &\quad \times \left(\sum_{i=0}^m c_i x_{*ik-1}(t) \right)^{s_{k-1}} - \frac{1}{\left(\sum_{i=0}^m c_i y^{[i]}(t) \right)^{s+1}} \left(\sum_{i=0}^m c_i y_{*i1}(t) \right)^{s_1} \\ &\quad \times \left(\sum_{i=0}^m c_i y_{*i2}(t) \right)^{s_2} \cdots \left. \left(\sum_{i=0}^m c_i y_{*ik-1}(t) \right)^{s_{k-1}} \right] \Bigg| \\ &\leq \delta(\xi - \delta)^{-2} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left(\sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \\ &\quad + (\xi - \delta)^{-2} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left(\sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \\ &\quad + \sum_{k=2}^n \max_{t \in I} \sum \frac{(k-1)!s!}{s_1!s_2! \cdots s_{k-1}!1!^{s_1}2!^{s_2} \cdots (k-1)!^{s_{k-1}}} \\ &\quad \times \left[\left| \left(\sum_{i=0}^m c_i x^{[i]}(t) \right)^{-s-1} - \left(\sum_{i=0}^m c_i y^{[i]}(t) \right)^{-s-1} \right| \left(\sum_{i=0}^m |c_i| |x_{*i1}(t)| \right)^{s_1} \cdots \right. \\ &\quad \times \left(\sum_{i=0}^m |c_i| |x_{*ik-1}(t)| \right)^{s_{k-1}} + \left| \left(\sum_{i=0}^m c_i y^{[i]}(t) \right)^{-s-1} \right| \\ &\quad \times \left| \left(\sum_{i=0}^m c_i x_{*i1}(t) \right)^{s_1} - \left(\sum_{i=0}^m c_i y_{*i1}(t) \right)^{s_1} \right| \end{aligned}$$

$$\begin{aligned}
& \times \left(\sum_{i=0}^m |c_i| |x_{*i2}(t)| \right)^{s_2} \cdots \left(\sum_{i=0}^m |c_i| |x_{*ik-1}(t)| \right)^{s_{k-1}} \\
& + \cdots + \left| \left(\sum_{i=0}^m c_i y^{[i]}(t) \right)^{-s-1} \left(\sum_{i=0}^m |c_i| |y_{*i1}(t)| \right)^{s_1} \cdots \left(\sum_{i=0}^m |c_i| |y_{*ik-2}(t)| \right)^{s_{k-2}} \right. \\
& \times \left. \left| \left(\sum_{i=0}^m c_i x_{*ik-1}(t) \right)^{s_{k-1}} - \left(\sum_{i=0}^m c_i y_{*ik-1}(t) \right)^{s_{k-1}} \right| \right] \\
& \leq (\xi - \delta)^{-2} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left(|c_0| + \sum_{i=1}^m i |c_i| \right) \|x - y\| \\
& + \sum_{k=2}^n \sum \frac{(k-1)!s!}{s_1!s_2! \cdots s_{k-1}! 1! 2!^{s_1} 2!^{s_2} \cdots (k-1)!^{s_{k-1}}} \\
& \times \left[(s+1)(\xi - \delta)^{-s-2} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-s-2} \left(\sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \right. \\
& \times \left(\sum_{i=0}^m |c_i| H_{i1} \right)^{s_1} \cdots \left(\sum_{i=0}^m |c_i| H_{ik-1} \right)^{s_{k-1}} \\
& + s_1 (\xi - \delta)^{-s-1} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left(\sum_{i=0}^m |c_i| H_{i1} \right)^{s_1-1} \\
& \times \left(\sum_{i=0}^m |c_i| H_{i2} \right)^{s_2+1} \cdots \left(\sum_{i=0}^m |c_i| H_{ik-1} \right)^{s_{k-1}} \left(\sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \\
& + \cdots + s_{k-1} (\xi - \delta)^{-s-1} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left(\sum_{i=0}^m |c_i| H_{i1} \right)^{s_1} \cdots \\
& \times \left(\sum_{i=0}^m |c_i| H_{ik-2} \right)^{s_{k-2}} \left(\sum_{i=0}^m |c_i| H_{ik-1} \right)^{s_{k-1}-1} \left(\sum_{i=0}^m c_i H_{ik} \right) \\
& \times \left. \left(\sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \right] \\
& + \delta^{n+1} (\xi - \delta)^{-2} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left(|c_0| + \sum_{i=1}^m i |c_i| \right) \|x^{(n)} - y^{(n)}\|,
\end{aligned} \tag{2.19}$$

where $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$.

Moreover, we can find some constants P_k such that

$$\begin{aligned}
& \sum_{k=2}^n \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!1!^{s_1}2!^{s_2}\cdots (k-1)!^{s_{k-1}}} \\
& \times \left[(s+1)(\xi-\delta)^{-s-2} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-s-2} \left(\sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \times \right. \\
& \quad \times \left(\sum_{i=0}^m |c_i| H_{i1} \right)^{s_1} \cdots \left(\sum_{i=0}^m |c_i| H_{ik-1} \right)^{s_{k-1}} \\
& \quad + s_1(\xi-\delta)^{-s-1} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left(\sum_{i=0}^m |c_i| H_{i1} \right)^{s_1-1} \\
& \quad \times \left(\sum_{i=0}^m |c_i| H_{i2} \right)^{s_2+1} \cdots \left(\sum_{i=0}^m |c_i| H_{ik-1} \right)^{s_{k-1}} \left(\sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \\
& \quad + \cdots + s_{k-1}(\xi-\delta)^{-s-1} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left(\sum_{i=0}^m |c_i| H_{i1} \right)^{s_1} \cdots \\
& \quad \times \left(\sum_{i=0}^m |c_i| H_{ik-2} \right)^{s_{k-2}} \left(\sum_{i=0}^m |c_i| H_{ik-1} \right)^{s_{k-1}-1} \left(\sum_{i=0}^m |c_i| H_{ik} \right) \\
& \quad \times \left. \left(\sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \right] \\
& \leq \sum_{k=1}^{n-1} P_k(\xi, \delta, c_i, H_{ij}) \|x^{(k)} - y^{(k)}\|,
\end{aligned} \tag{2.20}$$

where

$$P_k(\xi, \delta, c_i, H_{ij}) = P(\xi, \delta; c_1, \dots, c_m; H_{11}, \dots, H_{1k+1}; \dots; H_{m1}, \dots, H_{mk+1};) \tag{2.21}$$

are the positive constants depend on ξ, δ, c_i , and H_{ij} , $i = 1, \dots, m$; $j = 1, \dots, k+1$. Then

$$\begin{aligned}
\|Tx - Ty\|_n & \leq (\xi - \delta)^{-2} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left(|c_0| + \sum_{i=1}^m |c_i| \right) \|x - y\| \\
& \quad + \sum_{k=1}^{n-1} P_k(\xi, \delta, c_i, H_{ij}) \|x^{(k)} - y^{(k)}\| \\
& \quad + \delta^{n+1} (\xi - \delta)^{-2} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left(|c_0| + \sum_{i=1}^m |c_i| \right) \|x^{(n)} - y^{(n)}\| \\
& \leq \Gamma \|x - y\|_n.
\end{aligned} \tag{2.22}$$

Here,

$$\Gamma = \max \left\{ (\xi - \delta)^{-2} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left(|c_0| + \sum_{i=1}^m i |c_i| \right); \max_{1 \leq k \leq n-1} \{P_k(\xi, \delta, c_i, H_{ij})\}; \right. \\ \left. \delta^{n+1} (\xi - \delta)^{-2} \left(|c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left(|c_0| + \sum_{i=1}^m i |c_i| \right) \right\}, \quad (2.23)$$

$k = 1, \dots, n-1$. So we can say that T is continuous.

It is easy to see that X is closed and convex. We now show that X is a relatively compact subset of $C^n(I, I)$. For any $x = x(t) \in X$,

$$\|x\|_n \leq \|x\| + \sum_{k=1}^n \left\| x^{(k)} \right\| \leq |\xi| + \delta + 1 + \sum_{k=2}^n M_k. \quad (2.24)$$

Next, for any t_1, t_2 in I , we have

$$|x(t_1) - x(t_2)| \leq |t_1 - t_2|. \quad (2.25)$$

Hence, X is bounded in $C^n(I, I)$ and equicontinuous on I , and by the Arzela-Ascoli theorem, we know X is relatively compact in $C^n(I, I)$, since $C^n(I, I)$ is the subset of $C^n(I, R)$, and we can say that X is relatively compact in $C^n(I, R)$.

From Schauder's fixed-point theorem, we conclude that

$$x(t) = \xi + \int_{\xi}^t \frac{1}{c_0 x^{[0]}(s) + c_1 x^{[1]}(s) + \dots + c_m x^{[m]}(s)} ds, \quad (2.26)$$

for some $x = x(t)$ in X . By differentiating both sides of the above equality, we see that x is the desired solution of (1.6). This completes the proof. $\square$

Theorem 2.2. Let $I = [\xi - \delta, \xi + \delta]$, where ξ and δ satisfy (2.4), then (1.6) has a unique solution in

$$X(\xi; \xi_0, \dots, \xi_n; 1, M_2, \dots, M_{n+1}; I), \quad (2.27)$$

provided the conditions (2.6)–(2.9) hold and $\Gamma < 1$ in (2.23).

Proof. Since $\Gamma < 1$, we see that T defined by (2.10) is contraction mapping on the close subset X of $C^n(I, I)$. Thus, the fixed point x in the proof of Theorem 2.1 must be unique. This completes the proof. $\square$

Remark 2.3. By Theorem 2.1 or Theorem 2.2, the existence and uniqueness of smooth solutions of an iterative functional differential equation of the form (1.6) can be obtained. If $n \rightarrow +\infty$, we can also find that the solution is C^∞ -smooth.

Now, we will show that the conditions in Theorem 2.1 do not self-contradict. Consider the following equation:

$$x'(t) = \frac{1}{t + (1/2)x(t) + (1/4)x(x(t))}, \quad (2.28)$$

where $c_0 = 1$, $c_1 = (1/2)$, $c_2 = (1/4)$, and $\xi \geq 4$. Moreover, we take $0 < \delta \leq \xi - 4$. Then, (2.4) is satisfied, and ξ, δ define the interval $I = [\xi - \delta, \xi + \delta]$. Now, take $\xi_0 = \xi$,

$$\begin{aligned} \xi_1 &= \frac{4}{7}\xi^{-1}, \\ \xi_2 &= -\frac{16}{49}\xi^{-2}\left(1 + \frac{1}{2}\xi_1 + \frac{1}{4}\xi_1^2\right), \\ \xi_3 &= \frac{4}{343}\xi^{-3}\left(4 + 2\xi_1 + \xi_1^2\right)^2 - \frac{4}{49}\xi^{-2}\xi_2\left(2 + \xi_1 + \xi_1^2\right), \end{aligned} \quad (2.29)$$

then (2.6) and (2.7) are satisfied.

Finally, if we take

$$M_1 = 1, \quad M_2 = 28(\xi - \delta)^{-2}, \quad M_3 = 392(\xi - \delta)^{-3} + 16(\xi - \delta)^{-2}M_2 \quad (2.30)$$

as positive, and

$$M_4 = 8232(\xi - \delta)^{-4} + 576(\xi - \delta)^{-3}M_2 + 8(\xi - \delta)^{-2}(6M_2^2 + 5M_3), \quad (2.31)$$

then (2.8) and (2.9) are satisfied.

Thus, we have shown that when $\xi_0, \dots, \xi_3$ and $M_1, \dots, M_4$ are defined as above, then there will be a solution for (2.28) in $X(\xi; \xi_0, \dots, \xi_3; 1, \dots, M_4; I)$.

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