Research Article

Smooth Solutions of a Class of Iterative Functional Differential Equations

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By Faà di Bruno’s formula, using the fixed-point theorems of Schauder and Banach, we study the existence and uniqueness of smooth solutions of an iterative functional differential equation

\[ x'(t) = \frac{1}{c_0 x^{[0]}(t) + c_1 x^{[1]}(t) + \cdots + c_m x^{[m]}(t)}. \]

1. Introduction

There has been a lot of monographs and research articles to discuss the kinds of solutions of functional differential equations since the publication of Jack Hale’s paper [1]. Several papers discussed the iterative functional differential equations of the form

\[ x'(t) = H\left(x^{[0]}(t), x^{[1]}(t), \ldots, x^{[m]}(t)\right), \tag{1.1} \]

where \( x^{[0]}(t) = t, \ x^{[1]}(t) = x(t), \ x^{[k]}(t) = x(x^{[k-1]}(t)), \ k = 2, \ldots, m. \) More specifically, Eder [2] considered the functional differential equation

\[ x'(t) = x^{[2]}(t) \tag{1.2} \]

and proved that every solution either vanishes identically or is strictly monotonic. Furthermore, Fečkan [3] and Wang [4] studied the equation

\[ x'(t) = f\left(x^{[2]}(t)\right) \tag{1.3} \]
with different conditions. Staněk [5] considered the equation

$$x'(t) = x(t) + x^{[2]}(t)$$  \hfill (1.4)

and obtained every solution either vanishes identically or is strictly monotonic. Si and his coauthors [6, 7] studied the following equations:

$$x'(t) = x^{[m]}(t),$$  \hfill (1.5)

$$x'(t) = \frac{1}{x^{[m]}(t)},$$  \hfill (1.6)

and established sufficient conditions for the existence of analytic solutions. Especially in [8, 9], the smooth solutions of the following equations:

$$x'(t) = \sum_{j=1}^{m} a_j x^{[j]}(t) + F(t),$$  \hfill (1.7)

$$x'(t) = \sum_{j=1}^{m} a_j(t) x^{[j]}(t) + F(t),$$  \hfill (1.8)

have been studied by the fixed-point theorems of Schauder and Banach.

A smooth function is taken to mean one that has a number of continuous derivatives and for which the highest continuous derivative is also Lipschitz. Let $x \in C^n$ if $x', \ldots, x^{(n)}$ are continuous, $C^n(I, I)$ is the set in which $x \in C^n$ and maps a closed interval $I$ into $I$. As in [9], we, using the same symbols, denote the norm

$$\|x\|_n = \sum_{k=0}^{n} \|x^{(k)}\|, \quad \|x\| = \max_{t \in I} \{|x(t)|\},$$  \hfill (1.9)

then $C^n(I, R)$ with $\| \cdot \|_n$ is a Banach space, and $C^n(I, I)$ is a subset of $C^n(I, R)$. For given $M_i > 0$ ($i = 1, 2, \ldots, n + 1$), let

$$\Omega(M_1, \ldots, M_{n+1}; I) = \left\{ x \in C^n(I, I) : \left| x^{(i)}(t) \right| \leq M_i, i = 1, 2, \ldots, n; \right. \quad \begin{cases} x^{(i)}(t_1) - x^{(i)}(t_2) & \leq M_{n+1}|t_1 - t_2|, \quad t, t_1, t_2 \in I. \end{cases}$$  \hfill (1.10)

For convenience, we will make use of the notation

$$x_{ij}(t) = x^{(i)}(x^{[j]}(t)), \quad x_{ijk}(t) = \left(x^{[j]}(t)\right)^{(k)},$$  \hfill (1.11)
where \(i, j, \) and \(k\) are nonnegative integers. Let \(I\) be a closed interval in \(R\). By induction, we may prove that

\[
x_{i,j,k}(t) = P_{j,k}(x_{i,0}(t); \ldots; x_{i,j-1}(t); \ldots; x_{k,0}(t); \ldots; x_{k,j-1}(t)),
\]

(1.11)

\[
\beta_{j,k} = P_{j,k}\left(\begin{array}{c}
\delta, \xi, x^{(1)}(\xi), \ldots, x^{(k)}(\xi)
\end{array}\right),
\]

(1.12)

\[
H_{j,k} = P_{j,k}\left(\begin{array}{c}
\delta, \xi, x^{(1)}(\xi), \ldots, x^{(k)}(\xi)
\end{array}\right),
\]

(1.13)

where \(P_{j,k}\) is a uniquely defined multivariate polynomial with nonnegative coefficients. The proof can be found in [8].

In order to seek a solution \(x(t)\) of (1.6), in \(C^n(I, I)\) such that \(\xi\) is a fixed point of the function \(x(t)\), that is, \(x(\xi) = \xi\), it is natural to seek an interval \(I\) of the form \([\xi - \delta, \xi + \delta]\) with \(\delta > 0\).

Let us define

\[
X(\xi_0; \xi_n; 1, M_2, \ldots, M_{n+1}; I) = \left\{ x \in \Omega(1, M_2, \ldots, M_{n+1}; I) : x(\xi) = \xi_0, x^{(i)}(\xi) = \xi_i, i = 1, 2, \ldots, n \right\}.
\]

(1.14)

2. Smooth Solutions of (1.6)

In this section, we will prove the existence theorem of smooth solutions for (1.6). First of all, we have the inequalities in the following for all \(x(t), y(t) \in X\):

\[
\left| x^{(j)}(t_1) - x^{(j)}(t_2) \right| \leq |t_1 - t_2|, \quad t_1, t_2 \in I, \quad j = 0, 1, \ldots, m,
\]

(2.1)

\[
\left\| x^{(j)} - x^{(j)} \right\| \leq j \left\| x - y \right\|, \quad j = 1, \ldots, m,
\]

(2.2)

\[
\left\| x - y \right\| \leq \delta^n \left\| x^{(n)} - y^{(n)} \right\|
\]

(2.3)

and the proof can be found in [9].

**Theorem 2.1.** Let \(I = [\xi - \delta, \xi + \delta]\), where \(\xi\) and \(\delta\) satisfy

\[
\xi \geq \frac{1}{|c_0| - \sum_{i=1}^m |c_i|}, \quad 0 < \delta \leq \frac{1}{|c_0| - \sum_{i=1}^m |c_i|},
\]

(2.4)

where \(|c_0| > \sum_{i=1}^m |c_i|\), then (1.6) has a solution in

\[
X(\xi_0; \xi_n; 1, M_2, \ldots, M_{n+1}; I),
\]

(2.5)
provided the following conditions hold:

(i) 

\[
\xi_1 = \xi^{-1} \left( \sum_{i=0}^{m} c_i \right)^{-1},
\]

\[
\xi_k = \sum \frac{(-1)^s (k-1)! s!}{s_1!s_2! \cdots s_k!} \left( \sum_{i=0}^{m} c_i \right)^{-s-1} \left( \sum_{i=0}^{m} c_i \beta_{i1} \right)^{s_1} \times \left( \sum_{i=0}^{m} c_i \beta_{i2} \right)^{s_2} \cdots \left( \sum_{i=0}^{m} c_i \beta_{i(k-1)} \right)^{s_{k-1}},
\]

where \( k = 2, \ldots, n \), and the sum is over all nonnegative integer solutions of the Diophantine equation \( s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1 \) and \( s = s_1 + s_2 + \cdots + s_{k-1} \).

(ii) 

\[
\sum \frac{(k-1)! s!}{s_1!s_2! \cdots s_k!} (\xi - \delta)^{-s-1} \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-s-1} \left( \sum_{i=0}^{m} |c_i| H_{i1} \right)^{s_1} \times \left( \sum_{i=0}^{m} |c_i| H_{i2} \right)^{s_2} \cdots \left( \sum_{i=0}^{m} |c_i| H_{i(k-1)} \right)^{s_{k-1}} \leq M_k, \quad k = 2, \ldots, n,
\]

where \( s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1 \) and \( s = s_1 + s_2 + \cdots + s_{k-1} \).

(iii) 

\[
(\xi - \delta)^{-s-1} \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-s-2} \left( \sum_{i=0}^{m} |c_i| H_{i1} \right)^{s_1+1} \times \left( \sum_{i=0}^{m} |c_i| H_{i2} \right)^{s_2} \cdots \left( \sum_{i=0}^{m} |c_i| H_{i(k-1)} \right)^{s_{k-1}}
\]

\[
\times \sum \frac{(n-1)! s!}{s_1!s_2! \cdots s_{n-1}!} (\xi - \delta)^{-s} \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-s} \left( \sum_{i=0}^{m} |c_i| H_{i1} \right)^{s_1} \times \left( \sum_{i=0}^{m} |c_i| H_{i2} \right)^{s_2} \cdots \left( \sum_{i=0}^{m} |c_i| H_{i(n-1)} \right)^{s_{n-1}}
\]

\[
+ \cdots + s_{n-1} (\xi - \delta)^{-s-1} \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-s-1} \left( \sum_{i=0}^{m} |c_i| H_{i1} \right)^{s_1} \times \left( \sum_{i=0}^{m} |c_i| H_{i2} \right)^{s_2} \cdots \left( \sum_{i=0}^{m} |c_i| H_{i(n-2)} \right)^{s_{n-2}}
\]

\[
\times \left( \sum_{i=0}^{m} |c_i| H_{i(n-1)} \right)^{s_{n-1}} \left( \sum_{i=0}^{m} |c_i| H_{in} \right)^{s_{n-1}} \leq M_{n+1},
\]

where \( s_1 + 2s_2 + \cdots + (n-1)s_{n-1} = n-1 \) and \( s = s_1 + s_2 + \cdots + s_{n-1} \).
Proof. Define an operator $T$ from $X$ into $C^n(I, I)$ by

$$ (Tx)(t) = \xi + \int_\xi^t \frac{1}{c_0 x^{[0]}(s) + c_1 x^{[1]}(s) + \cdots + c_m x^{[m]}(s)} ds. \quad (2.10) $$

We will prove that for any $x \in X$, $Tx \in X$,

$$ |(Tx)(t) - \xi| = \left| \int_\xi^t \frac{1}{\sum_{i=0}^m c_i x^{[i]}(s)} ds \right| \leq \left( (\xi - \delta) \left( |c_0| - \sum_{i=1}^m |c_i| \right) \right)^{-1} |t - \xi| \leq \delta, \quad (2.11) $$

where the second inequality is from (2.4) and $x(I) \subseteq I$. Thus, $(Tx)(I) \subseteq I$.

It is easy to see that

$$ (Tx)'(t) = \frac{1}{\sum_{i=0}^m c_i x^{[i]}(t)}, \quad (2.12) $$

and by Faà di Bruno’s formula, for $k = 2, \ldots, n$, we have

$$ (Tx)^{(k)}(t) = \left( \frac{1}{\sum_{i=0}^m c_i x^{[i]}(t)} \right)^{(k-1)} \sum_{s_1! \cdots s_{k-1}!} \frac{(-1)^s}{s_1! \cdots s_{k-1}!} \frac{1}{\left( \sum_{i=0}^m c_i x^{[i]}(t) \right)^{s_1 + 1}} \left( \int_\xi^t \frac{1}{\sum_{i=0}^m c_i x^{[i]}(s)} ds \right)^{s_1} \times \left( \frac{\sum_{i=0}^m c_i x^{[i]}(t)}{2!} \right)^{s_2} \cdots \left( \frac{\sum_{i=0}^m c_i x^{[i]}(t)}{k-1!} \right)^{s_{k-1}} \sum_{s_1! \cdots s_{k-1}!} \left( \sum_{i=0}^m c_i x^{[i]}(t) \right)^{s_1} \times \left( \sum_{i=0}^m c_i x^{[i]}(t) \right)^{s_2} \cdots \left( \sum_{i=0}^m c_i x^{[i]}(t) \right)^{s_{k-1}}, \quad (2.13) $$

where the sum is over all nonnegative integer solutions of the Diophantine equation $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$. 
Furthermore, note \((Tx)(\xi) = \xi\), by (2.6) and (2.7),

\[
(Tx)^{(0)}(\xi) = \frac{1}{s!} \sum_{i=0}^{m} c_i x_i(t) = \frac{1}{s!} \sum_{i=0}^{m} c_i = \xi,
\]

\[
(Tx)^{(k)}(\xi) = \sum_{s_1! \cdots s_{k-1}!} \frac{(-1)^s (k-1)! s!}{s_1! \cdots s_{k-1}!} \left[ \frac{1}{s!} \sum_{i=0}^{m} c_i x_i(t) \right]^{s_1} \left[ \frac{1}{s!} \sum_{i=0}^{m} c_i x_i(t) \right]^{s_2} \cdots \left[ \frac{1}{s!} \sum_{i=0}^{m} c_i x_i(t) \right]^{s_{k-1}}
\]

\[
\leq \sum_{s_1! \cdots s_{k-1}!} \frac{(k-1)! s!}{s_1! \cdots s_{k-1}!} \left[ \frac{1}{s!} \sum_{i=0}^{m} c_i x_i(t) \right]^{s_1} \left[ \frac{1}{s!} \sum_{i=0}^{m} c_i x_i(t) \right]^{s_2} \cdots \left[ \frac{1}{s!} \sum_{i=0}^{m} c_i x_i(t) \right]^{s_{k-1}}
\]

\[
\leq \sum_{s_1! \cdots s_{k-1}!} \frac{(k-1)! s!}{s_1! \cdots s_{k-1}!} \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-s_1} \left( \sum_{i=0}^{m} |c_i| H_{i1} \right)^{s_1}
\]

where \(s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k - 1\) and \(s = s_1 + s_2 + \cdots + s_{k-1}\). Thus, \((Tx)^{(k)}(\xi) = \xi_k\) for \(k = 0, 1, \ldots, n\),

\[
|\langle Tx \rangle(t)| = \left| \frac{1}{s!} \sum_{i=0}^{m} c_i x_i(t) \right| \leq \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-1} \leq 1 = M_1.
\]
Finally,

\[
\left| (Tx)^{(m)}(t_1) - (Tx)^{(m)}(t_2) \right| \\
\leq \sum_{s_1!s_2! \cdots s_{n-1}!^m} \sum_{2!^{s_2} \cdots (n-1)!^{s_{n-1}}} \left( \left( \sum_{i=0}^{m} c_i x_{i1}(t_1) \right)^{s_1} - \left( \sum_{i=0}^{m} c_i x_{i1}(t_2) \right)^{s_1} \right) \left( \sum_{i=0}^{m} c_i x_{i2}(t_1) \right)^{s_2} \cdots \left( \sum_{i=0}^{m} c_i x_{i(n-1)}(t_1) \right)^{s_{n-1}} \\
\times \left( \left( \sum_{i=0}^{m} c_i x_{i1}(t_2) \right)^{s_1} - \left( \sum_{i=0}^{m} c_i x_{i1}(t_2) \right)^{s_1} \right) \left( \sum_{i=0}^{m} c_i x_{i2}(t_2) \right)^{s_2} \cdots \left( \sum_{i=0}^{m} c_i x_{i(n-1)}(t_2) \right)^{s_{n-1}} \\
\leq \sum_{s_1!s_2! \cdots s_{n-1}!^m} \sum_{2!^{s_2} \cdots (n-1)!^{s_{n-1}}} \left( \left( \sum_{i=0}^{m} c_i x_{i1}(t_1) \right)^{s_1} - \left( \sum_{i=0}^{m} c_i x_{i1}(t_2) \right)^{s_1} \right) \left( \sum_{i=0}^{m} c_i x_{i2}(t_1) \right)^{s_2} \cdots \left( \sum_{i=0}^{m} c_i x_{i(n-1)}(t_1) \right)^{s_{n-1}} \\
\times \left( \left( \sum_{i=0}^{m} c_i x_{i1}(t_2) \right)^{s_1} - \left( \sum_{i=0}^{m} c_i x_{i1}(t_2) \right)^{s_1} \right) \left( \sum_{i=0}^{m} c_i x_{i2}(t_2) \right)^{s_2} \cdots \left( \sum_{i=0}^{m} c_i x_{i(n-1)}(t_2) \right)^{s_{n-1}} \\
\leq \sum_{s_1!s_2! \cdots s_{n-1}!^m} \sum_{2!^{s_2} \cdots (n-1)!^{s_{n-1}}} \left( \left( \sum_{i=0}^{m} c_i x_{i1}(t_1) \right)^{s_1} - \left( \sum_{i=0}^{m} c_i x_{i1}(t_2) \right)^{s_1} \right) \left( \sum_{i=0}^{m} c_i x_{i2}(t_1) \right)^{s_2} \cdots \left( \sum_{i=0}^{m} c_i x_{i(n-1)}(t_1) \right)^{s_{n-1}} \\
\times \left( \left( \sum_{i=0}^{m} c_i x_{i1}(t_2) \right)^{s_2} + \cdots + \left( \sum_{i=0}^{m} c_i x_{i1}(t_2) \right)^{s_1-1} \right) \\
\times \left( \sum_{i=0}^{m} c_i x_{i2}(t_1) \right)^{s_2} \cdots \left( \sum_{i=0}^{m} c_i x_{i(n-2)}(t_1) \right)^{s_{n-2}} \\
\times \left( \sum_{i=0}^{m} c_i x_{i(n-1)}(t_1) \right)^{s_{n-1}} - \left( \sum_{i=0}^{m} c_i x_{i(n-1)}(t_2) \right)^{s_{n-1}} \right] \\
\times (s+1)(\xi - \delta)^{-s-2} \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-s-2} \left( \sum_{i=0}^{m} |c_i| H_{i1} \right)^{s_1+1} \left( \sum_{i=0}^{m} |c_i| H_{i2} \right)^{s_2} \\
\times \left( \sum_{i=0}^{m} |c_i| H_{i1} \right)^{s_2+1} \left( \sum_{i=0}^{m} |c_i| H_{i2} \right)^{s_3} \cdots \left( \sum_{i=0}^{m} |c_i| H_{i(n-1)} \right)^{s_{n-1}} \\
+ \cdots + s_{n-1}(\xi - \delta)^{-s-1} \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-s-1} \left( \sum_{i=0}^{m} |c_i| H_{i1} \right)^{s_1} \left( \sum_{i=0}^{m} |c_i| H_{i2} \right)^{s_{n-2}} \\
\times \left( \sum_{i=0}^{m} |c_i| H_{i(n-1)} \right)^{s_{n-1}-1} \left| t_1 - t_2 \right|, \\
\right.

(2.17)
where \( s_1 + 2s_2 + \cdots + (k - 1)s_{k-1} = k - 1 \) and \( s = s_1 + s_2 + \cdots + s_{k-1} \). By (2.9), we see that

\[
\left| (Tx)^{(n)}(t_1) - (Tx)^{(n)}(t_2) \right| \leq M_{n+1}|t_1 - t_2|.
\]

(2.18)

Now, we can say that \( T \) is an operator from \( X \) into itself.

Next, we will show that \( T \) is continuous. Let \( x, y \in X \), then

\[
\|Tx - Ty\|_n = \|Tx - Ty\| + \|(Tx)' - (Ty)'\| + \sum_{k=2}^{n} \| (Tx)^{(k)} - (Ty)^{(k)} \|
\]

\[
= \max_{t \in I} \left| \int_t^1 \left( \frac{1}{\sum_{i=0}^m c_i x_i[s]} - \frac{1}{\sum_{i=0}^m c_i y_i[s]} \right) ds \right|
\]

\[
+ \max_{t \in I} \left| \int_t^1 \left( \frac{1}{\sum_{i=0}^m c_i x_i[t]} - \frac{1}{\sum_{i=0}^m c_i y_i[t]} \right) ds \right|
\]

\[
+ \sum_{k=2}^{n} \max_{t \in I} \left| \sum_{s_1!s_2!\cdots s_{k-1}!2s_k \cdots (k - 1)^{s_{k-1}}} \left( \frac{1}{\sum_{i=0}^m c_i x_i[s_1]} \right)^{s_1} \left( \sum_{i=0}^m c_i x_i[s_2] \right)^{s_2} \cdots \left( \frac{1}{\sum_{i=0}^m c_i y_i[s_1]} \right)^{s_1} \left( \sum_{i=0}^m c_i y_i[s_2] \right)^{s_2} \cdots \right|
\]

\[
\leq \delta (\xi - \delta)^{-2} \left( |c_0| - \sum_{s=1}^m |c_i| \right)^{-2} \left( \sum_{i=0}^m |c_i| \| x_i[s] - y_i[s] \| \right)
\]

\[
+ (\xi - \delta)^{-2} \left( |c_0| - \sum_{s=1}^m |c_i| \right)^{-2} \left( \sum_{i=0}^m |c_i| \| x_i[s] - y_i[s] \| \right)
\]

\[
+ \sum_{k=2}^{n} \max_{t \in I} \sum_{s_1!s_2!\cdots s_{k-1}!2s_k \cdots (k - 1)^{s_{k-1}}} \left( \frac{1}{\sum_{i=0}^m c_i x_i[s_1]} \right)^{s_1} \left( \sum_{i=0}^m c_i x_i[s_2] \right)^{s_2} \cdots \left( \frac{1}{\sum_{i=0}^m c_i y_i[s_1]} \right)^{s_1} \left( \sum_{i=0}^m c_i y_i[s_2] \right)^{s_2} \cdots \right|
\]

\[
\times \left[ \left( \frac{1}{\sum_{i=0}^m c_i x_i[t]} - \frac{1}{\sum_{i=0}^m c_i y_i[t]} \right)^{s_k} \left( \sum_{i=0}^m c_i \| x_i[s_k] \| \right)^{s_k} \cdots \left( \frac{1}{\sum_{i=0}^m c_i x_i[t]} - \frac{1}{\sum_{i=0}^m c_i y_i[t]} \right)^{s_k} \left( \sum_{i=0}^m c_i \| x_i[s_k] \| \right)^{s_k} \cdots \right]
\]

\[
\times \left( \sum_{i=0}^m |c_i| \| x_{s_i}[t] \| \right)^{s_1} + \left( \sum_{i=0}^m |c_i| \| x_{s_i}[t] \| \right)^{s_2} \cdots \left( \sum_{i=0}^m |c_i| \| x_{s_i}[t] \| \right)^{s_k} \cdots \right|
\]

\[
\times \left( \sum_{i=0}^m c_i x_{s_i}[t] \right)^{s_1} - \left( \sum_{i=0}^m c_i y_{s_i}[t] \right)^{s_1} \right|
\]
\[
\begin{align*}
&\times \left( \sum_{i=0}^{m} |c_i||x_{i2}(t)| \right)^{s_2} \cdots \left( \sum_{i=0}^{m} |c_i||x_{i(k-1)}(t)| \right)^{s_{k-1}} \\
&+ \cdots + \left( \sum_{i=0}^{m} c_i y^{[i]}(t) \right)^{-s-1} \left( \sum_{i=0}^{m} |c_i||y_{i1}(t)| \right)^{s_1} \cdots \left( \sum_{i=0}^{m} |c_i||y_{i(k-2)}(t)| \right)^{s_{k-2}} \\
&\times \left( \sum_{i=0}^{m} c_i x_{i(k-1)}(t) \right)^{s_{k-1}} - \left( \sum_{i=0}^{m} c_i y_{i(k-1)}(t) \right)^{s_{k-1}} \\
&\leq (\xi - \delta)^{-2} \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-2} \left( |c_0| + \sum_{i=1}^{m} |c_i| \right) \| x - y \| \\
&+ \sum_{k=2}^{n} \sum_{s_1!s_2! \cdots s_{k-1}!2!^{s_2} \cdots (k-1)!^{s_{k-1}}} (k-1)!^{s_{k-1}} \\
&\times \left( s + 1 \right) (\xi - \delta)^{-s-2} \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-s-2} \left( \sum_{i=0}^{m} |c_i||x^{[i]} - y^{[i]}| \right) \\
&\times \left( \sum_{i=0}^{m} |c_i||H_{i1}| \right)^{s_1} \cdots \left( \sum_{i=0}^{m} |c_i||H_{i(k-1)}| \right)^{s_{k-1}} \\
&+ s_1 (\xi - \delta)^{-s-1} \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-s-1} \left( \sum_{i=0}^{m} |c_i||H_{i1}| \right)^{s_1} \cdots \\
&\times \left( \sum_{i=0}^{m} |c_i||H_{i2}| \right)^{s_{k-1}+1} \cdots \left( \sum_{i=0}^{m} |c_i||H_{i(k-1)}| \right)^{s_{k-1}} \left( \sum_{i=0}^{m} |c_i||x^{[i]} - y^{[i]}| \right) \\
&+ \cdots + s_{k-1} (\xi - \delta)^{-s-1} \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-s-1} \left( \sum_{i=0}^{m} |c_i||H_{i1}| \right)^{s_1} \cdots \\
&\times \left( \sum_{i=0}^{m} |c_i||H_{i(k-2)}| \right)^{s_{k-2}} \left( \sum_{i=0}^{m} |c_i||H_{i(k-1)}| \right)^{s_{k-1}+1} \left( \sum_{i=0}^{m} |c_i|H_{ik} \right) \\
&\times \left( \sum_{i=0}^{m} |c_i||x^{[i]} - y^{[i]}| \right) \\
&+ \delta^{n+1} (\xi - \delta)^{-2} \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-2} \left( |c_0| + \sum_{i=1}^{m} |c_i| \right) \| x^{(n)} - y^{(n)} \|,
\end{align*}
\]

(2.19)

where \( s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k - 1 \) and \( s = s_1 + s_2 + \cdots + s_{k-1} \).
Moreover, we can find some constants $P_k$ such that

\[
\sum_{k=1}^{n} \sum_{s_1!s_2! \cdots s_{k-1}!1^{s_k}2!^{s_k-1} \cdots (k-1)!^{s_k-1}} \frac{(k-1)!s!}{s_1!s_2! \cdots s_{k-1}!1^{s_k}2!^{s_k-1} \cdots (k-1)!^{s_k-1}}
\times \left[(s+1)(\xi - \delta)^{-s-2}\left(|c_0| - \sum_{i=1}^{m} |c_i| \right)^{-s-2}\left(\sum_{i=0}^{m} |c_i| \|x^{[i]} - y^{[i]}\| \right) \times \right.
\times \left(\sum_{i=0}^{m} |c_i| H_{i1}\right) \cdots \left(\sum_{i=0}^{m} |c_i| H_{ik-1}\right)
\left. + \delta_{j=1}^{s_{k-1}}(\xi - \delta)^{-s-1}\left(|c_0| - \sum_{i=1}^{m} |c_i| \right)^{-s-1}\left(\sum_{i=0}^{m} |c_i| H_{i1}\right) \cdots \right.
\times \left(\sum_{i=0}^{m} |c_i| H_{i2}\right) \cdots \left(\sum_{i=0}^{m} |c_i| H_{ik-1}\right)
\times \left(\sum_{i=0}^{m} |c_i| H_{ik}\right)
\times \left(\sum_{i=0}^{m} |c_i| H_{i} \right) \right]
\leq \sum_{k=1}^{n-1} P_k(\xi, \delta, c_i, H_{ij}) \|x^{(k)} - y^{(k)}\|
\]

where

\[
P_k(\xi, \delta, c_i, H_{ij}) = P(\xi, \delta; c_1, \ldots, c_m; H_{11}, \ldots, H_{1k+1}; \ldots; H_{m1}, \ldots, H_{mk+1})
\]

are the positive constants depend on $\xi, \delta, c_i$, and $H_{ij}, i = 1, \ldots, m; j = 1, \ldots, k + 1$. Then

\[
\|Tx - Ty\| \leq (\xi - \delta)^{-2} \left(|c_0| - \sum_{i=1}^{m} |c_i| \right)^{-2} \left(|c_0| + \sum_{i=1}^{m} |c_i| \right) \|x - y\|
\]

\[
+ \sum_{k=1}^{n-1} P_k(\xi, \delta, c_i, H_{ij}) \|x^{(k)} - y^{(k)}\|
\]

\[
+ \delta^{n+1}(\xi - \delta)^{-2} \left(|c_0| - \sum_{i=1}^{m} |c_i| \right)^{-2} \left(|c_0| + \sum_{i=1}^{m} |c_i| \right) \|x^{(n)} - y^{(n)}\|
\]

\[
\leq \Gamma \|x - y\|_n.
\]
Here,\[\Gamma = \max \left\{ (\xi - \delta)^{-2} \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-2} \left( |c_0| + \sum_{i=1}^{m} i |c_i| \right), \max_{1 \leq k \leq n-1} \left\{ P_k (\xi, \delta, c, H_{ij}) \right\} \right\};\]
\[\delta^{n+1} (\xi - \delta)^{-2} \left( |c_0| - \sum_{i=1}^{m} |c_i| \right)^{-2} \left( |c_0| + \sum_{i=1}^{m} i |c_i| \right) \right\},\] \tag{2.23}
k = 1, \ldots, n - 1. So we can say that \(T\) is continuous.

It is easy to see that \(X\) is closed and convex. We now show that \(X\) is a relatively compact subset of \(C^n(I, I)\). For any \(x = x(t) \in X\),
\[\|x\|_n \leq \|x\| + \sum_{k=1}^{n} \left\| x^{(k)} \right\| \leq |\xi| + \delta + 1 + \sum_{k=2}^{n} M_k.\] \tag{2.24}

Next, for any \(t_1, t_2 \in I\), we have
\[|x(t_1) - x(t_2)| \leq |t_1 - t_2|.\] \tag{2.25}

Hence, \(X\) is bounded in \(C^n(I, I)\) and equicontinuous on \(I\), and by the Arzela-Ascoli theorem, we know \(X\) is relatively compact in \(C^n(I, I)\), since \(C^n(I, I)\) is the subset of \(C^n(I, R)\), and we can say that \(X\) is relatively compact in \(C^n(I, R)\).

From Schauder’s fixed-point theorem, we conclude that
\[x(t) = \xi + \int_{\xi}^{t} \frac{1}{c_0 x^{[0]}(s) + c_1 x^{[1]}(s) + \cdots + c_m x^{[m]}(s)} ds,\] \tag{2.26}
for some \(x = x(t)\) in \(X\). By differentiating both sides of the above equality, we see that \(x\) is the desired solution of (1.6). This completes the proof. \(\square\)

**Theorem 2.2.** Let \(I = [\xi - \delta, \xi + \delta]\), where \(\xi\) and \(\delta\) satisfy (2.4), then (1.6) has a unique solution in
\[X(\xi; \xi_0, \ldots, \xi_m; 1, M_{2, \ldots, M_{n+1}; I}),\] \tag{2.27}
provided the conditions (2.6)–(2.9) hold and \(\Gamma < 1\) in (2.23).

**Proof.** Since \(\Gamma < 1\), we see that \(T\) defined by (2.10) is contraction mapping on the close subset \(X\) of \(C^n(I, I)\). Thus, the fixed point \(x\) in the proof of Theorem 2.1 must be unique. This completes the proof. \(\square\)

**Remark 2.3.** By Theorem 2.1 or Theorem 2.2, the existence and uniqueness of smooth solutions of an iterative functional differential equation of the form (1.6) can be obtained. If \(n \to +\infty\), we can also find that the solution is \(C^n\)-smooth.
Now, we will show that the conditions in Theorem 2.1 do not self-contradict. Consider the following equation:

$$x'(t) = \frac{1}{t + (1/2)x(t) + (1/4)x'(t)},$$

(2.28)

where $c_0 = 1, \quad c_1 = (1/2), \quad c_2 = (1/4)$, and $\xi \geq 4$. Moreover, we take $0 < \delta \leq \xi - 4$. Then, (2.4) is satisfied, and $\xi, \delta$ define the interval $I = [\xi - \delta, \xi + \delta]$. Now, take $\xi_0 = \xi$,

$$\xi_1 = \frac{4}{\xi} - 1,$$

$$\xi_2 = -\frac{16}{49} \xi^{-2} \left( 1 + \frac{1}{2} \xi_1 + \frac{1}{4} \xi_1^2 \right),$$

$$\xi_3 = \frac{4}{343} \xi^{-3} \left( 4 + 2 \xi_1 + \xi_1^2 \right)^2 - \frac{4}{49} \xi^{-2} \xi_2 \left( 2 + \xi_1 + \xi_1^2 \right),$$

(2.29)

then (2.6) and (2.7) are satisfied.

Finally, if we take

$$M_1 = 1, \quad M_2 = 28(\xi - \delta)^{-2}, \quad M_3 = 392(\xi - \delta)^{-3} + 16(\xi - \delta)^{-2}M_2$$

(2.30)

as positive, and

$$M_4 = 8232(\xi - \delta)^{-4} + 576(\xi - \delta)^{-3}M_2 + 8(\xi - \delta)^{-2} \left( 6M_2^2 + 5M_3 \right),$$

(2.31)

then (2.8) and (2.9) are satisfied.

Thus, we have shown that when $\xi_0, \ldots, \xi_3$ and $M_1, \ldots, M_4$ are defined as above, then there will be a solution for (2.28) in $X(\xi; \xi_0, \ldots, \xi_3; 1, \ldots, M_4; I)$.

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**References**


