Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 956359, 13 pages
doi:10.1155/2012/956359

Research Article

An Algebraic Criterion of Zero Solutions of Some Dynamic Systems

Ying Wang,[1] Baodong Zheng,[1] and Chunrui Zhang[2]

[1] *Department of Mathematics, Harbin Institute of Technology, Harbin, Heilongjiang 150001, China*
[2] *Department of Mathematics, Northeast Forestry University, Harbin, Heilongjiang 150040, China*

Correspondence should be addressed to Baodong Zheng, zbd@hit.edu.cn

Received 19 October 2012; Revised 7 December 2012; Accepted 7 December 2012

Academic Editor: Allan Peterson

We establish some algebraic results on the zeros of some exponential polynomials and a real coefficient polynomial. Based on the basic theorem, we develop a decomposition technique to investigate the stability of two coupled systems and their discrete versions, that is, to find conditions under which all zeros of the exponential polynomials have negative real parts and the moduli of all roots of a real coefficient polynomial are less than 1.

1. Introduction

For an ordinary (delay) differential equation, the trivial solution is asymptotically stable if and only if all roots of the corresponding characteristic equation of the linearized system have negative real parts while the moduli of all roots of a real coefficient polynomial less than 1 mean the trivial solution is asymptotically stable for the difference equation. However, it is difficult to obtain the expression of the characteristic equation corresponding to the linearized systems. Special cases of the characteristic equation have been discussed by many authors. For example, Bellman and Cooke [1], Boese [2], Kuang [3], and Ruan and Wei [4–8] studied some exponential polynomials and used the results to investigate the stability and bifurcations for some systems. The well-known Jury criterion can be used to determine the moduli of the roots of a real coefficient polynomial less than one [9, 10], but the calculation is prolixly.

The purpose of this paper is to provide a new algebraic criterion of zero for some exponential polynomials and a real coefficient polynomial.

2. Some Algebraic Results

Let V be a linear space over a number field F and W a subspace of V. A complement space of W in V is a linear space U of V such that $V = W \oplus U$. A vector γ is said to be the projection of a vector α along W at U if $\alpha = \beta + \gamma$, where $\beta \in W$ and $\gamma \in U$.

Let $\mathcal{A}$ be a linear transformation on V and W a subspace of V. A subspace W is said to be a $\mathcal{A}$ invariant subspace if $\mathcal{A}\alpha \in W$ for any $\alpha \in W$. Let U be a k-complement space of $\mathcal{A}$ for W, if

(1) W is a $\mathcal{A}$ invariant subspace;

(2) U is a complement space of W in V, that is, $V = W \oplus U$;

(3) for any $\alpha \in U$, the projection of $(\mathcal{A} - kI_V)\alpha$ along W at U is always 0.

Let $\mathcal{A}$, $\mathcal{B}$ be two linear transformations on V. We consider that $\mathcal{A}$ and $\mathcal{B}$ are (k,l) concordant if there exist a nontrivial invariant subspaces W of $\mathcal{A}$ and $\mathcal{B}$, and a complement space U of W in V, such that

(1) at least one of constraints of $\mathcal{A}, \mathcal{B}$ on W is a scalar transformation lI_W;

(2) U is a k-complement space of $\mathcal{A}$ or $\mathcal{B}$ for W.

Let $A, B \in \mathbf{R}^{n\times n}$. We say that A and B are (k,l) concordant if linear transformations $\mathcal{A}$ and $\mathcal{B}$ of $\mathbf{R}^n$

$$\mathcal{A}: \mathbf{R}^n \to \mathbf{R}^n,\ X \mapsto AX, \text{ for all } X \in \mathbf{R}^n,$$
$$\mathcal{B}: \mathbf{R}^n \to \mathbf{R}^n,\ X \mapsto BX, \text{ for all } X \in \mathbf{R}^n$$

are concordant.

For example, I_n and a reducible matrix $A_{n\times n}$ are always in concordance.

Theorem 2.1. *Let $A, B \in \mathbf{R}^{n\times n}$ be (k,l) concordant. Then there exist an invertible matrix P, such that*

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ O & kI_{n_2} \end{pmatrix}, \qquad PBP^{-1} = \begin{pmatrix} lI_{n_1} & B_{12} \\ O & B_{22} \end{pmatrix}, \tag{2.1}$$

or

$$PAP^{-1} = \begin{pmatrix} lI_{n_1} & A_{12} \\ O & kI_{n_2} \end{pmatrix}, \qquad PBP^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ O & B_{22} \end{pmatrix}, \tag{2.2}$$

or

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}, \qquad PBP^{-1} = \begin{pmatrix} lI_{n_1} & B_{12} \\ O & kI_{n_2} \end{pmatrix}, \tag{2.3}$$

or

$$PAP^{-1} = \begin{pmatrix} lI_{n_1} & A_{12} \\ O & A_{22} \end{pmatrix}, \qquad PBP^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ O & kI_{n_2} \end{pmatrix}, \tag{2.4}$$

where $n_1 > 0, n_2 > 0, n_1 + n_2 = n$.

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be the base of W and $\alpha_{r+1}, \ldots, \alpha_n$ of U. If the constraints of $\mathcal{A}$ on W are lI_W and U is a k-complement space of $\mathcal{A}$ for W, then $A\alpha_i = l\alpha_i,\ i = 1, \ldots, r$ and there exists $a_{1j}, \ldots, a_{rj}$ making $A\alpha_j - k\alpha_j = a_{1j}\alpha_1 + \cdots + a_{rj}\alpha_r$, that is, $A\alpha_j = a_{1j}\alpha_1 + \cdots + a_{rj}\alpha_r + k\alpha_j,\ j = r+1, \ldots, n$. So

$$A(\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n) = (\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n) \begin{pmatrix} l & \cdots & 0 & a_{1,r+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & l & a_{r,r+1} & \cdots & a_{rn} \\ & & 0 & k & \cdots & 0 \\ & & & & \ddots & \vdots \\ & & & & & k \end{pmatrix}. \tag{2.5}$$

Because a subspace W is a $\mathcal{B}$ invariant subspace, we can write $B\alpha_i = b_{1i}\alpha_1 + \cdots + b_{ri}\alpha_r,\ i = 1, \ldots, r,\ B\alpha_i = b_{1j}\alpha_1 + \cdots + b_{rj}\alpha_r + \cdots + b_{nj}\alpha_n,\ j = r+1, \ldots, n$. So

$$B(\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n) = (\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n) \begin{pmatrix} b_{11} & \cdots & b_{1r} & b_{1,r+1} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ b_{r1} & \cdots & b_{rr} & b_{r,r+1} & \cdots & b_{rn} \\ & & 0 & b_{r+1,r+1} & \cdots & b_{r+1,n} \\ & & & \vdots & \ddots & \vdots \\ & & & b_{n,r+1} & \cdots & b_{nn} \end{pmatrix}. \tag{2.6}$$

If the constraint of $\mathcal{A}$ on W is lI_W and U is a k-complement space of $\mathcal{B}$ for W, then $B\alpha_i = l\alpha_i,\ i = 1, \ldots, r$. $A\alpha_j = a_{1j}\alpha_1 + \cdots + a_{rj}\alpha_r + a_{r+1,j}\alpha_{r+1} + \cdots + a_{nj}\alpha_n,\ j = r+1, \ldots, n$. So

$$A(\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n) = (\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n) \begin{pmatrix} l & \cdots & 0 & a_{1,r+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & l & a_{r,r+1} & \cdots & a_{rn} \\ & & 0 & a_{r+1,r+1} & \cdots & a_{r+1,n} \\ & & & \vdots & \ddots & \vdots \\ & & & a_{n,r+1} & \cdots & a_{nn} \end{pmatrix}. \tag{2.7}$$

Because a subspace W is a $\mathcal{B}$ invariant subspace, we can write $B\alpha_i = b_{1i}\alpha_1 + \cdots + b_{ri}\alpha_r$, $i = 1,\ldots,r$. Because $\alpha_j \in U$ $(j = r+1,\ldots,n)$, $B\alpha_j - k\alpha_j \in W$, then there exists $b_{1j},\ldots,b_{rj}$ making $B\alpha_j = b_{1j}\alpha_1 + \cdots + b_{rj}\alpha_r + k\alpha_j$, $j = r+1,\ldots,n$. So

$$B(\alpha_1,\ldots,\alpha_r,\alpha_{r+1},\ldots,\alpha_n) = (\alpha_1,\ldots,\alpha_r\alpha_{r+1},\ldots,\alpha_n)\begin{pmatrix} b_{11} & \cdots & b_{1r} & b_{1,r+1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ b_{r1} & \cdots & b_{rr} & b_{r,r+1} & \cdots & b_{r,n} \\ & & 0 & k & \cdots & 0 \\ & & & & \ddots & \vdots \\ & & & & & k \end{pmatrix}. \tag{2.8}$$

Other situations are similar. □

3. Algebraic Criterion of Zero for Some Exponential Polynomials

From Section 2, we have the following theorem on the roots of exponential polynomial.

Theorem 3.1. *Let $A, B \in \mathbf{R}^n$ be (k,l) concordant, τ a constant, $\tau > 0$, $f(\lambda)$ a polynomial about λ. If λ_0 is a root of*

$$\left| f(\lambda)I_n - A - Be^{-\lambda\tau} \right| = 0, \tag{3.1}$$

then

$$u = f(\lambda_0) - ae^{-\lambda_0\tau} \tag{3.2}$$

is an eigenvalue of A, or

$$v = \left(f(\lambda_0) - a\right)e^{\lambda_0\tau} \tag{3.3}$$

is an eigenvalue of B, where $a = k$ or l.

Proof. Without loss of generality, assume that

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ O & kI_{n_2} \end{pmatrix}, \quad PBP^{-1} = \begin{pmatrix} lI_{n_1} & B_{12} \\ O & B_{22} \end{pmatrix}. \tag{3.4}$$

Because λ_0 is a root of

$$\begin{gathered} \left| f(\lambda)I_n - A - Be^{-\lambda\tau} \right| = 0, \\ \begin{vmatrix} f(\lambda_0)I_{n_1} - A_{11} - lI_{n_1}e^{-\lambda_0\tau} & * \\ O & f(\lambda_0)I_{n_2} - kI_{n_2} - B_{22}e^{-\lambda_0\tau} \end{vmatrix} = 0, \end{gathered} \tag{3.5}$$

then

$$\left|f(\lambda_0)I_{n_1} - A_{11} - lI_{n_1}e^{-\lambda_0\tau}\right| = 0 \quad \text{or} \quad \left|f(\lambda_0)I_{n_2} - kI_{n_2} - B_{22}e^{-\lambda_0\tau}\right| = 0. \tag{3.6}$$

If $|f(\lambda_0)I_{n_1} - A_{11} - lI_{n_1}e^{-\lambda_0\tau}| = 0$, then $u = f(\lambda_0) - le^{-\lambda_0\tau}$ is an eigenvalue of A.

If $|f(\lambda_0)I_{n_2} - kI_{n_2} - B_{22}e^{-\lambda_0\tau}| = 0$, then $v = (f(\lambda_0) - k)e^{\lambda_0\tau}$ is an eigenvalue of B.

As a application of Theorem 3.1, consider a BAM neural network model with delays:

$$\begin{aligned} \dot{X} &= -aX + f(Y(t-\tau_1)), \\ \dot{Y} &= -bY + g(X(t-\tau_2)), \end{aligned} \tag{3.7}$$

where $X \in \mathbf{R}^n, Y \in \mathbf{R}^n, a > 0, b > 0$. Assume that $f, g \in C^1$, and $f(0) = 0, g(0) = 0$.

Under the hypothesis, the origin O is an equilibrium of (3.7), and the linearization of system (3.7) at the origin O is

$$\begin{aligned} \dot{X} &= -aX + B_1Y(t-\tau_1), \\ \dot{Y} &= -bY + B_2X(t-\tau_2), \end{aligned} \tag{3.8}$$

where B_1, B_2 are Jacobi matrices.

The associated characteristic equation of (3.8) is

$$\begin{vmatrix} \lambda I_n + aI_n & -B_1e^{-\lambda\tau_1} \\ -B_2e^{-\lambda\tau_2} & \lambda I_n + aI_n \end{vmatrix} = 0. \tag{3.9}$$

Since $\lambda = -a, \lambda = -b$ have no influence on the stability of system (3.7), then let $\lambda \neq -a, \lambda \neq -b$. We have

$$\left|(\lambda + a)(\lambda + b)I_n - e^{-\lambda(\tau_1+\tau_2)}B_1B_2\right| = 0. \tag{3.10}$$

Let $u_1, u_2, \ldots, u_r$ be eigenvalues of B_1B_2. From Theorem 3.1, we have

$$(\lambda + a)(\lambda + b) - u_ke^{-\lambda\tau} = 0, \quad (k = 1, 2, \ldots, n), \tag{3.11}$$

where $\tau = \tau_1 + \tau_2$, that is,

$$\lambda^2 + (a + b)\lambda + ab - u_ke^{-\lambda\tau} = 0. \tag{3.12}$$

Consider

$$\lambda^2 + (a + b)\lambda + ab - u_ke^{-\lambda\tau} = 0, \quad u_k = c_k + d_ki. \tag{3.13}$$

□

Lemma 3.2. *If* $\max_{1\le k\le n}|u_k| > ab$, *then* (3.12) *has a root* $iw_0^{(k)}$ $(w_0 > 0)$, *and*

$$w_0^{(k)} = \frac{1}{\sqrt{2}}\left[-\left(a^2+b^2\right)+\sqrt{\left((a^2-b^2)^2+4|u_k|\right)}\right]^{1/2}, \tag{3.14}$$

$\tau_0^{(k)}$ *are determined by*

$$\begin{aligned}\left[w_0^k\right]^2 &= c_k\cos w_0^{(k)}\tau - d_k\sin w_0^{(k)}\tau,\\ a+b &= c_k\sin w_0^{(k)}\tau - d_k\cos w_0^{(k)}\tau.\end{aligned} \tag{3.15}$$

Proof. Let iw $(w > 0)$ be a root of (3.13), then

$$-w^2 + i(a+b)w + ab - (c_k + d_k i)(\cos w\tau + i\sin w\tau) = 0. \tag{3.16}$$

Separating the real and imaging parts, the roots can be obtained. □

Lemma 3.3. $\mathrm{Re}(d\lambda/d\tau)|_{\tau=\tau_j^{(k)},w=w_0^{(k)}} > 0$.

Proof. Differentiating both sides of (3.13) with respect to τ gives

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + a + b - u_k e^{-\lambda\tau}}{u_k\lambda e^{-\lambda\tau}}, \tag{3.17}$$

that is,

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_j^{(k)},w=w_0^{(k)}} = \frac{\left[w_0^{(k)}\right]^2}{\Delta}\left[a^2+b^2+2\left[w_0^{(k)}\right]^2\right] > 0, \tag{3.18}$$

where $\Delta = (a+b)^2[w_0^{(k)}]^2 + [w_0^{(k)}]^2[[w_0^{(k)}]^2 - ab]^2$. □

Theorem 3.4. *Let* $\tau_0 = \min\{\tau_0^{(1)}, \tau_0^{(2)}, \ldots, \tau_0^{(n)}\}$.

(1) *If* $\max_{1\le k\le n}|u_k| < ab$, *then the zero solution of* (3.7) *is absolutely stable.*

(2) *If* $\min_{1\le k\le n}|u_k| \ge ab, \max_{1\le k\le n}\mathrm{Re}\,u_k < ab$, *then the zero solution of* (3.7) *is asymptotically stable when* $\tau \in [0,\tau_0)$ *and unstable when* $\tau > \tau_0$, *and* (3.7) *undergoes a Hopf bifurcation at the origin 0 when* $\tau = \tau_j^{(k)}$, $j = 0,1,\ldots$, $k = 1,2,\ldots,n$, *where* $\tau = \tau_j^{(k)}$ *is defined in Lemma 3.2.*

(3) *If* $\min_{1\le k\le n}\mathrm{Re}\,u_k \ge ab$, *then the zero solution of* (3.7) *is unstable for all* $\tau \ge 0$.

Proof. For $\tau = 0$, (3.13) becomes

$$\lambda^2 + (a+b)\lambda + ab - (c_k + d_k i) = 0. \tag{3.19}$$

Let $\lambda = \alpha + i\beta$ be a root of (3.19). Separating the real and imaginary parts, we can obtain

$$\begin{gathered} \alpha^2 - \beta^2 + (a+b)\alpha + ab - c_k = 0, \\ 2\alpha\beta + (a+b)\beta - d_k = 0. \end{gathered} \tag{3.20}$$

Hence, $\alpha = (-(a+b) \pm \sqrt{(a+b)^2 - 4(ab - c_k)})/2$. If $c_k < 0$, then $\alpha < 0$. Using Lemmas 3.2 and 3.3 the conclusions follow. □

4. Algebraic Criterion of Zero for Some Real Polynomial

Theorem 4.1. *Let A, B be (k,l) concordant. If $f(z)$ be a polynomial about z, z_0 is a root of*

$$\left| z^m \left[f(z) I - A \right] - B \right| = 0, \tag{4.1}$$

then

$$u = f(z_0) - a z_0^{-m} \tag{4.2}$$

is an eigenvalue of A, or

$$v = z_0^m \left[f(z_0) - a \right] \tag{4.3}$$

is an eigenvalue of B, where $a = k$ or l.

Proof. Let us assume that

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ O & kI_{n_2} \end{pmatrix}, \qquad PBP^{-1} = \begin{pmatrix} lI_{n_1} & B_{12} \\ O & B_{22} \end{pmatrix}. \tag{4.4}$$

Because z_0 is a root of

$$\begin{gathered} \left| z^m \left[f(z) I - A \right] - B \right| = 0, \\ \begin{vmatrix} z_0^m \left[f(z_0) I_{n_1} - A_{11} \right] - lI_{n_1} & * \\ O & z_0^m \left[f(z_0) I_{n_2} - kI_{n_2} \right] - B_{22} \end{vmatrix} = 0, \end{gathered} \tag{4.5}$$

then

$$\left| z_0^m \left[f(z_0) I_{n_1} - A_{11} \right] - lI_{n_1} \right| = 0 \quad \text{or} \quad \left| z_0^m \left[f(z_0) I_{n_2} - kI_{n_2} \right] - B_{22} \right| = 0. \tag{4.6}$$

If $|z_0^m[f(z_0)I_{n_1} - A_{11}] - lI_{n_1}| = 0$, then $u = f(z_0) - az_0^{-m}$ is an eigenvalue of A.

If $|z_0^m[f(z_0)I_{n_2} - kI_{n_2}] - B_{22}| = 0$, then $v = z_0^m[f(z_0) - a]$ is an eigenvalue of B.

For the application of Theorem 4.1, consider the discretization of BAM neural network (3.7). Let $U(t) = X(t-\tau_2), V(t) = Y(t), \tau = (\tau_1 + \tau_2)$, then (3.7) can be rewritten as

$$\begin{aligned} \dot{U}(t) &= -aU(t) + f(V(t-\tau)), \\ \dot{V}(t) &= -bV(t) + g(U(t)). \end{aligned} \tag{4.7}$$

Let $M(t) = U(t\tau), N(t) = V(t\tau)$, then (4.7) can be rewritten as

$$\begin{aligned} \dot{M}(t) &= -a\tau M(t) + \tau f(N(t-1)), \\ \dot{N}(t) &= -b\tau N(t) + \tau g(M(t)). \end{aligned} \tag{4.8}$$

Let $h = 1/m, m \in N_+$, using an Euler method to (4.8), we obtain

$$\begin{aligned} M_{n+1} &= M_n - ha\tau M_n + h\tau f(N_{n-m}), \\ N_{n+1} &= N_n - hb\tau N_n + h\tau g(M_n). \end{aligned} \tag{4.9}$$

Let $Z_n = (M_n^T, N_n^T, N_{n-1}^T, \ldots, N_{n-m}^T)$. Using the notation, (4.9) can be expressed as

$$Z_{n+1} = F(Z_n, \tau), \tag{4.10}$$

where $F = (F_{-1}^T, F_0^T, F_1^T, \ldots, F_m^T)$ and

$$F_k = \begin{cases} (1-ha\tau)M_n + h\tau f(N_{n-m}), & k = -1, \\ (1-hb\tau)N_n + h\tau g(M_n), & k = 0, \\ N_{n-k}, & 1 \le k \le m. \end{cases} \tag{4.11}$$

It is clear that 0 is also an equilibrium of (4.10). The linearization of (4.10) at the origin 0 is

$$Z_{n+1} = \widetilde{A}(\tau)z_n, \tag{4.12}$$

$$\widetilde{A}(\tau) = \begin{pmatrix} (1-ha\tau)I_n & 0 & 0 & \cdots & 0 & h\tau B_1 \\ h\tau B_2 & (1-hb\tau)I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I_n & 0 \end{pmatrix}. \tag{4.13}$$

The characteristic equation of (4.12) is

$$\left| zI - \widetilde{A}(\tau) \right| = 0. \tag{4.14}$$

Since $|1 - ha\tau| < 1, |1 - hb\tau| < 1$, hence there is no influence on the stability of (4.10). Let $z \neq 1 - ha\tau, z \neq 1 - hb\tau$, then (4.14) can be rewritten as

$$\left| z^m(z - 1 + ha\tau) z^m(z - 1 + hb\tau) I_n - (h\tau)^2 B_1 B_2 \right| = 0. \tag{4.15}$$

Let u_k $(1 \leq k \leq n)$ be an eigenvalue of $B_1 B_2$. Using Theorem 4.1, we have

$$[z^m(z - 1 + ha\tau)][z^m(z - 1 + hb\tau)] I_r - (h\tau)^2 u_k = 0. \tag{4.16}$$

It is clear that the stability of system (4.10) is determined by the distribution of the roots of (4.16). Next, we consider a special case of system (3.7)

$$\dot{X} = -aX + f(X(t - \tau)), \tag{4.17}$$

where $X \in \mathbf{R}^n$, which is a Hopfiled neural network with delay. Its discrete system is

$$M_{n+1} = (1 - ha\tau) M_n + h\tau f(M_{n-m}). \tag{4.18}$$

The linear part is

$$M_{n+1} = (1 - ha\tau) M_n + h\tau B M_{n-m}. \tag{4.19}$$

It is clear that we only need to discuss the distribution of the roots

$$z^m(z - 1 + ha\tau) = h\tau u_k, \tag{4.20}$$

where u_k $(1 \leq k \leq n)$ is an eigenvalue of B. □

Lemma 4.2. *Let* $\max_{1 \leq k \leq n} \operatorname{Re} u_k < a$. *There exists a* $\tilde{\tau} > 0$ *such that for* $0 < \tau < \tilde{\tau}$, *all roots of* (4.20) *have moduli less than one.*

Proof. When $\tau = 0$, (4.20) has an m-fold root $z = 0$ and a simple root $z = 1$ at $\tau = 0$. Consider the root $z(t)$ such that $z(0) = 1$. This root depends continuously on τ and is a differential function of τ:

$$\left. \frac{d|z|^2}{d\tau} \right|_{z=1,\tau=0} = \left. \left(z \frac{d\tilde{z}}{d\tau} + \tilde{z} \frac{dz}{d\tau} \right) \right|_{z=1,\tau=0} = 2(-a + \operatorname{Re} u_k) h < 0. \tag{4.21}$$

Hence, $|z(t)| < 1$ for all sufficiently small $\tau > 0$. Thus, all roots of (4.20) lie in $|z(t)| < 1$ for sufficiently small positive τ and existence of the maximal $\tilde{\tau}$ follows. □

Lemma 4.3. *Assume that the seep size h is sufficiently small. Let* $\max_{1 \leq k \leq n} |u_k| < 1$, *then there are no roots of* (4.20) *with moduli one for all* $\tau > 0$.

Proof. Let $z = e^{iw^*/m}$ be a root of (4.20) when $\tau = \tau^*$, then

$$\begin{aligned} \cos\frac{m+1}{m}w^* - \left(1 - ha\tau_k^*\right)\cos w^* &= h\tau_k^* c_k, \\ \sin\frac{m+1}{m}w^* - \left(1 - hb\tau_k^*\right)\sin w^* &= h\tau_k^* d_k, \end{aligned} \tag{4.22}$$

where $u_k = c_k + id_k$, hence

$$\cos\frac{w^*}{m} = 1 + \frac{\left(a^2 - |u_k|^2\right) h^2(\tau^*)^2}{2(1 - ha\tau^*)}. \tag{4.23}$$

Let $|u_k| < a$, for sufficiently small $h > 0$, $|\cos(w^*/m)| > 1$, which yields a contradiction. The proof is complete. □

Lemma 4.4. *If the seep size h is sufficiently small, then*

$$\left.\frac{d|z|^2}{d\tau}\right|_{\tau=\tau_k^*, w=w^*} > 0, \tag{4.24}$$

where τ^*, w^*satisfy (4.22).

Proof. Consider the following equation:

$$\begin{aligned} \left.\frac{d|z|^2}{d\tau}\right|_{\tau=\tau_k^*, w=w^*} &= \left.\left(z\frac{d\tilde{z}}{d\tau} + \tilde{z}\frac{dz}{d\tau}\right)\right|_{\tau=\tau_k^*, w=w^*} \\ &= \frac{2}{\tau} \times \frac{(m+1)(1-\cos(w^*/m)) + m\left(1 - \tau_k^* ha\right)(1-\cos(w^*/m))}{(m+1)^2 + \left(1 - ha\tau_k^*\right)^2 + 2m(m+1)\left(1 - ha\tau_k^*\right)\cos(w^*/m)} > 0. \end{aligned} \tag{4.25}$$

□

Using Lemmas 4.1–4.4 and Theorem 1 of [11], we have the following results.

Theorem 4.5. *(1) If* $\max_{1\le k\le n}|u_k| < a$, *then the zero solution of* (4.10) *is unstable for* $\tau \ge 0$.

(2) If $\min_{1\le k\le n}|u_k| \ge a$, $\max_{1\le k\le n}u_k < a$, *then the zero solution of* (4.18) *is asymptotically stable when* $\tau \in (0, \tau^*)$, *where* $\tau^* = \min\{\tau_1^*, \tau_2^*, \ldots, \tau_n^*\}$, *and* (4.10) *undergoes a Naimark-Sacker bifurcation.*

(3) If $\min_{1\le k\le n}\operatorname{Re} u_k \ge a$, *then the zero solution of* (4.10) *is unstable for all* $\tau \ge 0$.

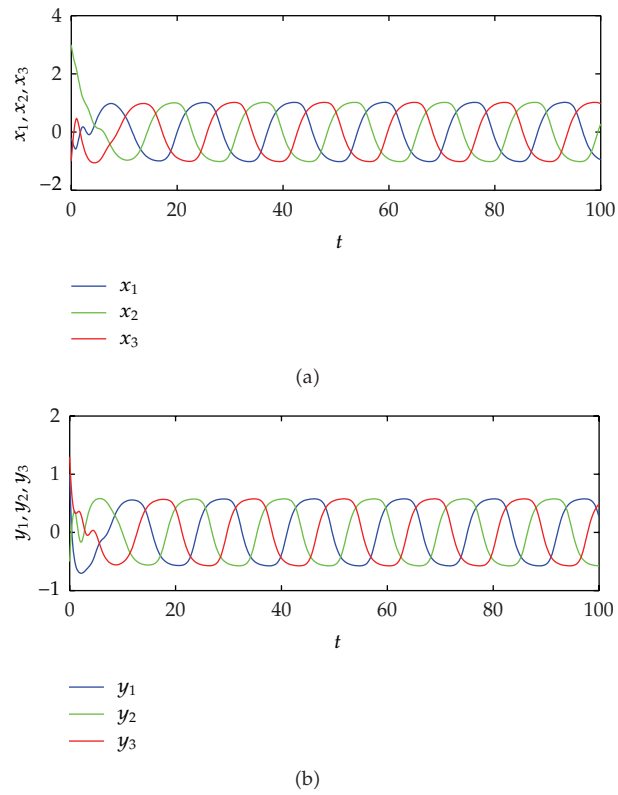

Figure 1: Two phase-locked oscillations: $x_1(t) = x_2(t + P_T/3) = x_3(t + 2P_T/3)$, $y_1(t) = y_2(t + P_T/3) = y_3(t + 2P_T/3)$.

5. Computer Simulation

To illustrate the analytical results found, let us consider the following particular case of (3.7):

$$\begin{aligned}
\dot{x}_1 &= -x_1 + 2\tanh\big(y_2(t-\tau_1)\big),\\
\dot{x}_2 &= -x_2 + 2\tanh\big(y_3(t-\tau_1)\big),\\
\dot{x}_3 &= -x_3 + 2\tanh\big(y_1(t-\tau_1)\big),\\
\dot{y}_1 &= -2y_1 - 1.5\tanh(x_2(t-\tau_2)),\\
\dot{y}_2 &= -2y_2 - 1.5\tanh(x_3(t-\tau_2)),\\
\dot{y}_3 &= -2y_3 - 1.5\tanh(x_1(t-\tau_2)),
\end{aligned} \tag{5.1}$$

where τ_1, τ_2 are parameters.

Using the conclusions of Theorem 3.4, the phase-locked periodic solutions appear with $\tau_1 + \tau_2 = 1.731$. See Figure 1.

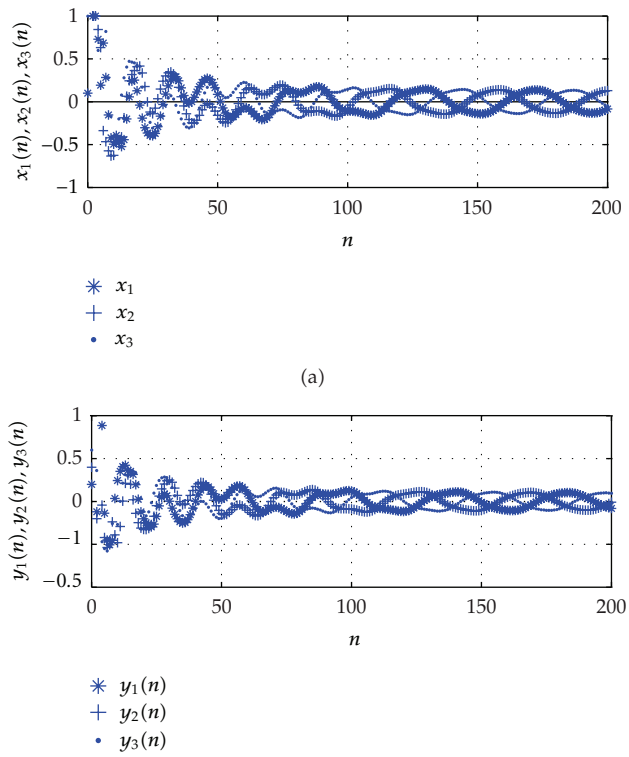

Figure 2: Two phase-locked oscillations: $x_1(n) = x_2(n + P_T/3) = x_3(n + 2P_T/3)$, $y_1(n) = y_2(n + P_T/3) = y_3(n + P_T/3)$.

For the special case of (4.8), we have the following equations:

$$\begin{aligned}
x_{1,n+1} &= (1-\tau h)x_{1,n} + 2\tau h \tanh(y_{2,n-m}),\\
x_{2,n+1} &= (1-\tau h)x_{2,n} + 2\tau h \tanh(y_{3,n-m}),\\
x_{3,n+1} &= (1-\tau h)x_{3,n} + 2\tau h \tanh(y_{1,n-m}),\\
y_{1,n+1} &= (1-2\tau h)y_{1,n} - 1.5\tau h \tanh(x_{2,n-m}),\\
y_{2,n+1} &= (1-2\tau h)y_{2,n} - 1.5\tau h \tanh(x_{3,n-m}),\\
y_{3,n+1} &= (1-2\tau h)y_{3,n} - 1.5\tau h \tanh(x_{1,n-m}),
\end{aligned} \tag{5.2}$$

where $\tau = 1.731, m = 3, h = 1/m$. The phase-locked periodic solutions appear. See Figure 2.

Acknowledgment

This research was supported by the National Natural Science Foundation of China (no. 10871056).

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