Research Article

f -Orthomorphisms and f -Linear Operators on the Order Dual of an f -Algebra

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We consider the f -orthomorphisms and f -linear operators on the order dual of an f -algebra. In particular, when the f -algebra has the factorization property (not necessarily unital), we prove that the orthomorphisms, f -orthomorphisms, and f -linear operators on the order dual are precisely the same class of operators.

1. Introduction

Let A be an f -algebra with ${}^\circ(A^\sim) = \{0\}$. Recall that we can define a multiplication on $(A^\sim)_n^\sim$, the order continuous part of the order bidual of A , with respect to which $(A^\sim)_n^\sim$ can also be made an f -algebra. This is done in three steps:

- (1) $A \times A^\sim \rightarrow A^\sim$
 $(a, f) \mapsto f \cdot a : (f \cdot a)(b) = f(ab)$ for $b \in A$,
- (2) $(A^\sim)_n^\sim \times A^\sim \rightarrow A^\sim$
 $(F, f) \mapsto F \cdot f : (F \cdot f)(a) = F(f \cdot a)$ for $a \in A$,
- (3) $(A^\sim)_n^\sim \times (A^\sim)_n^\sim \rightarrow (A^\sim)_n^\sim$
 $(F, G) \mapsto F \cdot G : (F \cdot G)(f) = F(G \cdot f)$ for $f \in A^\sim$.

With the so-called Arens multiplication defined in step (3), $(A^\sim)_n^\sim$ is an Archimedean (and hence commutative) f -algebra. Moreover, if A has a multiplicative unit, then $(A^\sim)_n^\sim = (A^\sim)^\sim$, the whole order bidual of A . The mapping $V : (A^\sim)_n^\sim \rightarrow \text{Orth}(A^\sim)$ defined by $V(F) = V_F$ for all $F \in (A^\sim)_n^\sim$, where $V_F(f) = F \cdot f$ for every $f \in A^\sim$, is an algebra and Riesz isomorphism. See [1, 2] for details.

Let A be an f -algebra. A Riesz space L with $0(L^-) = \{0\}$ is said to be an (left) f -module over A (cf. [2, 3]) if L is a left module over A and satisfies the following two conditions:

- (i) for each $a \in A^+$ and $x \in L^+$, we have $ax \in L^+$,
- (ii) if $x \perp y$, then for each $a \in A$, we have $a \cdot x \perp y$.

When A is an f -algebra with unit e , saying L is a unital f -module over A implies that the left multiplication satisfies $e \cdot x = x$ for all $x \in L$. From Corollary 2.3 in [2], we know that if L is an f -module over A , then L^- is an f -module over A (and $(A^-)^\sim_n$). The f -module L over A with unit e is said to be *topologically full* with respect to A if for two arbitrary vectors x, y satisfying $0 \leq y \leq x$ in L , there exists a net $0 \leq a_\alpha \leq e$ in A such that $a_\alpha \cdot x \rightarrow y$ in $\sigma(L, L^-)$. If L is topologically full with respect to A , then L^- is topologically full with respect to $(A^-)^\sim_n$ [2, Proposition 3.12].

Let A be a unital f -algebra, and, L, M be f -modules over A . $T \in L_b(L, M)$ is called an f -linear operator if $T(a \cdot x) = a \cdot Tx$ for each $a \in A$ and $x \in L$. The collection of all f -linear operators will be denoted by $L_b(L, M; A)$. For each $x \in L$ and $f \in L^-$, we can define $\psi_{x,f} \in A^-$ by $\psi_{x,f}(a) = f(a \cdot x)$ for all $a \in A$. Let $S(x) := \{\psi_{x,f} : f \in L^-\}$. Then $S(x)$ is an order ideal in A^- [2]. $T \in L_b(L, M)$ is said to be an f -orthomorphism if $S(Tx) \subseteq S(x)$ for each $x \in L$. The collection of all f -orthomorphisms will be denoted by $\text{Orth}(L, M; A)$. Turan [2] showed that $\text{Orth}(L, M; A) = L_b(L, M; A)$ whenever M is topologically full with respect to A .

Clearly, A^- is an f -module over the f -algebras A and $(A^-)^\sim_n$, respectively. If A is unital, then A is topologically full with respect to itself ([2, Proposition 2.6]). From the above remarks we know that A^- is topologically full with respect to $(A^-)^\sim_n$, and hence, the f -orthomorphisms and f -linear operators are precisely the same class of operators, that is,

$$\text{Orth}(A^-, A^-; (A^-)^\sim_n) = L_b(A^-, A^-; (A^-)^\sim_n). \quad (*)$$

An f -algebra A is said to be *square-root closed* whenever for any $a \in A$ there exists $b \in A$ such that $|a| = b^2$. An immediate example is that a uniformly complete f -algebra with unit element is square-root closed [4]. However, a square-root closed f -algebra is not necessarily unital. For instance, c_0 , with the familiar coordinatewise operations and ordering, is a square-root closed f -algebra without unit. We recall that an Archimedean f -algebra A is said to have the *factorization property* if, given $a \in A$, there exist $b, c \in A$ such that $a = bc$. It should be noted that if A is unital or square-root closed, then A has the factorization property.

In this paper, we do not have to assume that the f -algebras are unital. We modify the definition of the f -orthomorphism introduced by Turan [2, Definition 3.7] and consider the f -orthomorphisms and f -linear operators on the order dual of an f -algebra. In particular, when the f -algebra with separating order dual has the factorization property, we prove that the orthomorphisms, f -orthomorphisms, and f -linear operators on the order dual are precisely the same class of operators, that is, the above equality (*) still holds.

Our notions are standard. For the theory of Riesz spaces, positive operators, and f -algebras, we refer the reader to the monographs [5–7].

2. f -Orthomorphisms on the Order Dual

Let A be an f -algebra with separating order dual (and hence A Archimedean!) and $f \in A^-$. We consider the mapping $T_f : (A^-)^\sim_n \rightarrow A^-$ defined by $T_f(F) = F \cdot f$ for all $F \in (A^-)^\sim_n$.

It should be noted that the mapping $V : (A^\sim)_n^\sim \rightarrow \text{Orth}(A^\sim)$ defined by $V(F) = V_F$ for all $F \in (A^\sim)_n^\sim$, where $V_F(f) = F \cdot f$ for every $f \in A^\sim$, is an algebra and Riesz isomorphism (cf. [2, Proposition 2.2]).

Theorem 2.1. *For $0 \leq f \in A^\sim$, T_f is an interval preserving lattice homomorphism.*

Proof. Clearly, T_f is linear and positive. Since the mapping V is a lattice homomorphism and $V_F, V_G \in \text{Orth}(A^\sim)$ for $F, G \in (A^\sim)_n^\sim$, we have

$$\begin{aligned}
 T_f(F \vee G) &= (F \vee G) \cdot f = V_{F \vee G}(f) \\
 &= (V(F \vee G))(f) \\
 &= (V(F) \vee V(G))(f) \\
 &= (V(F)(f)) \vee (V(G)(f)) \\
 &= F \cdot f \vee G \cdot f \\
 &= T_f(F) \vee T_f(G).
 \end{aligned} \tag{2.1}$$

Hence, T_f is a lattice homomorphism.

Next, we show that T_f is an interval preserving operator. We identify x with its canonical image x'' in $(A^\sim)_n^\sim$ and denote the restriction of T_f to A by $T_f|_A$. Then

$$T_f|_A(x) = T_f(x'') = x'' \cdot f = f \cdot x. \tag{2.2}$$

Thus, for each $F \in (A^\sim)_n^\sim$ and $x \in A$, we see that

$$\left((T_f|_A)'(F) \right)(x) = F((T_f|_A)(x)) = F(f \cdot x) = (F \cdot f)(x) = (T_f(F))(x), \tag{2.3}$$

which implies that $(T_f|_A)'$ is the same as T_f on $(A^\sim)_n^\sim$. Since $(T_f|_A)'$ is interval preserving (cf. [5, Theorem 7.8]), T_f is likewise an interval preserving operator. $\square$

Corollary 2.2. *For $f \in A^\sim, F \in (A^\sim)_n^\sim$, one has $|F \cdot f| = |F| \cdot |f|$. Furthermore, if $f \perp g$ in $A^\sim$, $F \cdot f \perp G \cdot g$ holds for any $F, G \in (A^\sim)_n^\sim$.*

Proof. Since V_F is an orthomorphism on $A^\sim$, we have $V_F(f^+) \perp V_F(f^-)$ for each $f \in A^\sim$, that is, $F \cdot (f^+) \perp F \cdot (f^-)$. From Theorem 2.1, we know that

$$\begin{aligned}
 |F \cdot f| &= |F \cdot f^+| + |F \cdot f^-| \\
 &= |T_{f^+}(F)| + |T_{f^-}(F)| \\
 &= T_{f^+}(|F|) + T_{f^-}(|F|) \\
 &= |F| \cdot f^+ + |F| \cdot f^- = |F| \cdot |f|.
 \end{aligned} \tag{2.4}$$

Let $f \perp g$ in $A^\sim$. Then we have

$$\begin{aligned} |F \cdot f| \wedge |G \cdot g| &= |F| \cdot |f| \wedge |G| \cdot |g| \\ &\leq ((|F| + |G|) \cdot |f|) \wedge ((|F| + |G|) \cdot |g|) = 0, \end{aligned} \quad (2.5)$$

which implies that $F \cdot f \perp G \cdot g$ for all $F, G \in (A^\sim)_n^\sim$. $\square$

Following the above discussion, we now consider $R(f) = \{F \cdot f : F \in (A^\sim)_n^\sim\}$, the image of $(A^\sim)_n^\sim$ under T_f .

Corollary 2.3. *If A is an f -algebra and $f \in (A^\sim)$, then $R(f) = R(|f|)$, and $R(f)$ is an order ideal in $A^\sim$.*

Proof. First, since $T_{|f|}$ is an interval preserving lattice homomorphism, we can easily see that $R(|f|)$ is an order ideal in $A^\sim$. By Corollary 2.2 we conclude that $R(f) \subseteq R(|f|)$.

Now, to complete the proof we only need to prove that $R(|f|) \subseteq R(f)$. To this end, let $P_1 : A^\sim \rightarrow B_{f^+}$, $P_2 : A^\sim \rightarrow B_{f^-}$ be band projections, where B_{f^+} and B_{f^-} are the bands generated by f^+ and f^- in $A^\sim$, respectively. If $\pi = P_1 - P_2$, we have

$$\pi \in \text{Orth}(A^\sim), \quad \pi(f) = |f|, \quad \pi(|f|) = f. \quad (2.6)$$

In addition, $\pi(f) \cdot a = \pi(f \cdot a)$ for all $a \in A$ (cf. Theorem 3.1). Since π is an orthomorphism on $A^\sim$ and hence order continuous (cf. [5, Theorem 8.10]), we have $\pi'((A^\sim)_n^\sim) \subseteq (A^\sim)_n^\sim$. For all $a \in A$ and all $F \in (A^\sim)_n^\sim$, from

$$\begin{aligned} (F \cdot |f|)(a) &= (F \cdot \pi(f))(a) \\ &= F(\pi(f) \cdot a) \\ &= F(\pi(f \cdot a)) \\ &= (\pi'(F) \cdot f)(a), \end{aligned} \quad (2.7)$$

it follows that $F \cdot |f| = \pi'(F) \cdot f$ for all $F \in (A^\sim)_n^\sim$, which implies that $R(|f|) \subseteq R(f)$, as desired. $\square$

Next, we give a necessary and sufficient condition for $R(f) \perp R(g)$ when A has the factorization property. First, we need the following lemma.

Lemma 2.4. *Let A be an f -algebra with the factorization property, and $f \in A^\sim$. If $f \cdot x = 0$ for each $x \in A$, then $f = 0$.*

Proof. Since A has the factorization property, for each $a \in A^+$, there exist $x, y \in A$ such that $a = xy$. Hence, from

$$f(a) = f(xy) = (f \cdot x)(y) = 0, \quad (2.8)$$

it follows easily that $f = 0$ holds. $\square$

Theorem 2.5. Let A be an f -algebra with the factorization property. If $f, g \in A^\sim$, then $f \perp g$ if and only if $R(f) \perp R(g)$.

Proof. If $f \perp g$ in $A^\sim$, then it follows from Corollary 2.2 that $F \cdot f \perp G \cdot g$ for all $F, G \in (A^\sim)_n^\sim$. This implies that $R(f) \perp R(g)$.

Conversely, if $R(f)$ and $R(g)$ are disjoint, then for each $F \in ((A^\sim)_n^\sim)^+$ we have

$$\begin{aligned} F \cdot (|f| \wedge |g|) &= V_F(|f| \wedge |g|) \\ &= V_F(|f|) \wedge V_F(|g|) \\ &= F \cdot |f| \wedge F \cdot |g| \\ &= |F \cdot f| \wedge |F \cdot g| = 0. \end{aligned} \tag{2.9}$$

In particular, for any $x \in A$, its canonical image $x'' \in (A^\sim)_n^\sim$ also satisfies $x'' \cdot (|f| \wedge |g|) = (|f| \wedge |g|) \cdot x = 0$. By the preceding lemma, we have $|f| \wedge |g| = 0$, that is, $f \perp g$, as desired. $\square$

Now, we give the definition of the so-called f -orthomorphism.

Definition 2.6. Let A be an f -algebra and $T \in L_b(A^\sim)$. T is called an f -orthomorphism on $A^\sim$ if $R(Tf) \subseteq R(f)$ for each $f \in A^\sim$. The collection of all f -orthomorphisms on $A^\sim$ will be denoted by $\text{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$.

The next result deals with the relationship between the f -orthomorphisms and the orthomorphisms on the order dual of an f -algebra with the factorization property. Note that $\text{Orth}(A^\sim)$ is a band in $L_b(A^\sim)$.

Theorem 2.7. Let A be an f -algebra. Then $\text{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$ is a linear subspace of $L_b(A^\sim)$ and $\text{Orth}(A^\sim) \subseteq \text{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$.

If A , in addition, has the factorization property, then $\text{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim) = \text{Orth}(A^\sim)$.

Proof. First, we can easily see that $\text{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$ is a linear subspace of $L_b(A^\sim)$. To prove $\text{Orth}(A^\sim) \subseteq \text{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$, let $\pi \in \text{Orth}(A^\sim)$. We claim that $F \cdot \pi(f) = \pi'(F) \cdot f$ for all $F \in (A^\sim)_n^\sim$ and all $f \in A^\sim$. To this end, let $F \in (A^\sim)_n^\sim$, $f \in A^\sim$, and $x \in A$ be arbitrary. Since $(A^\sim)_n^\sim$ is a commutative f -algebra, by Theorem 3.1, we have

$$\begin{aligned} (\pi'(F) \cdot f)(x) &= \pi'(F)(f \cdot x) = F(\pi(f \cdot x)) = F(\pi(x'' \cdot f)) \\ &= F(x'' \cdot (\pi(f))) \\ &= (F \cdot x'')(\pi(f)) \\ &= (x'' \cdot F)(\pi(f)) \\ &= x''(F \cdot \pi(f)) = (F \cdot \pi(f))(x). \end{aligned} \tag{2.10}$$

Thus, $F \cdot \pi(f) = \pi'(F) \cdot f$. This implies that $R(\pi(f)) \subseteq R(f)$ for each $f \in A^\sim$, that is, $\text{Orth}(A^\sim) \subseteq \text{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$.

If A has the factorization property, we prove that $\text{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim) \subseteq \text{Orth}(A^\sim)$ holds. To this end, take $T \in \text{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$ and $f, g \in A^\sim$ satisfying $f \perp g$ in $A^\sim$. Then, it follows from Theorem 2.5 that $R(f) \perp R(g)$. Since $T \in \text{Orth}(A^\sim, A^\sim; (A^\sim)_n^\sim)$, we have

$R(T(f)) \subset R(f)$. Therefore, $R(T(f)) \perp R(g)$, which implies that $T(f) \perp g$, and hence T is an orthomorphism on $A^\sim$, as desired. $\square$

3. f -Linear Operators on the Order Dual

Let A be an f -algebra with separating order dual and $T \in L_b(A^\sim)$. Recall that T is called to be f -linear with respect to $(A^\sim)_n^\sim$ if $T(G \cdot f) = G \cdot T(f)$ for all $f \in A^\sim$ and $G \in (A^\sim)_n^\sim$. The set of all f -linear operators on $A^\sim$ will be denoted by $L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$. It follows from [3, Lemma 4.4] that $L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$ is a band in $L_b(A^\sim)$.

Theorem 3.1. *Let A be an f -algebra with separating order dual. Then $\text{Orth}(A^\sim) \subseteq L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$.*

Proof. Clearly $\text{Orth}(A^\sim)$ is commutative since $\text{Orth}(A^\sim)$ is an Archimedean f -algebra. To complete the proof, let $\pi \in \text{Orth}(A^\sim)$. We have

$$\pi(G \cdot f) = \pi(V_G(f)) = V_G(\pi(f)) = G \cdot (\pi(f)), \quad (3.1)$$

for all $f \in A^\sim$ and $G \in (A^\sim)_n^\sim$. Hence, $\pi \in L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$. $\square$

The following result deals with the order adjoint of an f -linear operator on the order dual of an f -algebra. It should be noted that the order adjoint of an order-bounded operator is order continuous (cf. [5, Theorem 5.8]).

Lemma 3.2. *Let $T \in L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$. Then the order adjoint T' of T satisfies $T'(F) \cdot f = F \cdot T(f)$ for all $F \in (A^\sim)_n^\sim$ and $f \in A^\sim$. In particular, $G \cdot T'(F) = T'(G \cdot F)$ for all $F, G \in (A^\sim)_n^\sim$.*

Proof. Since $T \in L_b(A^\sim, A^\sim; (A^\sim)_n^\sim)$, and $(A^\sim)_n^\sim$ is a commutative f -algebra, we have

$$\begin{aligned} (T'(F) \cdot f)(x) &= T'(F)(f \cdot x) = F(T(f \cdot x)) \\ &= F(T(x'' \cdot f)) \\ &= F(x'' \cdot (T(f))) \\ &= (F \cdot x'')(T(f)) \\ &= (x'' \cdot F)(T(f)) \\ &= x''(F \cdot T(f)) = (F \cdot T(f))(x), \end{aligned} \quad (3.2)$$

for all $F \in (A^\sim)_n^\sim$, $f \in A^\sim$, and $x \in A$, which implies that $T'(F) \cdot f = F \cdot T(f)$.

Let $F, G \in (A^\sim)_n^\sim$ be given. Then for $f \in A^\sim$, from

$$\begin{aligned} (G \cdot T'(F))(f) &= G(T'(F) \cdot f) = G(F \cdot T(f)) \\ &= (G \cdot F)(T(f)) \\ &= (T'(G \cdot F))(f), \end{aligned} \quad (3.3)$$

it follows that $G \cdot T'(F) = T'(G \cdot F)$. This completes the proof. $\square$

Theorem 3.3.

$$L_b(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}) \subseteq \text{Orth}(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}). \quad (3.4)$$

Proof. For $T \in L_b(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim})$, we know that $|T|$ is also f -linear with respect to $(A^{\sim})_n^{\sim}$. Assume that $0 \leq G \in (A^{\sim})_n^{\sim}$ and $f \in A^{\sim}$. So by Lemma 3.2, we have

$$0 \leq G \cdot (|T(f)|) \leq G \cdot (|T||f|) = (|T|'(G)) \cdot |f| = T_{|f|}(|T|'(G)). \quad (3.5)$$

Since $T_{|f|}$ is interval preserving, there exists $F \in (A^{\sim})_n^{\sim}$ such that $0 \leq F \leq |T|'(G)$ and $G \cdot (|T(f)|) = F \cdot |f|$. It is now immediate that $R(|T(f)|) \subseteq R(|f|)$, and hence, $L_b(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}) \subseteq \text{Orth}(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim})$, as desired. $\square$

Combining Theorems 3.1, 3.3, and 2.7, we have the following result.

Theorem 3.4. *If A is an f -algebra with separating order dual, then*

$$\text{Orth}(A^{\sim}) \subseteq L_b(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}) \subseteq \text{Orth}(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}). \quad (3.6)$$

In particular, if, in addition, A has the factorization property, then

$$\text{Orth}(A^{\sim}) = L_b(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}) = \text{Orth}(A^{\sim}, A^{\sim}; (A^{\sim})_n^{\sim}). \quad (3.7)$$

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