Research Article

Approximately Ternary Homomorphisms and Derivations on C*-Ternary Algebras

M. Eshaghi Gordji, 1 A. Ebadian, 2 N. Ghobadipour, 3 J. M. Rassias, 4 and M. B. Savadkouhi 5

1 Department of Mathematics, Semnan University, Semnan 35195-363, Iran
2 Department of Mathematics, Payame Noor University, Tabriz Branch, Tabriz, Iran
3 Department of Mathematics, Urmia University, Urmia, Iran
4 Section of Mathematics and Informatics, Pedagogical Department, National and Kapodistrian University of Athens, 4, Agamemnonos Street, Aghia Paraskevi, 15342 Athens, Greece
5 Department of Mathematics, Islamic Azad University, Mahdishahr Branch, Mahdishahr, Semnan, Iran

Correspondence should be addressed to M. Eshaghi Gordji, madjid.eshaghi@gmail.com and A. Ebadian, ebadian.ali@gmail.com

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We investigate the stability and superstability of ternary homomorphisms between C*-ternary algebras and derivations on C*-ternary algebras, associated with the following functional equation

\[ f\left(\frac{x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3x_3}{3}\right) + f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) = f(x_1). \]

1. Introduction

A C*-ternary algebra is a complex Banach space A, equipped with a ternary product \((x, y, z) \mapsto [x, y, z]\) of \(A^3\) into A, which is C-linear in the outer variables, conjugate C-linear in the middle variable, and associative in the sense that \([x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]\), and satisfies \(\|x, y, z\| \leq \|x\| \cdot \|y\| \cdot \|z\|\) and \(\|[x, x, x]\| = \|x\|^3\). If a C*-ternary algebra \((A, [\cdot, \cdot, \cdot])\) has an identity, that is, an element \(e \in A\) such that \(x = [x, e, e] = [e, e, x]\) for all \(x \in A\), then it is routine to verify that A, endowed with \(xoy := [x, e, y]\) and \(x^* := [e, x, e]\), is a unital C*-algebra. Conversely, if \((A, o)\) is a unital C*-algebra, then \(x, y, z := xoy^*oz\) makes A into a C*-ternary algebra. A C-linear mapping \(H : A \to B\) is called a C*-ternary algebra homomorphism if

\[ H([x, y, z]) = [H(x), H(y), H(z)], \]
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for all $x, y, z \in A$. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a $C^*$-ternary algebra derivation if

$$
\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)],
$$

for all $x, y, z \in A$.

Ternary structures and their generalization the so-called $n$-ary structures raise certain hopes in view of their applications in physics (see [1–8]). We say a functional equation $\zeta$ is stable if any function $g$ satisfying the equation $\zeta$ approximately is near to true solution of $\zeta$. Moreover, $\zeta$ is superstable if every approximately solution of $\zeta$ is an exact solution of it.


A generalized version of the theorem of Hyers for approximately additive maps was given by Rassias [11] in 1978 as follows.

**Theorem 1.1.** Let $f : E_1 \to E_2$ be a mapping from a normed vector space $E_1$ into a Banach space $E_2$ subject to the inequality:

$$
\|f(x + y) - f(x) - f(y)\| \leq e(|x|^p + \|y\|^p),
$$

for all $x, y \in E_1$, where $e$ and $p$ are constants with $e > 0$ and $p < 1$. Then, there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$
\|f(x) - T(x)\| \leq \frac{2e}{2 - 2^p} \|x\|^p,
$$

for all $x \in E_1$.

The stability phenomenon that was introduced and proved by Rassias is called Hyers-Ulam-Rassias stability. And then the stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [12–27]).

Throughout this paper, we assume that $A$ is a $C^*$-ternary algebra with norm $\| \cdot \|_A$ and that $B$ is a $C^*$-ternary algebra with norm $\| \cdot \|_B$. Moreover, we assume that $n_0 \in \mathbb{N}$ is a positive integer and suppose that $\mathbb{T}_{1/n_0} := \{e^{i\theta}, \ 0 \leq \theta \leq 2\pi/n_0\}$.

**2. Superstability**

In this section, first we investigate homomorphisms between $C^*$-ternary algebras. We need the following Lemma in the main results of the paper.

**Lemma 2.1.** Let $f : A \to B$ be a mapping such that

$$
\|f\left(\frac{x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3x_3}{3}\right) + f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right)\|_B \leq \|f(x_1)\|_B,
$$

for all $x_1, x_2, x_3 \in A$. Then $f$ is additive.
Proof. Letting $x_1 = x_2 = x_3 = 0$ in (2.1), we get
\[ \|3f(0)\|_B \leq \|f(0)\|_B. \] (2.2)

So $f(0) = 0$. Letting $x_1 = x_2 = 0$ in (2.1), we get
\[ \|f(-x_3) + f(x_3)\|_B \leq \|f(0)\|_B = 0, \] (2.3)
for all $x_3 \in A$. Hence $f(-x_3) = -f(x_3)$ for all $x_3 \in A$. Letting $x_1 = 0$ and $x_2 = 6x_3$ in (2.1), we get
\[ \|f(2x_3) - 2f(x_3)\|_B \leq \|f(0)\|_B = 0, \] (2.4)
for all $x_3 \in A$. Hence
\[ f(2x_3) = 2f(x_3), \] (2.5)
for all $x_3 \in A$. Letting $x_1 = 0$ and $x_2 = 9x_3$ in (2.1), we get
\[ \|f(3x_3) - f(x_3) - 2f(x_3)\|_B \leq \|f(0)\|_B = 0, \] (2.6)
for all $x_3 \in A$. Hence
\[ f(3x_3) = 3f(x_3), \] (2.7)
for all $x_3 \in A$. Letting $x_1 = 0$ in (2.1), we get
\[ \left\| f\left(\frac{x_2}{3}\right) + f(-x_3) + f\left(x_3 - \frac{x_2}{3}\right) \right\|_B \leq \|f(0)\|_B = 0, \] (2.8)
for all $x_2, x_3 \in A$. So
\[ f\left(\frac{x_2}{3}\right) + f(-x_3) + f\left(x_3 - \frac{x_2}{3}\right) = 0, \] (2.9)
for all $x_2, x_3 \in A$. Let $t_1 = x_3 - (x_2/3)$ and $t_2 = x_2/3$ in (2.9). Then
\[ f(t_2) - f(t_1 + t_2) + f(t_1) = 0, \] (2.10)
for all $t_1, t_2 \in A$, this means that $f$ is additive.

Now, we prove the first result in superstability as follows.
Theorem 2.2. Let $p \neq 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that

\begin{equation}
\left\| f \left( \frac{x_2 - x_1}{3} \right) + f \left( \frac{x_1 - 3\mu x_3}{3} \right) + \mu f \left( \frac{3x_1 + 3x_3 - x_2}{3} \right) \right\|_B \leq \| f(x_1) \|_B, \quad (2.11)
\end{equation}

\begin{equation}
\left\| f([x_1, x_2, x_3]) - [f(x_1), f(x_2), f(x_3)] \right\|_B \leq \theta \left( \| x_1 \|_A^3 + \| x_2 \|_A^3 + \| x_3 \|_A^3 \right), \quad (2.12)
\end{equation}

for all $\mu \in \mathbb{T}_{1/n}$ and all $x_1, x_2, x_3 \in A$. Then, the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism.

Proof. Assume $p > 1$.

Let $\mu = 1$ in (2.11). By Lemma 2.1, the mapping $f : A \to B$ is additive. Letting $x_1 = x_2 = 0$ in (2.11), we get

\begin{equation}
\left\| f(-\mu x_3) + \mu f(x_3) \right\|_B \leq \| f(0) \|_B = 0,
\end{equation}

for all $x_3 \in A$ and $\mu \in \mathbb{T}$. So

\begin{equation}
-f(\mu x_3) + \mu f(x_3) = f(-\mu x_3) + \mu f(x_3) = 0,
\end{equation}

for all $x_3 \in A$ and all $\mu \in \mathbb{T}$. Hence $f(\mu x_3) = \mu f(x_3)$ for all $x_3 \in A$ and all $\mu \in \mathbb{T}_{1/n}$. By same reasoning as proof of Theorem 2.2 of [28], the mapping $f : A \to B$ is $C$-linear. It follows from (2.12) that

\begin{equation}
\left\| f([x_1, x_2, x_3]) - [f(x_1), f(x_2), f(x_3)] \right\|_B
\end{equation}

\begin{equation}
= \lim_{n \to \infty} 8^n \left\| f \left( \frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n} \right) - [f \left( \frac{x_1}{2^n} \right), f \left( \frac{x_2}{2^n} \right), f \left( \frac{x_3}{2^n} \right)] \right\|_B
\end{equation}

\begin{equation}
\leq \lim_{n \to \infty} \frac{8^n \theta}{8np} \left( \| x_1 \|_A^3 + \| x_2 \|_A^3 + \| x_3 \|_A^3 \right) = 0,
\end{equation}

for all $x_1, x_2, x_3 \in A$. Thus,

\begin{equation}
f([x_1, x_2, x_3]) = [f(x_1), f(x_2), f(x_3)],
\end{equation}

for all $x_1, x_2, x_3 \in A$. Hence, the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism. Similarly, one obtains the result for the case $p < 1$.

Now, we establish the superstability of derivations on $C^*$-ternary algebras as follows.
Theorem 2.3. Let \( p \neq 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to A \) be a mapping satisfying (2.11) such that

\[
\| f([x_1, x_2, x_3]) - [f(x_1), x_2, x_3] - [x_1, f(x_2), x_3] - [x_1, x_2, f(x_3)] \|_A \\
\leq \theta \left( \|x_1\|_A^{3p} + \|x_2\|_A^{3p} + \|x_3\|_A^{3p} \right),
\]

for all \( x_1, x_2, x_3 \in A \). Then the mapping \( f : A \to A \) is a \( C^* \)-ternary derivation.

Proof. Assume \( p > 1 \).

By the Theorem 2.2, the mapping \( f : A \to A \) is \( C \)-linear. It follows from (2.17) that

\[
\| f([x_1, x_2, x_3]) - [f(x_1), x_2, x_3] - [x_1, f(x_2), x_3] - [x_1, x_2, f(x_3)] \|_A \\
= \lim_{n \to \infty} \left\| \frac{8^n}{8^n - 1} \right\| \frac{1}{2^n} \left( \frac{x_1}{2^n}, \frac{x_2}{2^n}, x_3 \right) \left( \frac{x_1}{2^n} \right) - \left( \frac{x_1}{2^n}, x_2, x_3 \right) \left( \frac{x_1}{2^n} \right) \|_A \\
\leq \lim_{n \to \infty} \frac{8^n}{8^n - 1} \left( \|x_1\|_A^{3p} + \|x_2\|_A^{3p} + \|x_3\|_A^{3p} \right) = 0,
\]

for all \( x_1, x_2, x_3 \in A \). So

\[
f([x_1, x_2, x_3]) = [f(x_1), x_2, x_3] + [x_1, f(x_2), x_3] + [x_1, x_2, f(x_3)]
\]

for all \( x_1, x_2, x_3 \in A \). Thus, the mapping \( f : A \to A \) is a \( C^* \)-ternary derivation. Similarly, one obtains the result for the case \( p < 1 \).

\[\square\]

3. Stability

First we prove the generalized Hyers-Ulam-Rassias stability of homomorphisms in \( C^* \)-ternary algebras.

Theorem 3.1. Let \( p > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to B \) be a mapping such that

\[
\| f \left( \frac{x_2 - x_1}{3} \right) + f \left( \frac{x_1 - 3\mu x_3}{3} \right) + \mu f \left( \frac{3x_1 + 3x_3 - x_2}{3} \right) - f(x_1) \|_B \\
\leq \theta \left( \|x_1\|_A^{3p} + \|x_2\|_A^{3p} + \|x_3\|_A^{3p} \right),
\]

\[
\| f([x_1, x_2, x_3]) - [f(x_1), f(x_2), f(x_3)] \|_B \leq \theta \left( \|x_1\|_A^{3p} + \|x_2\|_A^{3p} + \|x_3\|_A^{3p} \right),
\]

for all \( x_1, x_2, x_3 \in A \).
for all $\mu \in T_{1/m}$, and all $x_1, x_2, x_3 \in A$. Then there exists a unique $C^*$-ternary homomorphism $H : A \rightarrow B$ such that

$$\|H(x_1) - f(x_1)\|_B \leq \frac{\theta(1 + 2^p)\|x_1\|_A^p}{1 - 3^{1-p}},$$

(3.3)

for all $x_1 \in A$.

**Proof.** Let us assume $\mu = 1, x_2 = 2x_1$ and $x_3 = 0$ in (3.1). Then we get

$$\left\| 3f\left(\frac{x_1}{3}\right) - f(x_1) \right\|_B \leq \theta(1 + 2^p)\|x_1\|_A^p,$$

(3.4)

for all $x_1 \in A$. So by induction, we have

$$\left\| 3^n f\left(\frac{x_1}{3^n}\right) - f(x_1) \right\|_B \leq \theta(1 + 2^p)\|x_1\|_A^p \sum_{i=0}^{n-1} 3^{i(1-p)},$$

(3.5)

for all $x_1 \in A$. Hence

$$\left\| 3^{n+m} f\left(\frac{x_1}{3^{n+m}}\right) - 3^m f\left(\frac{x_1}{3^m}\right) \right\|_B \leq \theta(1 + 2^p)\|x_1\|_A^p \sum_{i=0}^{n-1} 3^{i(1-p)} \sum_{i=m}^{n+m-1} 3^{i(1-p)},$$

(3.6)

for all nonnegative integers $m$ and $n$ with $n \geq m$, and all $x_1 \in A$. It follows that the sequence $\{3^n f(x_1/3^n)\}$ is a Cauchy sequence for all $x_1 \in A$. Since $B$ is complete, the sequence $\{3^n f(x_1/3^n)\}$ converges. Thus, one can define the mapping $H : A \rightarrow B$ by

$$H(x_1) := \lim_{n \to \infty} 3^n f\left(\frac{x_1}{3^n}\right),$$

(3.7)

for all $x_1 \in A$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (3.6), we get (3.3). It follows from (3.1) that

$$\left\| H\left(\frac{x_2 - x_1}{3}\right) + H\left(\frac{x_1 - 3\mu x_3}{3}\right) + \mu H\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) - H(x_1) \right\|_B$$

$$= \lim_{n \to \infty} \left\| f\left(\frac{x_2 - x_1}{3^{n+1}}\right) + f\left(\frac{x_1 - 3\mu x_3}{3^{n+1}}\right) + f\left(\frac{3x_1 + 3x_3 - x_2}{3^{n+1}}\right) - f\left(\frac{x_1}{3^n}\right) \right\|_B$$

$$\leq \lim_{n \to \infty} \frac{3^n \theta}{3^np} \left(\|x_1\|^p_A + \|x_2\|^p_A + \|x_3\|^p_A\right) = 0,$$
for all $\mu \in \mathbb{T}_{1/n_0}$, and all $x_1, x_2, x_3 \in A$. So

$$H\left(\frac{x_2 - x_1}{3}\right) + H\left(\frac{x_1 - 3\mu x_3}{3}\right) + \mu H\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) = H(x_1),$$  \hspace{1cm} (3.9)$$

for all $\mu \in \mathbb{T}_{1/n_0}$ and all $x_1, x_2, x_3 \in A$. By the same reasoning as proof of Theorem 2.2 of [28], the mapping $H : A \rightarrow B$ is $\mathbb{C}$-linear.

Now, let $H' : A \rightarrow B$ be another additive mapping satisfying (3.3). Then, we have

$$\|H(x_1) - H'(x_1)\|_B = 3^n \|H\left(\frac{x_1}{3^n}\right) - H'\left(\frac{x_1}{3^n}\right)\|_B$$

$$\leq 3^n \left(\|H\left(\frac{x_1}{3^n}\right) - f\left(\frac{x_1}{3^n}\right)\|_B + \|H'\left(\frac{x_1}{3^n}\right) - f\left(\frac{x_1}{3^n}\right)\|_B\right)$$

$$\leq \frac{2 \cdot 3^n \theta(1 + 2^p)}{3^{np}(1 - 3^{1-p})} \|x\|_{A'}^p$$

which tends to zero as $n \rightarrow \infty$ for all $x_1 \in A$. So we can conclude that $H(x_1) = H'(x_1)$ for all $x_1 \in A$. This proves the uniqueness of $H$.

It follows from (3.2) that

$$\|H([x_1, x_2, x_3]) - [H(x_1), H(x_2), H(x_3)]\|_B$$

$$= \lim_{n \rightarrow \infty} 27^n \|f\left(\frac{[x_1, x_2, x_3]}{3^n \cdot 3^n \cdot 3^n}\right) - f\left(\frac{x_1}{3^n}\right), f\left(\frac{x_2}{3^n}\right), f\left(\frac{x_3}{3^n}\right)\|_B$$

$$\leq \lim_{n \rightarrow \infty} \frac{27^n \theta}{27^n \cdot 27^n} \left(\|x_1\|_A^{3p} + \|x_2\|_A^{3p} + \|x_3\|_A^{3p}\right) = 0,$$  \hspace{1cm} (3.11)$$

for all $x_1, x_2, x_3 \in A$.

Thus, the mapping $H : A \rightarrow B$ is a unique $C^*$-ternary homomorphism satisfying (3.3).

\begin{theorem}
Let $p < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \rightarrow B$ be a mapping satisfying (3.1) and (3.2). Then, there exists a unique $C^*$-ternary homomorphism $H : A \rightarrow B$ such that

$$\|H(x_1) - f(x_1)\|_B \leq \frac{\theta(1 + 2^p)\|x_1\|_{A'}^p}{3^{1-p} - 1},$$  \hspace{1cm} (3.12)$$

for all $x_1 \in A$.
\end{theorem}

\begin{proof}
The proof is similar to the proof of Theorem 3.1.
\end{proof}

Now, we prove the generalized Hyers-Ulam-Rassias stability of derivations on $C^*$-ternary algebras.
Theorem 3.3. Let $p > 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to A$ be a mapping such that

$$\|f\left(\frac{x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3\mu x_3}{3}\right) + \mu f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) - f(x_1)\|_A \leq \theta \left(\|x_1\|_A^p + \|x_2\|_A^p + \|x_3\|_A^p\right),$$

(3.13)

for all $\mu \in \mathbb{T}^1_{1/n}$, and all $x_1, x_2, x_3 \in A$. Then, there exists a unique $C^*$-ternary derivation $D : A \to A$ such that

$$\|D(x_1) - f(x_1)\|_A \leq \frac{\theta (1 + 2^p)\|x_1\|_A^p}{1 - 3^{1-p}},$$

(3.15)

for all $x_1 \in A$.

Proof. By the same reasoning as in the proof of the Theorem 3.1, there exists a unique $C$-linear mapping $D : A \to A$ satisfying (3.15). The mapping $D : A \to A$ is defined by

$$D(x_1) := \lim_{n \to \infty} 3^n f\left(\frac{x_1}{3^n}\right),$$

(3.16)

for all $x_1 \in A$. It follows from (3.14) that

$$\|D([x_1, x_2, x_3]) - [D(x_1), x_2, x_3] - [x_1, D(x_2), x_3] - [x_1, x_2, D(x_3)]\|_A$$

$$= \lim_{n \to \infty} 2^7\|\left[x_1, x_2, x_3\right] - \left[f\left(\frac{x_1}{3^n}\right), x_2, x_3\right] - \left[x_1, f\left(\frac{x_2}{3^n}\right), x_3\right] - \left[x_1, x_2, f\left(\frac{x_3}{3^n}\right)\right]\|_A$$

$$\leq \lim_{n \to \infty} \frac{27\theta}{27^n} \left(\|x_1\|_A^p + \|x_2\|_A^p + \|x_3\|_A^p\right) = 0,$$

(3.17)

for all $x_1, x_2, x_3 \in A$. So

$$D([x_1, x_2, x_3]) = [D(x_1), x_2, x_3] + [x_1, D(x_2), x_3] + [x_1, x_2, D(x_3)]$$

(3.18)

for all $x_1, x_2, x_3 \in A$.

Thus, the mapping $D : A \to A$ is a unique $C^*$-ternary derivation satisfying (3.15). $\square$
Theorem 3.4. Let $p < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (3.13) and (3.14). Then, there exists a unique $C^*$-ternary derivation $D : A \to A$ such that

$$\|D(x_1) - f(x_1)\|_A \leq \frac{\theta(1 + 2^p)\|x_1\|^p_1}{3^{1-p} - 1},$$

for all $x_1 \in A$.

Proof. The proof is similar to the proof of Theorems 3.1 and 3.3.

4. Conclusions

In this paper, we have analyzed some detail $C^*$-ternary algebras and derivations on $C^*$-ternary algebras, associated with the following functional equation:

$$f\left(\frac{x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3x_3}{3}\right) + f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) = f(x_1).$$

(4.1)

A detailed study of how we can have the generalized Hyers-Ulam-Rassias stability of homomorphisms and derivations on $C^*$-ternary algebras is given.

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