Research Article

The Relationship between Two Kinds of Generalized Convex Set-Valued Maps in Real Ordered Linear Spaces

Zhi-Ang Zhou

College of Mathematics and Statistics, Chongqing University of Technology, Chongqing 400054, China

Correspondence should be addressed to Zhi-Ang Zhou; zhi_ang@163.com

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1. Introduction

In optimization theory, the generalized convexity of set-valued maps plays an important role. Corley [1] introduced the cone convexity of set-valued maps. To extend the cone convexity of set-valued maps, some authors [2–5] introduced new generalized convexity such as cone convexlikeness, cone subconvexlikeness, generalized cone subconvexlikeness, nearly cone subconvexlikeness, and ic-cone-convexlikeness. The above generalized convexity set-valued maps mentioned were defined in topological spaces. Recently, Li [6] introduced the cone subconvexlike set-valued map based on the algebraic interior in linear spaces. Very recently, Hernández et al. [7] have defined the cone subconvexlikeness of the set-valued map characterized by the relative algebraic interior. Xu and Song [8] gave the relationship between ic-cone convexity and nearly cone subconvexlikeness in locally convex spaces. In this paper, we will extend the results obtained by Xu and Song [8] from locally convex spaces to linear spaces.

This paper is organized as follows. In Section 2, we give some preliminaries, including notations and lemmas. In Section 3, we obtain the relationship between ic-cone convexity and nearly cone subconvexlikeness in linear spaces. Our results generalize and improve the ones obtained by Xu and Song [8].

2. Preliminaries

In this paper, we always suppose that $A$ is a nonempty set and $Y$ is a real ordered linear space. Let $0$ denote the zero element for every space. Let $K$ be a nonempty subset in $Y$. The affine hull of $K$ is defined as $\text{aff}(K) := \{k \mid k = \sum_{i=1}^{n} \lambda_{i}k_{i}, \forall i \in \{1, 2, \ldots, n\}, k_{i} \in K, \lambda_{i} \in \mathbb{R}, \sum_{i=1}^{n} \lambda_{i} = 1\}$. The generated cone of $K$ is defined as $\text{cone}(K) := \{\lambda k \mid k \in K, \lambda \geq 0\}$. Write $\text{cone}_{+}(K) := \{\lambda k \mid k \in K, \lambda > 0\}$. Clearly, $\text{cone}(K) = \text{cone}_{+}(K) \cup \{0\}$. $K$ is called a cone if and only if $\lambda K \subseteq K$ for any $\lambda \geq 0$. Note that some authors defined the cone in the following way: $K$ is a cone if and only if $\lambda K \subseteq K$ for every $\lambda > 0$ [5]. It is possible that $0 \notin K$ if $K$ is a cone in the sense of the latter definition. Moreover, if $K$ is a cone in the sense of the former definition, then $K \cup \{0\}$ is a cone in the sense of the former definition. In this paper, if not specially specified, we suppose that all the cones mentioned are defined in the sense of the former definition. $K$ is called a convex set if and only if

$$\lambda k_{1} + (1-\lambda) k_{2} \in K, \quad \forall \lambda \in [0, 1], \forall k_{1}, k_{2} \in K. \quad (1)$$

Clearly, a cone $K$ is convex if and only if $K + K \subseteq K$. $K$ is said to be nontrivial if and only if $K \neq \{0\}$ and $K \neq Y$.

From now on, we suppose that $C$ is a nontrivial convex cone in $Y$ and $C_{+}$ satisfies the condition $C = C_{+} \cup \{0\}$. We recall the following well-known concepts.
**Definition 1** (see [9]). Let $K$ be a nonempty subset in $Y$. The algebraic interior of $K$ is the set

$$\text{cor}(K) := \left\{ k \in K \mid \forall h \in Y, \exists \lambda' > 0, \forall \lambda \in [0, \lambda'], k + \lambda h \in K \right\}. \quad (2)$$

**Definition 2** (see [10]). Let $K$ be a nonempty subset in $Y$. The relative algebraic interior of $K$ is the set

$$\text{icr}(K) := \left\{ k \in K \mid \forall h \in Y, \exists \lambda' > 0, \forall \lambda \in [0, \lambda'], k + \lambda h \in K \right\}. \quad (3)$$

**Remark 3.** Clearly, $\text{cor}(K) \subseteq \text{icr}(K)$. Moreover, if $\text{cor}(K) \neq \emptyset$, then $\text{cor}(K) = \text{icr}(K)$.

**Definition 4** (see [12]). A set-valued map $F: A \rightarrow Y$ is called nearly $C$-subconvexlike on $A$ if and only if $\text{vcl}(\text{cone}(F(A) + C))$ is a convex set in $Y$.

**Definition 5** (see [13]). A set-valued map $F: A \rightarrow Y$ becomes $\text{icr}(F(A) + C)$ is a convex set in $Y$.

**Remark 4.** When the set-valued map $F: A \rightarrow Y$ becomes $\text{icr}(F(A) + C)$ is a convex set in $Y$. The conclusions of Lemma 10 are true when $C$ is replaced by $C_+$.  

**Theorem 12.** Let $F: A \rightarrow Y$ be a set-valued map on $A$. Then $F$ is $\text{icr}(F(A) + C_+)$-convexlike on $A$, then $F$ is nearly $C$-subconvexlike on $A$.

**Proof.** Since $F$ is $\text{icr}(F(A) + C_+)$-convexlike on $A$, $\text{cor}(\text{cone}_+(F(A) + C_+))$ is a convex set in $Y$ and $\text{cone}_+(F(A) + C_+) \subseteq \text{vcl}(\text{cor}(\text{cone}_+(F(A) + C_+)))$, which implies that

$$\text{vcl}(\text{cone}_+(F(A) + C_+)) \subseteq \text{vcl}(\text{cor}(\text{cone}_+(F(A) + C_+))). \quad (9)$$

Using the convexity of $\text{cor}(\text{cone}_+(F(A) + C_+))$ and (b) of Lemma 9, we have

$$\text{vcl}(\text{vcl}(\text{cor}(\text{cone}_+(F(A) + C_+)))) \subseteq \text{vcl}(\text{cor}(\text{cone}_+(F(A) + C_+))). \quad (10)$$

It follows from (9) and (10) that

$$\text{vcl}(\text{cor}(\text{cone}_+(F(A) + C_+))) \subseteq \text{vcl}(\text{cor}(\text{cone}_+(F(A) + C_+))). \quad (11)$$

Clearly,

$$\text{vcl}(\text{cor}(\text{cone}_+(F(A) + C_+))) \subseteq \text{vcl}(\text{cor}(\text{cone}_+(F(A) + C_+))). \quad (12)$$
By (11) and (12), we obtain
\[ vcl(\text{cone}_+(F(A) + C_+)) = vcl(\text{cor}(\text{cone}_+(F(A) + C_+))). \] (13)

Since \( \text{cor}(\text{cone}_+(F(A) + C_+)) \) is a convex set in \( Y \), it follows from (13) and (a) of Lemma 9 that \( vcl(\text{cone}_+(F(A) + C_+)) \) is a convex set in \( Y \). Using Lemma 8, we have
\[ vcl(\text{cone}_+(F(A) + C)) = vcl(\text{cone}_+(F(A) + C)) \cup vcl(0). \] (14)

Now, we prove that
\[ vcl(\text{cone}_+(F(A))) \subseteq vcl(\text{cone}_+(F(A) + C_+)). \] (15)

Let \( y \in vcl(\text{cone}_+(F(A))) \). Then, \( \exists h \in Y \), for all \( \lambda > 0 \), \( \exists \lambda \in [0, \lambda'] \), and we have
\[ y + \lambda h \in \text{cone}_+(F(A)). \] (16)

Take \( c \in C_+ \) in \( Y \). By (16), \( \exists h + c \in Y \), for all \( \lambda > 0 \), \( \exists \lambda \in [0, \lambda'] \), and we have
\[ y + \lambda (h + c) \in \text{cone}_+(F(A)) + C_+. \] (17)

which implies \( y \in vcl(\text{cone}_+(F(A)) + C_+) \). Therefore, (15) holds. It follows from (14) and (15) that
\[ vcl(\text{cone}_+(F(A) + C)) = vcl(\text{cone}_+(F(A))) + C_+. \] (18)

Since \( vcl(\text{cone}_+(F(A) + C)) \) is a convex set in \( Y \), it follows from (18) that \( vcl(\text{cone}(F(A) + C)) \) is a convex set in \( Y \). Therefore, \( F \) is nearly \( C_+ \)-subconvexlike on \( A \).

Remark 13. If \( Y \) is a locally convex space or a finite dimensional linear space, then the condition icr(\( \text{cor}(\text{cone}_+(F(A) + C_+)) \)) \( \neq 0 \) can be dropped. Thus, Theorem 12 generalizes Theorem 3.2 in [8] from locally convex spaces to linear spaces.

The following example shows that the converse of Theorem 12 is not true.

Example 14. Let \( Y = \mathbb{R}^2 \), \( C = \{(y_1, y_2) \mid y_1 \geq 0, y_2 = 0\} \), \( C_+ = \{(y_1, y_2) \mid y_1 > 0, y_2 = 0\} \), and \( A = \{(1, 0), (0, 1)\} \). The set-valued map \( F : A \rightrightarrows Y \) is defined as follows:
\[ F(1, 0) = \{(y_1, y_2) \mid 1 \leq y_1 \leq 2 - y_2, 0 \leq y_2 \leq 2\}, \]
\[ F(0, 1) = \{(y_1, y_2) \mid 1 \leq y_1 \leq 2 + y_2, 0 \leq y_2 \leq 2\}. \] (19)

It is easy to check that icr(\( \text{cor}(\text{cone}_+(F(A) + C_+)) \)) \( \neq 0 \). Moreover, \( vcl(\text{cone}(F(A) + C)) \) is a convex set in \( Y \). Therefore, \( F \) is nearly \( C_+ \)-subconvexlike on \( A \). However, \( \text{cor}(\text{cone}_+(F(A) + C_+)) \) is not a convex set in \( Y \). Therefore, \( F \) is not ic-C_+ -convexlike on \( A \).

In Theorem 12, we do not suppose that \( \text{cor}(C) \neq 0 \). If \( \text{cor}(C) \neq 0 \), we have the following result.

Theorem 15. Let \( F : A \rightrightarrows Y \) be a set-valued map on \( A \). If \( \text{cor}(C) \neq 0 \), then \( F \) is ic-C_+ -convexlike on \( A \) if and only if \( F \) is nearly \( C_+ \)-subconvexlike on \( A \).

Proof. Necessity. Suppose that \( F \) is ic-C_+ -convexlike on \( A \). Clearly,
\[ \text{icr}(\text{cor}(\text{cone}_+(F(A) + C_+))) = \text{icr}(\text{cor}(\text{cone}_+(F(A) + C_+))). \] (20)

Since \( \text{cor}(C) \neq 0 \), \( \text{cor}(C) \neq 0 \). It follows from Lemma 10 that
\[ \text{cor}(\text{cor}(\text{cone}_+(F(A) + C_+))) = \text{cor}(\text{cone}_+(F(A) + C_+)) \] (21)

which implies that
\[ \text{icr}(\text{cor}(\text{cone}_+(F(A) + C_+))) \neq 0. \] (22)

By (20) and (22), we have icr(\( \text{cor}(\text{cone}_+(F(A) + C_+)) \)) \( \neq 0 \). Since \( F \) is ic-C_+ -convexlike on \( A \), it follows from Theorem 12 that \( F \) is nearly \( C_+ \)-subconvexlike on \( A \).

Sufficiency. We suppose that \( F \) is nearly \( C_+ \)-subconvexlike on \( A \). Since \( \text{cor}(C) \neq 0 \), it follows from Lemma 10 and (18) that
\[ \text{cor}(\text{cone}_+(F(A) + C_+)) = \text{cor}(\text{cone}_+(F(A))) + C_+ \]
\[ = \text{cor}(\text{cone}_+(F(A) + C_+)) \]
\[ = \text{cor}(\text{cone}_+(F(A) + C_+)) \] (23)

Since \( F \) is nearly \( C_+ \)-subconvexlike on \( A \), cor(\( vcl(\text{cone}(F(A) + C)) \)) is a convex set in \( Y \). Hence, \( \text{cor}(\text{cone}_+(F(A) + C_+)) \) is a convex set in \( Y \).

Clearly,
\[ \text{cone}_+(F(A) + C_+ \subseteq vcl(\text{cone}_+(F(A) + C_+)) \]
\[ \subseteq vcl(\text{cone}_+(F(A) + C_+)). \] (24)

Since \( \text{cor}(C) \neq 0 \) implies \( \text{cor}(\text{cone}_+(F(A) + C_+)) \neq 0 \), \( \text{cor}(vcl(\text{cone}_+(F(A) + C_+))) \neq 0 \). By the near \( C_+ \)-subconvexlikeliness of \( F \), it is easy to check that \( vcl(\text{cone}(F(A) + C_+)) \) is a convex set in \( Y \). It follows from (c) of Lemma 9 that
\[ vcl(\text{cone}_+(F(A) + C_+)) = vcl(\text{cor}(\text{cone}_+(F(A) + C_+))). \] (25)
By Lemma 10, we have
\[
\text{vcl}(\text{cor}(\text{vcl}(\text{cone}_+ (F(A) + C_+)))) = \text{vcl}(\text{cor}(\text{cone}_+ (F(A) + C_+))).
\]  
(26)

By (24), (25), and (26), we have \( \text{cone}_+ (F(A) + C_+) \subseteq \text{vcl}(\text{cor}(\text{cone}_+ (F(A) + C_+))). \) Therefore, \( F \) is ic-\( C_+ \)-convexlike on \( A \).


Remark 17. Xu and Song used Lemma 2.2 in [8] to prove Theorems 3.1 and 3.2 in [8]. However, in this paper, our methods are different from those in [8].

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References


