Research Article

On an Invariant Set in the Heat Conductivity Problem with Time Lag

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The problems of weak and strong invariance of a constant multivalued mapping with respect to the heat conductivity equation with time lag are studied. Sufficient conditions of weak and strong invariance of a given multivalued mapping are obtained.

1. Introduction

1.1. Related Works. There are many theoretical and practical problems in control problems with distributed parameters where known methods do not work to solve them. The typical examples of such problems are conservation of temperature of a volume within admissible bounds and deviation from undesirable states.

Note that the works [1–7] were devoted to the problems of invariance of given sets for the controlled systems. In these works, some results on construction of core of liveness, the maximal weak invariant subset of the given set was obtained for the control system.

In the paper [3], a family of trajectories, which is kept to be within the given set until a certain time (viability), is analytically described for control systems given by differential inclusions. The paper [8] deals with the problem of diffusion process control by boundary control.

However, all the works mentioned above relate to control systems with concentrated parameters. In the papers [9, 10], weak and strong invariance of the given set with respect to a system with distributed parameters were studied. Alimov [11], Albeverio and Alimov [12] studied interesting applied control problems on heat distribution by convectors in a volume.

The present paper deals with the problems of weak and strong invariance of given multivalued mapping for the 3rd heat conductivity boundary value problem with time lag. In the equation of this problem, the control parameter appears

on the right hand side. We obtain conditions which can be easily checked to determine the invariance of the given constant multivalued mapping.

1.2. Preliminaries. First of all, we recall some definitions. A bounded region $\Omega (\subset \mathbb{R}^n)$ is referred to as the region with piecewise smooth boundary if its boundary $\Gamma = \overline{\Omega} \setminus \Omega$ can be represented as follows: $\Gamma = \sum_{j=1}^N \bar{\Gamma}_j$, where $\Gamma_j \subset \Gamma$ is an open set with respect to the topology on Γ induced by the topology in $\mathbb{R}^n$. Moreover, each Γ_j is a connected surface of class C^1 ; that is, for any point $x_0 \in \Gamma_j$ there exists a ball $U_\epsilon(x_0)$ of radius $\epsilon > 0$ such that the piece $\Gamma_j \cap U_\epsilon(x_0)$ of the surface is given by the equation of the form $x_k = f_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$, where $f_k(\cdot) \in C^1$ and $1 \leq k \leq n$.

Let Ω be a region in $\mathbb{R}^n$ with piecewise smooth boundary. We will use the letter A to denote the following differential operator [13, 14]:

$$A(x, D)\varphi = - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial \varphi}{\partial x_j} \right), \quad (1)$$

where the functions $a_{ij}(x) \in C^1(\Omega) \cap C(\overline{\Omega})$ satisfy the conditions $a_{ij}(x) = a_{ji}(x)$, $x \in \Omega$, and

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \gamma \sum_{i=1}^n \xi_i^2, \quad (2)$$

for all $(\xi_1, \xi_2, \dots, \xi_n) \in R^n$. The inequality (2) is called the condition of uniform ellipticity of the operator A defined by (1). The domain of the operator A is $C^2(\Omega)$, which is the space of functions that are twice continuously differentiable in Ω and continuous on $\Omega \cup \partial\Omega$.

Define the operator P by the equation

$$P\varphi = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial \varphi}{\partial x_j} \cos(l, x_i) + k(x) \varphi, \quad x \in \partial\Omega, \quad (3)$$

where l denotes the upward unit normal vector on $\partial\Omega$ and $k(x)$ is a given positive continuous function defined on $\partial\Omega$.

Definition 1. The number λ for which the following boundary value problem:

$$\begin{aligned} A(x, D)\varphi &= \lambda\varphi, \quad x \in \Omega, \\ P\varphi &= 0, \quad x \in \partial\Omega, \end{aligned} \quad (4)$$

has nonzero solution $\varphi_\lambda(x)$, $x \in \Omega$ is called eigenvalue of this boundary value problem, and the solution $\varphi_\lambda(x)$ is called eigenfunction of the boundary value problem.

Since problem (4) is homogeneous, we assume that

$$\int_{\Omega} \varphi_\lambda^2 dx = 1. \quad (5)$$

As the operator (1) is self-adjoint, then it has a discrete spectrum [13, 14]; that is,

(i) there exist a countable set of eigenvalues λ_k of the problem (4) such that

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots, \quad \lambda_n \rightarrow \infty, \quad n \rightarrow \infty, \quad (6)$$

(ii) for each eigenvalue λ_k , there are a finite number of eigenfunctions φ_k corresponding to it such that $\int_{\Omega} \varphi_i(x) \varphi_j(x) dx = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad (7)$$

(iii) the set of all eigenvalues $\{\varphi_k\}$ is complete (closed) in the space $L_2(\Omega)$; that is, any function f from the space $L_2(\Omega)$ can be uniquely represented in the form

$$f(x) = \sum_{k=1}^{\infty} f_k \varphi_k(x), \quad (8)$$

where the equality is understood in the following sense:

$$\int_{\Omega} \left[f(x) - \sum_{k=1}^N f_k \varphi_k(x) \right]^2 dx \rightarrow 0, \quad N \rightarrow \infty. \quad (9)$$

The Fourier coefficients f_k in the Fourier series (8) of $f(x)$ are defined by formula

$$f_k = \int_{\Omega} f(x) \varphi_k(x) dx. \quad (10)$$

Since the operator A is self-adjoint, in certain conditions on the functions a_{ij} , φ , and ψ , the Green formula is written in the form

$$\begin{aligned} & \int_{\Omega} (\psi A(x, D)\varphi - \varphi A(x, D)\psi) dx \\ &= \int_{\partial\Omega} (\psi P\varphi - \varphi P\psi) dS. \end{aligned} \quad (11)$$

Using these, we construct the following spaces depending on a parameter. Let r be any nonnegative number. Denote

$$\ell_r = \left\{ \alpha = (\alpha_1, \alpha_2, \dots) \mid \sum_{i=1}^{\infty} \lambda_i^r \alpha_i^2 < \infty \right\}, \quad (12)$$

$$H_r(\Omega) = \left\{ f \in L_2(\Omega) \mid f = \sum_{i=1}^{\infty} \alpha_i \varphi_i, \alpha \in \ell_r \right\}.$$

We now define inner product and norm in the spaces $\ell_r, H_r = H_r(\Omega)$. Let

$$\alpha, \beta \in \ell_r, \quad f = \sum_{i=1}^{\infty} \alpha_i \varphi_i, \quad g = \sum_{i=1}^{\infty} \beta_i \varphi_i. \quad (13)$$

Set

$$\begin{aligned} (\alpha, \beta)_r &= \sum_{i=1}^{\infty} \lambda_i^r \alpha_i \beta_i, \quad \|\alpha\| = (\alpha, \alpha)^{1/2}_r, \\ (f, g)_r &= (\alpha, \beta)_r, \quad \|f\| = \|\alpha\|. \end{aligned} \quad (14)$$

It should be noted that $H_0(\Omega) = L_2(\Omega)$ and $H_r(\Omega) \subset H_s(\Omega)$ for all $0 \leq s \leq r$.

Denote by $C(0, T; H_r(\Omega))$ and $L_2(0, T; H_r(\Omega))$ the spaces of continuous functions and summable quadratically measurable functions defined on $[0, T]$ with the values in $H_r(\Omega)$, respectively, where T is a positive number.

1.3. Auxiliary Statements. Consider the following heat exchange control problem with lag:

$$\begin{aligned} & \frac{\partial z(x, t)}{\partial t} + A(x, D)z(x, t) \\ &= z_0(x, t-h) + u(x, t), \quad 0 < t \leq T, \end{aligned} \quad (15)$$

with boundary

$$Pz(x, t) = 0, \quad 0 \leq t \leq T, \quad x \in \partial\Omega, \quad (16)$$

and initial conditions

$$z(x, 0) = z_0(x, 0), \quad (17)$$

where $z_0(x, \cdot) \in X$; $X = \{z(x, \cdot) \mid z(x, t) \in H_r, -h \leq t \leq 0\}$. Here, $z(\cdot) = z(x, \cdot)$ and $u(\cdot) = u(x, \cdot)$ are abstract functions whose values at each $t > 0$ are unique elements of the space $H_r(\Omega)$; h is a positive fixed number and T is a positive number.

Further, we use the same letter A to denote the bounded operator that maps H_r into H_{r+2} and is defined by the following formula

$$A\varphi = A\left(\sum_{k=1}^{\infty} c_k \varphi_k\right) = \sum_{k=1}^{\infty} \lambda_k c_k \varphi_k, \quad (18)$$

where $\varphi = \sum_{k=1}^{\infty} c_k \varphi_k$. Clearly, we obtain from this that $\|A\varphi\|_{H_r} = \|f\|_{H_{r+2}}$.

The problem (15)–(17) is understood in the sense of the theory of generalized functions (the theory of distributions) with the values in H_r . We will look for the solution of the problem (15)–(17), which is continuous with respect to t and its values belong to one of the spaces H_r , and the initial and boundary conditions are considered as equality of elements of these spaces.

In the control problem (15)–(17), the control function $u(\cdot)$ is subjected to either constraint

$$\|u(t)\|_{H_r} \leq \rho, \quad 0 \leq t \leq T, \quad (19)$$

or

$$u(\cdot) \in L_2([0, h]; H_r), \quad \|u(\cdot)\|_{L_2([0, h]; H_r)} \leq \rho. \quad (20)$$

Accordingly, in this paper, we consider both of the problems (15)–(19) and (15)–(17), (20). The control function $u(\cdot)$ that satisfies either (19) or (20) is called admissible.

Assume that the problem (15)–(17) has a solution $z(t)$, $0 \leq t \leq h$, at some admissible control $u(\cdot)$. Then we have

$$\begin{aligned} z(t) &= \sum_{k=1}^{\infty} z_k(t) \varphi_k(x); \\ z_k(t) &= \int_{\Omega} z(t) \varphi_k(x) dx. \end{aligned} \quad (21)$$

It follows from (15) that

$$\begin{aligned} \int_{\Omega} \left(\frac{\partial z(t)}{\partial t} + Az(x) - z_0(t-h) - u(t) \right) \varphi_k(x) dx &\equiv 0, \\ k &= 1, 2, \dots \end{aligned} \quad (22)$$

Also, by definition of the functions $z(t)$, $\varphi_k(x)$ we obtain

$$\begin{aligned} Pz(t) &= 0, \quad 0 \leq t \leq h, \quad x \in \partial\Omega, \\ P\varphi_k &= 0, \quad A\varphi_k = \lambda_k \varphi_k. \end{aligned} \quad (23)$$

Therefore, in view of (11) we have

$$\int_{\Omega} \varphi_k(x) Az(t) dx = \lambda_k z_k(t). \quad (24)$$

Denoting

$$\begin{aligned} z_{0k}(t-h) &= \int_{\Omega} z_0(t-h) \varphi_k(x) dx, \\ u_k(t) &= \int_{\Omega} u(t) \varphi_k(x) dx \end{aligned} \quad (25)$$

and using the fact that

$$\int_{\Omega} \frac{\partial z(t)}{\partial t} \varphi_k(x) dx = \frac{dz_k(t)}{dt}, \quad (26)$$

we deduce from formulas (22) and (24) that

$$\begin{aligned} \dot{z}_k(t) + \lambda_k z_k(t) &= z_{0k}(t-h) + u_k(t), \\ 0 \leq t \leq h, \quad k &= 1, 2, \dots \end{aligned} \quad (27)$$

Since the function (21) must satisfy the initial condition $z(0) = z_0(0)$, we have

$$z_k(0) = z_{0k}^0 = \int_{\Omega} z_0(0) \varphi_k(x) dx, \quad k = 1, 2, \dots \quad (28)$$

From this and (27) we see that

$$\begin{aligned} z_k(t) &= e^{-\lambda_k t} z_{0k}^0 + \int_0^t e^{-\lambda_k(t-\tau)} (z_{0k}(\tau-h) + u_k(\tau)) d\tau, \\ k &= 1, 2, \dots \end{aligned} \quad (29)$$

Thus, (21) and (29) define the formal solution of the problem (15)–(17) on $[0, h]$ which can be written as follows:

$$\begin{aligned} z(t) &= \sum_{k=1}^{\infty} \left[e^{-\lambda_k t} z_{0k}^0 + \int_0^t e^{-\lambda_k(t-\tau)} (z_{0k}(\tau-h) + u_k(\tau)) d\tau \right] \\ &\quad \times \varphi_k(x). \end{aligned} \quad (30)$$

The following assumption will be needed throughout the paper.

Assumption 2. Let functions $z(\cdot)$ of the set X satisfy the following conditions:

$$\sum_{k=1}^{\infty} \lambda_k^r \int_0^t |z_{0k}(\tau-h)|^2 d\tau \leq b^2, \quad (31)$$

$$\sum_{k=1}^{\infty} \lambda_k^r |z_{0k}(0)|^2 \leq b^2,$$

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda_k^r \int_0^t e^{-\lambda_k(t-\tau)} |z_{0k}(\tau-h)|^2 d\tau \\ \leq b^2 \int_0^t e^{-\lambda_1(t-\tau)} d\tau, \quad 0 \leq t \leq h. \end{aligned} \quad (32)$$

Proposition 3. For any function $f(t)$, $\|f(t)\|_{H_r} \leq M$, $0 \leq t \leq T$, the following inequality holds:

$$\sum_{k=1}^{\infty} \lambda_k^r \left(\int_0^t e^{-\lambda_k(t-\tau)} f_k(\tau) d\tau \right)^2 \leq a_1^2(t) M^2; \quad (33)$$

$$a_1(t) = \frac{1 - e^{-\lambda_1 t}}{\lambda_1}, \quad 0 \leq t \leq T,$$

where $\lambda_1, \lambda_2, \dots$ are the eigenvalues of the operator A arranged in decreasing order, M is a positive number, and $f_k(\cdot)$ is the Fourier coefficient of the function $f(t)$.

Proof. Indeed, for $0 \leq t \leq T$, using the Cauchy-Schwartz inequality and the fact that

$$\|f(\tau)\|_{H_r}^2 = \sum_{k=1}^{\infty} \lambda_k^r f_k^2(\tau) \leq M^2, \quad 0 \leq \tau \leq t \leq T, \quad (34)$$

we obtain

$$\begin{aligned} & \sum_{k=1}^{\infty} \lambda_k^r \left(\int_0^t e^{-\lambda_k(t-\tau)} f_k(\tau) d\tau \right)^2 \\ &= \sum_{k=1}^{\infty} \lambda_k^r \left(\int_0^t e^{-(\lambda_k/2)(t-\tau)} \cdot e^{-(\lambda_k/2)(t-\tau)} f_k(\tau) d\tau \right)^2 \\ &\leq \sum_{k=1}^{\infty} \lambda_k^r \int_0^t e^{-\lambda_1(t-\tau)} d\tau \int_0^t e^{-\lambda_1(t-\tau)} f_k^2(\tau) d\tau \\ &\leq \int_0^t e^{-\lambda_1(t-\tau)} d\tau \int_0^t e^{-\lambda_1(t-\tau)} \cdot \sum_{k=1}^{\infty} \lambda_k^r f_k^2(\tau) d\tau \\ &\leq a_1(t) \int_0^t e^{-\lambda_1(t-\tau)} M^2 d\tau = a_1^2(t) M^2. \end{aligned} \quad (35)$$

□

Theorem 4. Let $u(t)$, $0 \leq t \leq h$ be an admissible control that satisfies the condition (19), and let Assumption 2 be satisfied. Then there exists a unique solution of the problem (15)–(17) in the space $C([0, h]; H_r(\Omega))$.

Proof. To prove the theorem we use the formal representation (30) of the solution of the problem (15)–(17). By definition of the norm in $H_r(\Omega)$ for $0 \leq t \leq h$, we have

$$\begin{aligned} \|z(t)\|_{H_r}^2 &= \sum_{k=1}^{\infty} \lambda_k^r |z_k(t)|^2 \\ &= \sum_{k=1}^{\infty} \lambda_k^r \left(e^{-\lambda_k t} z_{0k}^0 + \int_0^t e^{-\lambda_k(t-\tau)} f_k(\tau) d\tau \right)^2, \end{aligned} \quad (36)$$

where $f_k(\tau) = z_{0k}(\tau - h) + u_k(\tau)$.

We use the Cauchy-Schwartz inequality to obtain the following chain of relations:

$$\begin{aligned} & \sum_{k=1}^{\infty} \lambda_k^r \left(\int_0^t e^{-\lambda_k(t-\tau)} f_k(\tau) d\tau \right)^2 \\ &\leq \sum_{k=1}^{\infty} \lambda_k^r \left(\int_0^t e^{-(\lambda_k(t-\tau))/2} e^{-(\lambda_k(t-\tau))/2} f_k(\tau) d\tau \right)^2 \\ &\leq \sum_{k=1}^{\infty} \lambda_k^r \int_0^t e^{-\lambda_k(t-\tau)} d\tau \int_0^t e^{-\lambda_k(t-\tau)} f_k^2(\tau) d\tau \\ &\leq \sum_{k=1}^{\infty} \lambda_k^r \frac{1 - e^{-\lambda_k t}}{\lambda_k} \int_0^t e^{-\lambda_k(t-\tau)} f_k^2(\tau) d\tau. \end{aligned} \quad (37)$$

Therefore,

$$\begin{aligned} & \sum_{k=1}^{\infty} \lambda_k^r \frac{1 - e^{-\lambda_k t}}{\lambda_k} \int_0^t e^{-\lambda_k(t-\tau)} f_k^2(\tau) d\tau \\ &\leq a_1(t) \sum_{k=1}^{\infty} \lambda_k^r \int_0^t e^{-\lambda_k(t-\tau)} (z_{0k}(\tau - h) + u_k(\tau))^2 d\tau. \end{aligned} \quad (38)$$

Next, we expand the integrand on the right hand side of the last inequality; then we use the Cauchy-Schwartz inequality and after that we use Assumption 2 to obtain

$$\sum_{k=1}^{\infty} \lambda_k^r \left(\int_0^t e^{-\lambda_k(t-\tau)} f_k(\tau) d\tau \right)^2 \leq (b + \rho)^2 a_1^2(t). \quad (39)$$

Expanding the square in (36), and then using (39) and Assumption 2, we arrive at

$$\|z(t)\|_{H_r}^2 \leq (e^{-\lambda_1 t} b + (b + \rho) a_1(t))^2. \quad (40)$$

Hence, $z(t) \in H_r(\Omega)$ for each $t \in [0, h]$.

We now show that the function $z(t)$, $0 \leq t \leq h$, is continuous with respect to the norm of the space H_r . We have

$$\begin{aligned} \|z(t + \delta) - z(t)\|_{H_r}^2 &= \sum_{k=1}^{\infty} \lambda_k^r |z_k(t + \delta) - z_k(t)|^2 \\ &= \sum_{k=1}^N \lambda_k^r |z_k(t + \delta) - z_k(t)|^2 \\ &\quad + \sum_{k=N+1}^{\infty} \lambda_k^r |z_k(t + \delta) - z_k(t)|^2, \end{aligned} \quad (41)$$

where the numbers δ and N will be chosen depending on any $\varepsilon > 0$.

By (29), We have

$$\begin{aligned} z_k(t + \delta) - z_k(t) &= z_{0k}^0 e^{-\lambda_k t} (e^{-\lambda_k \delta} - 1) \\ &\quad + \int_0^t e^{-\lambda_k(t-\tau)} (e^{-\lambda_k \delta} - 1) f_k(\tau) d\tau \\ &\quad + \int_t^{t+\delta} e^{-\lambda_k(t+\delta-\tau)} f_k(\tau) d\tau. \end{aligned} \quad (42)$$

Combining this with (41), we obtain

$$\begin{aligned} \|z(t+\delta) - z(t)\|_{H_r}^2 &\leq I_1 + I_2 + I_3; \\ I_1 &= 3 \sum_{k=1}^N \lambda_k^r \left| e^{-\lambda_k \delta} - 1 \right|^2 \left(\left| z_{0k}^0 \right|^2 + \left| \int_0^t e^{-\lambda_k(t-\tau)} f_k(\tau) d\tau \right|^2 \right) \\ I_2 &= 3 \sum_{k=N+1}^{\infty} \lambda_k^r \left(\left| z_{0k}^0 \right|^2 + \left| \int_0^t e^{-\lambda_k(t-\tau)} f_k(\tau) d\tau \right|^2 \right) \\ I_3 &= 3 \sum_{k=1}^{\infty} \lambda_k^r \left| \int_t^{t+\delta} e^{-\lambda_k(t+\delta-\tau)} f_k(\tau) d\tau \right|^2. \end{aligned} \quad (43)$$

Let ε be an arbitrary positive number. Since the series in (31) and (39) are convergent, one can choose the number N so that $I_2 < \varepsilon^2/3$. Then, we choose δ so that $I_1 < \varepsilon^2/3$. Further, similar to (39) we have

$$\begin{aligned} &\sum_{k=1}^{\infty} \lambda_k^r \left(\int_t^{t+\delta} e^{-\lambda_k(t+\delta-\tau)} f_k(\tau) d\tau \right)^2 \\ &\leq e^{-2\lambda_1 \delta} \sum_{k=1}^{\infty} \lambda_k^r \left(\int_t^{t+\delta} e^{-\lambda_k(t-\tau)} f_k(\tau) d\tau \right)^2 \\ &\leq e^{-2\lambda_1 \delta} \left((b + \rho) \int_t^{t+\delta} e^{-\lambda_k(t-\tau)} d\tau \right)^2. \end{aligned} \quad (44)$$

It is clear from these inequalities that we can choose δ so that $I_3 < \varepsilon^2/3$. From the estimations for I_1, I_2, I_3 obtained above we conclude according to (43) that $\|z(t+\delta) - z(t)\|_{H_r} < \varepsilon$, meaning that the function $z(t)$, $0 \leq t \leq h$, is continuous.

We now show that the problem (15)–(17) has a unique solution. We assume the contrary, that it has two different solutions $z'(t), z''(t)$, $0 \leq t \leq h$ for the same functions $z_0(\cdot) \in X$ and $u(\cdot)$. Then their difference $Z(t) = z'(t) - z''(t)$, $0 \leq t \leq h$, as a solution of corresponding homogeneous problem (15)–(17), can be represented in the form of the Fourier series (21), and the Fourier coefficients $Z_k(t)$, $0 \leq t \leq h$, of the function $Z(t)$, $0 \leq t \leq h$, are solutions of the following infinite system of differential equations:

$$\begin{aligned} \dot{z}_k(t) + \lambda_k z_k(t) &= 0, \quad z_k(0) = 0, \\ 0 \leq t \leq h, \quad k &= 1, 2, \dots \end{aligned} \quad (45)$$

Therefore, from the representation (29) we obtain $Z_k(t) = 0$ on $0 \leq t \leq h$. Consequently, $Z(t) = 0$, $0 \leq t \leq h$, a contradiction. This proves the theorem. $\square$

2. Main results

Consider heat exchange control problem with lag (15)–(17).

Definition 5. A set $W \subset R^1$ is referred to as the strong invariant set on the time interval $[0, T]$ with respect to the problem (15)–(19) if for any $z_0(t)$, $\|z_0(t)\|_{H_r} \in W$, and $u(t)$, $\|u(t)\|_{H_r} \leq \rho$, $t \in [0, T]$, the inclusion $\|z(t)\|_{H_r} \in W$ holds true on $[0, T]$.

Definition 6. A set $W \subset R^1$ is referred to as the weak invariant set on the time interval $[0, T]$ with respect to the problem (15)–(19) if for any $z_0(t)$, $\|z_0(t)\|_{H_r} \in W$ there exists $u(t)$, $\|u(t)\|_{H_r} \leq \rho$, $t \in [0, T]$, such that the inclusion $\|z(t)\|_{H_r} \in W$ holds true on $[0, T]$.

Further, strong and weak invariant sets of the form $W = [0, b]$ will be investigated, where b is a positive number. We will find relations among the parameters T, b, ρ , and λ_i in order to guarantee strong or weak invariance of the set W on the time interval $[0, T]$ with respect to the problem (15)–(19) (or (15)–(17), (20)).

Theorem 7. If $\lambda_1 \geq 1$, the set $W = [0, b]$ is strong invariant on $[0, T]$ with respect to the problem (15)–(19) if and only if

$$\rho \leq (\lambda_1 - 1)b. \quad (46)$$

Proof. Let (46) hold true and let $z_0(t)$, $\|z_0(t)\|_{H_r} \leq b$, $-h \leq t \leq 0$, and $u(t)$, $\|u(t)\|_{H_r} \leq \rho$, $0 \leq t \leq h$ be any functions. Using the same reasoning as in the proof of Theorem 4, we get from (40) that

$$\|z(t)\|_{H_r}^2 \leq \chi^2(t); \quad (47)$$

$$\chi(t) = e^{-\lambda_1 t} b + a_1(t)(b + \rho), \quad 0 \leq t \leq h.$$

Then

$$\begin{aligned} \chi(0) &= b, \\ \chi'(t) &= -\lambda_1 e^{-\lambda_1 t} b + e^{-\lambda_1 t} (b + \rho) \\ &= e^{-\lambda_1 t} (-\lambda_1 b + b + \rho) \leq 0. \end{aligned} \quad (48)$$

Hence, $\chi'(t) \leq 0$ meaning that the function $\chi(t)$ is nonincreasing. For this reason, $\|z(t)\|_{H_r} \leq b$, $0 \leq t \leq h$.

The same reasoning can be applied for the time interval $[h, 2h]$ to obtain $\|z(t)\|_{H_r} \leq b$. For any positive number $T > 0$, we can continue in this fashion to obtain

$$\|z(t)\|_{H_r} \leq b, \quad 0 \leq t \leq T; \quad (49)$$

that is, W is strong invariant.

We now turn to the proof of the second part of the theorem. Let the set W be strong invariant. Suppose, contrary to our claim, that the inequality (46) fails to hold. Then, clearly, the function $\chi(t)$ is increasing. Choose the functions $z_0(\cdot), u(\cdot)$ as follows:

$$\begin{aligned} z_{01}(t-h) &= \frac{b}{\lambda_1^{r/2}}, \quad u_1(t) = \frac{\rho}{\lambda_1^{r/2}}, \quad z_{0k}(t-h) = 0, \\ u_k(t) &= 0, \quad k \geq 2, \quad 0 \leq t \leq h. \end{aligned} \quad (50)$$

Then

$$\begin{aligned} &\|z(t)\|_{H_r}^2 \\ &= \lambda_1^r \left(e^{-\lambda_1 t} z_{01}^0 + \int_0^t e^{-\lambda_1(t-\tau)} (z_{01}(t-\tau) + u_1(\tau)) d\tau \right)^2 \\ &= \chi^2(t). \end{aligned} \quad (51)$$

Since the function $\chi(t)$ is increasing on $[0, h]$ and $\chi(0) = b$, therefore $t, \chi(t) > b$; that is, $\|z(t)\|_{H_r} > b$, which contradicts the strong invariance of the set W . This completes the proof of Theorem 7. $\square$

Theorem 8. Let T be any positive number. If $\lambda_1 \geq 1$, the set $W = [0, b]$ is weak invariant on $[0, T]$ with respect to the problem (15)–(19).

Proof. Let $\lambda_1 \geq 1$. Show that, under condition 9, the set $W = [0, b]$ is weak invariant with respect to the problem (15)–(17). Let $z_0(\cdot)$ be an arbitrary function from X . Set $u(\cdot) = 0$. Then we obtain from the representation (29) of the solution of the problem (27) that

$$\begin{aligned} \|z(t)\|_{H_r}^2 &= \sum_{k=1}^{\infty} \lambda_k^r \left(e^{-\lambda_k t} z_{0k}^0 + \int_0^t e^{-\lambda_k(t-\tau)} z_{0k}(\tau) d\tau \right)^2 \\ &\leq e^{-2\lambda_1 t} \sum_{k=1}^{\infty} \lambda_k^r |z_{0k}^0|^2 \\ &\quad + 2e^{-\lambda_1 t} \sum_{k=1}^{\infty} \lambda_k^r |z_{0k}^0| \int_0^t e^{-\lambda_k(t-\tau)} |z_{0k}(\tau)| d\tau \quad (52) \\ &\quad + \sum_{k=1}^{\infty} \lambda_k^r \left(\int_0^t e^{-\lambda_k(t-\tau)} z_{0k}^0(\tau) d\tau \right)^2 \\ &\leq e^{-2\lambda_1 t} b^2 + 2e^{-\lambda_1 t} b a_1(t) b + a_1^2(t) b^2 \\ &= (e^{-\lambda_1 t} + a_1(t))^2 b^2. \end{aligned}$$

Here, we used the following inequality:

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda_k^r |z_{0k}^0| \int_0^t e^{-\lambda_k(t-\tau)} |z_{0k}(\tau)| d\tau \\ \leq \sqrt{\sum_{k=1}^{\infty} \lambda_k^r |z_{0k}^0|^2} \sqrt{\sum_{k=1}^{\infty} \lambda_k^r \left(\int_0^t e^{-\lambda_k(t-\tau)} |z_{0k}(\tau)| d\tau \right)^2} \quad (53) \end{aligned}$$

and Assumption 2. Since the function

$$\xi(t) = e^{-\lambda_1 t} + \frac{1 - e^{-\lambda_1 t}}{\lambda_1}, \quad 0 \leq t \leq h \quad (54)$$

satisfies

$$\xi(0) = 1, \quad \xi'(t) = (1 - \lambda_1) e^{-\lambda_1 t} \leq 0, \quad (55)$$

we have $0 < \xi(t) \leq 1$. Then in view of (52) we conclude that $\|z(t)\|_{H_r} \leq b$, $0 \leq t \leq h$. Repeated application of this reasoning to the intervals $[jh, (j+1)h]$, $j = 1, 2, \dots$ enables us to conclude that $\|z(t)\|_{H_r} \leq b$, $0 \leq t \leq T$, and this completes the proof. $\square$

We now turn to the problem (15)–(17), (20).

Definition 9. The set $W \subset R^1$ is called weak invariant on $[0, T]$ with respect to the problem (15)–(17), (20) if for any $z_0(t)$, $\|z_0(t)\|_{H_r} \in W$, $t \in [-h, 0]$ there exists $u(t)$, $\|u(\cdot)\|_{L_2([0, T]; H_r)} \leq \rho$, such that the inclusion $\|z(t)\|_{H_r} \in W$ holds for all $t \in [0, T]$.

Definition 10. The set $W \subset R^1$ is called strong invariant on $[0, T]$ with respect to the problem (15)–(17), (20) if for any $z_0(t)$, $\|z_0(t)\|_{H_r} \in W$, $t \in [-h, 0]$ and $u(t)$, $\|u(\cdot)\|_{L_2([0, T]; H_r)} \leq \rho$ the inclusion $\|z(t)\|_{H_r} \in W$ holds for all $t \in [0, T]$.

Proposition 11. For any function $g(\tau)$, $0 \leq \tau \leq t \leq T$, for which $\|g(\cdot)\|_{L_2([0, T]; H_r)} \leq N$, the following inequality holds:

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda_k^r \left(\int_0^t e^{-\lambda_k(t-\tau)} g_k(\tau) d\tau \right)^2 &\leq a_2(t) N^2, \\ 0 \leq t \leq T; \quad (56) \\ a_2(t) &= \frac{1 - e^{-2\lambda_1 t}}{2\lambda_1}, \end{aligned}$$

where T a positive number.

The proof follows from the following chain of relations

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda_k^r \left(\int_0^t e^{-\lambda_k(t-\tau)} g_k(\tau) d\tau \right)^2 \\ \leq \sum_{k=1}^{\infty} \lambda_k^r \int_0^t e^{-2\lambda_k(t-\tau)} d\tau \cdot \int_0^t g_k^2(\tau) d\tau \quad (57) \\ \leq a_2(t) \sum_{k=1}^{\infty} \lambda_k^r \int_0^t g_k^2(\tau) d\tau \\ = a_2(t) \|g(\cdot)\|_{L_2([0, t]; H_r)}^2 \leq a_2(t) N^2; \end{aligned}$$

here we used the Cauchy-Schwartz inequality and the relation $e^{-\lambda_k t} \leq e^{-\lambda_1 t}$, $t \geq 0$, $k = 1, 2, \dots$

Theorem 12. If $\rho > 0$, then the set $W = [0, b]$ is not strong invariant with respect to the problem (15)–(17), (20) on $[0, T]$, where T is any number.

Proof. Let $\rho > 0$. To prove the theorem, we use the fact that modulus of the admissible control function can take any big values on small time interval. Consider the function

$$\eta(t) = e^{-\lambda_1 t} b + \frac{1 - e^{-\lambda_1 t}}{\lambda_1} b + e^{-\lambda_1 t} \sqrt{t} \rho, \quad t \geq 0. \quad (58)$$

Note that $\eta(0) = b$ and $\eta(t) > 0$ for all $t > 0$. Assuming $t > 0$, we have

$$\eta'(t) = e^{-\lambda_1 t} \left(b - \lambda_1 b - \lambda_1 \rho \sqrt{t} + \frac{\rho}{2\sqrt{t}} \right). \quad (59)$$

Due to the term $e^{-\lambda_1 t}(\rho/2\sqrt{t})$, we obtain that there exists t_1 , $0 < t_1 < h$, such that $\eta'(t) > 0$ whenever $0 < t \leq t_1$.

We now specify the initial function and admissible control function on $[0, h]$ as follows:

$$\begin{aligned} z_0(\tau - h) &= \frac{b}{\lambda_1^{r/2}} \varphi_1, & 0 \leq \tau \leq h, \\ u(\tau) &= \begin{cases} \frac{\rho}{\lambda_1^{r/2} \sqrt{t_1}} \varphi_1, & 0 \leq \tau < t_1, \\ 0, & t_1 \leq \tau \leq h. \end{cases} \end{aligned} \quad (60)$$

We observe that

$$\begin{aligned} \|z_0(\tau)\|_{H_r} &\leq b, & -h \leq \tau \leq 0, \\ \|u(\cdot)\|_{L_2([0, h]; H_r)} &\leq \rho. \end{aligned} \quad (61)$$

Then by the representation (30) we have

$$\begin{aligned} z(t) &= \left(e^{-\lambda_1 t} \frac{b}{\lambda_1^{r/2}} + \int_0^t e^{-\lambda_1(t-\tau)} \left(\frac{b}{\lambda_1^{r/2}} + \frac{\rho}{\lambda_1^{r/2} \sqrt{t_1}} \right) d\tau \right) \varphi_1, \\ &0 \leq t \leq t_1. \end{aligned} \quad (62)$$

Hence

$$\begin{aligned} \|z(t)\|_{H_r}^2 &= \lambda_1^r \left(e^{-\lambda_1 t} \frac{b}{\lambda_1^{r/2}} + \int_0^t e^{-\lambda_1(t-\tau)} \left(\frac{b}{\lambda_1^{r/2}} + \frac{\rho}{\lambda_1^{r/2} \sqrt{t_1}} \right) d\tau \right)^2, \\ &0 \leq t \leq t_1. \end{aligned} \quad (63)$$

This implies that

$$\begin{aligned} \|z(t_1)\|_{H_r} &= e^{-\lambda_1 t_1} b + b \int_0^{t_1} e^{-\lambda_1(t_1-\tau)} d\tau \\ &\quad + \int_0^{t_1} e^{-\lambda_1(t_1-\tau)} \frac{\rho}{\sqrt{t_1}} d\tau \\ &\geq e^{-\lambda_1 t_1} b + b \int_0^{t_1} e^{-\lambda_1(t_1-\tau)} d\tau \\ &\quad + \int_0^{t_1} e^{-\lambda_1 t_1} \frac{\rho}{\sqrt{t_1}} d\tau = \eta(t_1). \end{aligned} \quad (64)$$

Since $\eta(0) = b$ and $\eta'(t) > 0$ for $0 < t \leq t_1$, then $\eta(t_1) > b$. Therefore $\|z(t_1)\|_{H_r} > b$ which shows that the set $W = [0, b]$ is not strong invariant, and the proof of Theorem 12 is complete. $\square$

Theorem 13. If $\lambda_1 \geq 1$, then the set $W = [0, b]$ is weak invariant with respect to the problem (15)–(17), (20) on $[0, h]$.

The proof of the theorem is similar to that of Theorem 8.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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