Research Article

Generalized Difference $\lambda$-Sequence Spaces Defined by Ideal Convergence and the Musielak-Orlicz Function

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1. Introduction

Throughout the paper $\omega$, $\ell_\infty$, $c$, $c_0$, and $\ell_p$ denote the classes of all, bounded, convergent, null, and $p$-absolutely summable sequences of complex numbers. The sets of natural numbers and real numbers will be denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. Many authors studied various sequence spaces using normed or seminormed linear spaces. In this paper, using an infinite matrix of complex numbers and the notion of ideal, we aimed to introduce some new sequence spaces with respect to generalized difference operator $\Delta^m$ on $\lambda$-sequences and the Musielak-Orlicz function in $n$-normed linear spaces. By an ideal we mean a family $I \subset 2^Y$ of subsets of a nonempty set $Y$ satisfying the following: (i) $\phi \in I$; (ii) $A, B \in I$ imply $A \cup B \in I$; (iii) $A \in I$, $B \subset A$ imply $B \in I$, while an admissible ideal $I$ of $Y$ further satisfies $\{x\} \in I$ for each $x \in Y$. The notion of ideal convergence was introduced first by Kostyrko et al. [1] as a generalization of statistical convergence. The concept of 2-normed spaces was initially introduced by Gähler [2] in the 1960s, while that of $n$-normed spaces can be found in [3]; this concept has been studied by many authors; see for instance [4–7]. The notion of ideal convergence in a 2-normed space was initially introduced by Gürdal [8]. Later on, it was extended to $n$-normed spaces by Gürdal and Şahiner [9]. Given that $I \subset 2^\mathbb{N}$ is a nontrivial ideal in $\mathbb{N}$, the sequence $(x_n)_{n \in \mathbb{N}}$ in a normed space $(X, \| \cdot \|)$ is said to be $I$-convergent to $x \in X$, if, for each $\epsilon > 0$,

$$A(\epsilon) = \{ n \in \mathbb{N} : \| x_n - x \| \geq \epsilon \} \in I.$$ (1)

A sequence $(x_k)$ in a normed space $(X, \| \cdot \|)$ is said to be $I$-bounded if there exists $L > 0$ such that

$$\{ k \in \mathbb{N} : \| x_k \| > L \} \in I.$$ (2)

A sequence $(x_k)$ in a normed space $(X, \| \cdot \|)$ is said to be $I$-Cauchy if, for each $\epsilon > 0$, there exists a positive integer $m = m(\epsilon)$ such that

$$\{ k \in \mathbb{N} : \| x_k - x_m \| \geq \epsilon \} \in I.$$ (3)

In paper [10], the notion of $\lambda$-convergent and bounded sequences is introduced as follows: let $\lambda = (\lambda_j)_{j=1}^\infty$ be a strictly increasing sequence of positive real numbers tending to infinity; that is,

$$0 < \lambda_1 < \lambda_2 < \cdots , \lambda_j \rightarrow \infty \text{ as } j \rightarrow \infty.$$ (4)

We say that a sequence $x = (x_j) \in \omega$ is $\lambda$-convergent to the number $l \in \mathbb{C}$, called the $\lambda$-limit of $x$, if $\Lambda_j(x) \rightarrow l$ as $j \rightarrow \infty$, where

$$\Lambda_j(x) = \frac{1}{j} \sum_{r=1}^{j} (\lambda_r - \lambda_{r-1}) x_r, \quad j \in \mathbb{N}. \quad (5)$$

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The class of all sequences \((\lambda_j)\) satisfying this property is denoted by \(\Lambda\).

In particular, we say that \(x\) is a \(\lambda\)-null sequence if \(\lambda_j(x) \to 0\) as \(j \to \infty\). Further, we say that \(x\) is \(\lambda\)-bounded if \(\sup_j |\lambda_j(x)| < \infty\). Here and in the sequel, we will use the convention that any term with a zero subscript is equal to naught; for example, \(\lambda_0 = 0\) and \(x_0 = 0\). Now, it is well known [10] that if \(\lim_j x_j = a\) in the ordinary sense of convergence, then
\[
\lim_{j \to \infty} \left( \frac{1}{\lambda_j} \sum_{j=1}^N (\lambda_j - \lambda_{j-1}) |x_j - a| \right) = 0. \tag{6}
\]

This implies that
\[
\lim_{j \to \infty} \left| \lambda_j(x) - a \right| = \lim_{j \to \infty} \left( \frac{1}{\lambda_j} \sum_{j=1}^N (\lambda_j - \lambda_{j-1}) (x_j - a) \right) = 0, \tag{7}
\]
which yields that \(\lim \lambda_j(x) = a\) and hence \(x\) is \(\lambda\)-convergent to \(a\). We therefore deduce that the ordinary convergence implies the \(\lambda\)-convergence to the same limit.

An Orlicz function is a function \(M : [0, \infty) \to [0, \infty)\) which is continuous, nondecreasing, and convex with \(M(0) = 0\) and \(M(x) > 0\) for \(x > 0\) and \(M(x) \to \infty\) as \(x \to \infty\). If convexity of \(M\) is replaced by \(M(x + y) \leq M(x) + M(y)\), then it is called a modulus function, introduced by Nakano [11]. Ruckle [12] and Maddox [13] used the idea of a modulus function to construct some spaces of complex sequences. An Orlicz function \(M\) is said to satisfy the \(\Delta_2\)-condition for all values of \(x \geq 0\), if there exists a constant \(k > 0\), such that \(M(2x) \leq kM(x)\). The \(\Delta_2\)-condition is equivalent to \(M(lx) \leq kM(x)\) for all values of \(x\) and for \(l > 1\). Lindenstrauss and Tzafriri [14] used the idea of an Orlicz function to define the following sequence spaces:
\[
\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} \frac{M(|x(k)|)}{\rho} < \infty \right\}, \tag{8}
\]
which is a Banach space with the Luxemburg norm defined by
\[
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} \frac{M(|x(k)|)}{\rho} \leq 1 \right\}. \tag{9}
\]

The space \(\ell_M\) is closely related to the space \(\ell_p\), which is an Orlicz sequence space with \(M(x) = x^p\) for \(1 \leq p < \infty\). Recently different classes of sequences have been introduced using Orlicz functions. See [7, 9, 15–17].

A sequence \(M = (M_k)\) of Orlicz functions \(M_k\) for all \(k \in \mathbb{N}\) is called a Musielak-Orlicz function.

Kizmaz [18] defined the difference sequences \(\ell_\infty(\Delta), c(\Delta),\) and \(c_0(\Delta)\) as follows.
\[
\Delta(x) = \{ x = (x_k) : (\Delta x_k) \in Z \}. \tag{10}
\]

The space \(\ell_\infty(\Delta)\) is closely related to the space \(\ell_p\), which is an Orlicz sequence space with \(M(x) = x^p\) for \(1 \leq p < \infty\). Recently different classes of sequences have been introduced using Orlicz functions. See [7, 9, 15–17].

The space \(\ell_M\) is closely related to the space \(\ell_p\), which is an Orlicz sequence space with \(M(x) = x^p\) for \(1 \leq p < \infty\). Recently different classes of sequences have been introduced using Orlicz functions. See [7, 9, 15–17].
defines an \((n - 1)\)-norm on \(X\) with respect to \(a_1, a_2, a_3, \ldots, a_n\) and this is known as the derived \((n - 1)\)-norm. The standard \((n)\)-norm on \(X\), a real inner product space of dimension \(d \geq n\), is as follows:

\[
\|x_1, x_2, \ldots, x_n\| = \frac{\left( \left\| x_1, x_1 \right\|^2 \left\| x_1, x_2 \right\|^2 \cdots \left\| x_1, x_n \right\|^2 \right)^{1/2}}{\left( \left\| x_2, x_1 \right\|^2 \left\| x_2, x_2 \right\|^2 \cdots \left\| x_2, x_n \right\|^2 \right)^{1/2}},
\]

(13)

where \(\langle \cdot, \cdot \rangle\) denotes the inner product on \(X\). If we take \(X = \mathbb{R}^n\), then

\[
\|x_1, x_2, \ldots, x_n\|_E = \|x_1, x_2, \ldots, x_n\|,
\]

(14)

For \(n = 1\), this \(n\)-norm is the usual norm \(\|x_1\| = \sqrt{(x_1, x_1)}\).

**Definition 1.** A sequence \((x_k)\) in an \(n\)-normed space is said to be convergent to \(x \in X\) if

\[
\lim_{k \to \infty} \left\| (z_1, z_2, \ldots, z_{n-1}, x_k - x) \right\|_n = 0,
\]

\[
\forall z_1, z_2, \ldots, z_{n-1} \in X.
\]

(15)

**Definition 2.** A sequence \((x_k)\) in an \(n\)-normed space is called Cauchy (with respect to \(n\)-norm) if

\[
\lim_{k, j \to \infty} \left\| (z_1, z_2, \ldots, z_{n-1}, x_k - x_j) \right\|_n = 0,
\]

\[
\forall z_1, z_2, \ldots, z_{n-1} \in X.
\]

(16)

If every Cauchy sequence in \(X\) converges to an \(x \in X\), then \(X\) is said to be complete (with respect to the \(n\)-norm). A complete \(n\)-normed space is called an \(n\)-Banach space.

**Definition 3.** A sequence \((x_k)\) in an \(n\)-normed space \((X, \| \cdot, \cdot, \ldots, \cdot \|)\) is said to be \(I\)-convergent to \(x_0 \in X\) with respect to \(n\)-norm, if, for each \(\epsilon > 0\), the set

\[
\{ k \in \mathbb{N} : \|x_k - x_0, z_1, z_2, \ldots, z_{n-1}\| \geq \epsilon, \ \forall z_1, z_2, \ldots, z_{n-1} \in I \}
\]

(17)

is included in \(I\).

**Definition 4.** A sequence \((x_k)\) in an \(n\)-normed space \((X, \| \cdot, \cdot, \ldots, \cdot \|)\) is said to be \(I\)-Cauchy if, for each \(\epsilon > 0\), there exists a positive integer \(m = m(\epsilon)\) such that the set

\[
\{ k \in \mathbb{N} : \|x_k - x_m, z_1, z_2, \ldots, z_{n-1}\| \geq \epsilon, \ \forall z_1, z_2, \ldots, z_{n-1} \in I \}
\]

(18)

is included in \(I\).

Let \(x = (x_k)\) be a sequence; then \(S(x)\) denotes the set of all permutations of the elements of \((x_k)\); that is, \(S(x) = (x_{\pi(n)}) : \pi \) is a permutation of \(\mathbb{N}\).

**Definition 5.** A sequence space \(E\) is said to be symmetric if \(S(x) \subset E\) for all \(x \in E\).

**Definition 6.** A sequence space \(E\) is said to be normal (or solid) if \((\alpha, x_k) \in E\), whenever \((x_k) \in E\) and for all sequences \((\alpha, \delta)\) of scalars with \(|\alpha| \leq 1\) for all \(k \in \mathbb{N}\).

**Definition 7.** A sequence space \(E\) is said to be a sequence algebra if \(x, y \in E\); then \(x \cdot y = (x_2y_1, \ldots) \in E\).

**Lemma 8.** Every \(n\)-normed space is an \((n - r)\)-normed space for all \(r = 1, 2, 3, \ldots, n - 1\). In particular, every \(n\)-normed space is a normed space.

**Lemma 9.** On a standard \(n\)-normed space \(X\), the derived \((n - 1)\)-norm \(\| \cdot, \cdot, \ldots, \cdot \|_\infty\) defined with respect to the orthogonal set \(\{e_1, e_2, \ldots, e_n\}\) is equivalent to the standard \((n - 1)\)-norm \(\| \cdot, \cdot, \ldots, \cdot \|\). To be precise, one has

\[
\|x_1, x_2, \ldots, x_{n-1}\|_\infty \leq \|x_1, x_2, \ldots, x_{n-1}\| \leq \sqrt{n}\|x_1, x_2, \ldots, x_{n-1}\|_\infty,
\]

(19)

for all \(x_1, x_2, \ldots, x_{n-1} \in X\), where \(\|x_1, x_2, \ldots, x_{n-1}\|_\infty = \max_{1 \leq s \leq n} \|x_1, x_2, \ldots, x_n, e_s\|_1\).

For any bounded sequence \((p_n)\) of positive numbers, one has the following well known inequality: if \(0 \leq p_k \leq \sup_k p_k = G\) and \(D = \max(1, 2^{G-1})\), then \(\|a_n + b_n\|^{p_k} \leq D(\|a_n\|^{p_k} + \|b_n\|^{p_k})\), for all \(k\) and \(a_n, b_n \in C\).

### 3. Main Results

In this section, we define some new ideal convergent sequence spaces and investigate their linear topological structures. We find out some relations related to these sequence spaces. Let \(I\) be an admissible ideal of \(\mathbb{N}\), \(\mathcal{M} = \{M_j\}\) a Musielak-Orlicz function, and \(\Delta_m^s\) the forward generalized difference operator on the class of all sequences \((\lambda, i)\) satisfying the property \(\Lambda\) and an \(n\)-normed space \((X, \| \cdot, \cdot, \ldots, \cdot \|)\). Further, let \(\rho = (\rho_k)\) be any bounded sequence of positive real numbers; we will define the following sequence spaces:

\[
W[A, \mathcal{M}, \Delta_m^s, \Lambda, \rho, || \cdot, \cdot, \ldots, \cdot ||] = \left\{ x \in \omega(n - X) : \forall \epsilon > 0 \right. \left. x \times \left\{ k \in \mathbb{N} : \sum_{j=1}^{\infty} a_{kj} \right. \right. \left. \times \left( M_j \left( \left\| \Delta_m^s(\Lambda_j(x)) - 1 \right\| \right) \right) \geq \epsilon \right\} \in I,
\]
for some $\rho > 0, l \in X$ and each $z_1, z_2, \ldots, z_{n-1} \in X$,

$$W[A, \mathcal{M}, \Delta^t_m, \Lambda, p, \| \cdots \|_0^2]$$

$$= \left\{ x \in \omega(n - X) : \forall \varepsilon > 0$$

$$\times \left\{ k \in \mathbb{N} : \sum_{j=1}^{\infty} a_{kj}$$

$$\times \left[ M_j \left( \left\| \frac{\Delta^t_m(A_j(x))}{\rho} \right\| \right)^{p_j} \right] \right\}_{0}^{z_1, z_2, \ldots, z_{n-1}} \geq \varepsilon \right\} \subseteq I,$$

$$\text{for some } \rho > 0, \text{ and each } z_1, z_2, \ldots, z_{n-1} \in X \right\}.$$  

(20)

Let us consider a few special cases of the aforementioned sets.

(1) If $M_k(x) = M(x)$, for all $k \in \mathbb{N}$ then the previous classes of sequences are denoted by $W[A, M, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_0^2, W[A, M, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^1, W[A, M, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^\infty$, and $W[A, M, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^\infty$, respectively.

(2) If $p_k = 1$ for all $k \in \mathbb{N}$ then the previous classes of sequences are denoted by $W[A, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_0^2, W[A, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^1, W[A, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^\infty$, and $W[A, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^\infty$, respectively.

(3) If $M_k(x) = x$, for all $k \in \mathbb{N}$ and $x \in [0, \infty$], then the previous classes of sequences are denoted by $W[A, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_0^2, W[A, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^1, W[A, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^\infty$, and $W[A, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^\infty$, respectively.

(4) If we take $M_k(x) = M(x)$, for all $k \in \mathbb{N}$ and $A = (a_{kj})$ as

$$a_{kj} = \begin{cases} \frac{1}{k}, & k \geq j, \\ 0, & \text{otherwise}, \end{cases}$$

then we denote the previous classes of sequences by $W[C, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_0^2, W[C, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^1, W[C, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^\infty$, and $W[C, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^\infty$, respectively.

(5) If we take $M_k(x) = M(x)$ and $A = (a_{kj})$ as

$$a_{kj} = \begin{cases} \frac{1}{\Phi_k}, & j \in I_k = \{k - \phi_k + 1, k\}, \\ 0, & \text{otherwise}, \end{cases}$$

where $(\phi_k)$ is a nondecreasing sequence of positive numbers tending to $\omega$, $\phi_1 = 1$, and $\phi_{k+1} \leq \phi_k + 1$, then we denote the previous classes of sequences by $W[\Phi, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_0^2, W[\Phi, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^1, W[\Phi, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^\infty$, and $W[\Phi, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^\infty$, respectively.

(6) If $A = (a_{kj})$ as in (22), then we denote the previous classes of sequences by $W[\Phi, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_0^2, W[\Phi, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^1, W[\Phi, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^\infty$, and $W[\Phi, \mathcal{M}, \Delta^t_{m_k}, \Lambda, p, \| \cdots \|_c^\infty$, respectively.

And if $\lambda_j = j$ for all $j \in \mathbb{N}$, then the previous classes of sequences are denoted by $W[\Phi, \mathcal{M}, \Delta^t_{m_k}, C, p, \| \cdots \|_0^2, W[\Phi, \mathcal{M}, \Delta^t_{m_k}, C, p, \| \cdots \|_c^1, W[\Phi, \mathcal{M}, \Delta^t_{m_k}, C, p, \| \cdots \|_c^\infty$, and $W[\Phi, \mathcal{M}, \Delta^t_{m_k}, C, p, \| \cdots \|_c^\infty$, and they are a
generalization of the sequence spaces defined by Bakery et al. [22].

(7) By a lacunary \( \theta = (j_r) \), \( r = 0, 1, 2, \ldots \), where \( j_0 = 0 \), we will mean an increasing sequence of nonnegative integers with \( j_r - j_{r-1} \to \infty \) as \( r \to \infty \). The interval determined by \( \theta \) will be denoted by \( I_\theta = [j_{r-1}, j_r] \) and \( h_\theta = j_r - j_{r-1} \) and let \( A = (a_{kj}) \) as

\[
a_{kj} = \begin{cases} 
\frac{1}{h_r}, & j \in I_\theta = [j_{r-1}, j_r], \\
0, & \text{otherwise.}
\end{cases}
\]

Then we denote the previous classes of sequences by \( W[\theta, M, \Delta^\theta_m, \Lambda, p \| \cdot \cdot \cdot \|_1^l, W[\theta, M, \Delta^\theta_m, \Lambda, p \| \cdot \cdot \cdot \|_l^\infty \) and \( W[\theta, M, \Delta^\theta_m, \Lambda, p \| \cdot \cdot \cdot \|_l^\infty \) respectively.

(8) If \( M_k(x) = M(x) \) for all \( k \in \mathbb{N} \), \( A = I \), and \( \lambda_j = j \), then the previous classes of sequences are denoted by \( W[M, \Delta^\theta_m, C, p \| \cdot \cdot \cdot \|_1^l, W[M, \Delta^\theta_m, C, p \| \cdot \cdot \cdot \|_l^\infty \) respectively.

(9) If \( s = 1 \), then the previous classes of sequences are denoted by \( W[A, M, \Delta^\theta_m, \Lambda, p \| \cdot \cdot \cdot \|_1^l, W[M, \Delta^\theta_m, C, p \| \cdot \cdot \cdot \|_l^\infty \) respectively.

(10) If \( m = 1 \), then the previous classes of sequences are denoted by \( W[A, M, \Delta^\theta_m, \Lambda, p \| \cdot \cdot \cdot \|_1^l, W[M, \Delta^\theta_m, C, p \| \cdot \cdot \cdot \|_l^\infty \) respectively.

**Theorem 10.** The spaces \( W[A, \mathcal{M}, \Delta^\theta_m, \Lambda, p \| \cdot \cdot \cdot \|_1^l, W[A, \mathcal{M}, \Delta^\theta_m, \Lambda, p \| \cdot \cdot \cdot \|_l^\infty \) and \( W[A, \mathcal{M}, \Delta^\theta_m, \Lambda, p \| \cdot \cdot \cdot \|_l^\infty \) are linear spaces.

**Proof.** We will prove the assertion for \( W[A, \mathcal{M}, \Delta^\theta_m, \Lambda, p \| \cdot \cdot \cdot \|_1^l \); the others can be proved similarly. Assume that \( x = (x_k), y = (y_k) \in W[A, \mathcal{M}, \Delta^\theta_m, \Lambda, p \| \cdot \cdot \cdot \|_1^l \), and \( \alpha, \beta \in \mathbb{C} \). Then, there exist \( \rho_1 \) and \( \rho_2 \) such that the sets

\[
\begin{align*}
\left\{ k \in \mathbb{N} : \sum_{j=1}^{\infty} a_{kj} x_j \leq \frac{\varepsilon}{2}, \right\} & \quad \in I, \\
\left\{ k \in \mathbb{N} : \sum_{j=1}^{\infty} a_{kj} y_j \geq \frac{\varepsilon}{2}, \right\} & \quad \in I,
\end{align*}
\]

where \( L = \max\{\|\alpha\rho_1 + |\alpha|\rho_1 + |\beta|\rho_2, |\beta|\rho_2/(|\alpha|\rho_1 + |\beta|\rho_2)\} \). On the other hand, from the above inequality we get

\[
\begin{align*}
\sum_{j=1}^{\infty} a_{kj} \left[ M_j \left( \frac{\Delta^\theta_m (\Lambda_j(y))}{\rho_2}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j} \\
\geq \frac{\varepsilon}{2} \quad \in I.
\end{align*}
\]

Since \( (X, \| \cdot \cdot \cdot \|) \) is an \( n \)-norm, \( \Delta^\theta_m \) and \( \Lambda_j \) are linear, and the Orlicz function \( M_j \) is convex for all \( j \in \mathbb{N} \), the following inequality holds:

\[
\begin{align*}
\sum_{j=1}^{\infty} a_{kj} \left[ M_j \left( \frac{\Delta^\theta_m (\Lambda_j(ax + \beta y))}{|\alpha|\rho_1 + |\beta|\rho_2}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j} \\
\leq D \sum_{j=1}^{\infty} a_{kj} \frac{|\alpha|\rho_1 + |\beta|\rho_2}{|\alpha|\rho_1 + |\beta|\rho_2} \\
\times \left[ M_j \left( \frac{\Delta^\theta_m (\Lambda_j(x))}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j} \\
\leq DL \sum_{j=1}^{\infty} a_{kj} \\
\times \left[ M_j \left( \frac{\Delta^\theta_m (\Lambda_j(y))}{\rho_2}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j},
\end{align*}
\]

(25)
Proof. Clearly \( g_\Delta (-x) = g_\Delta (x) \) and \( g_\Delta (\theta) = 0 \). Let \( x = (x_k) \) and \( y = (y_k) \in W[A, \mathcal{M}, \Delta^t_m, \Lambda, p, \ldots \cdot ||] \). Then, for \( \rho > 0 \) we set

\[
A_1 = \left\{ \rho : \sup_k \left[ \sum_{j=1}^{\infty} a_{kj} \left[ \frac{\Delta^t_m (\Lambda_j (x))}{\rho_1} \right]^{p_j} \right] \leq 1, \quad \text{for each } z_1, z_2, \ldots, z_{n-1} \in X \right\},
\]

\[
A_2 = \left\{ \rho : \sup_k \left[ \sum_{j=1}^{\infty} a_{kj} \left[ \frac{\Delta^t_m (\Lambda_j (y))}{\rho_2} \right]^{p_j} \right] \leq 1, \quad \text{for each } z_1, z_2, \ldots, z_{n-1} \in X \right\}.
\]

Let \( \rho_1 \in A_1 \), \( \rho_2 \in A_2 \), and \( \rho = \rho_1 + \rho_2 \); then we have

\[
\sup_k \left[ \sum_{j=1}^{\infty} a_{kj} \left[ \frac{\Delta^t_m (\Lambda_j (x+y))}{\rho} \right]^{p_j} \right]^{1/H} \leq 1, \quad \text{for some } \rho > 0,
\]

and each \( z_1, z_2, \ldots, z_{n-1} \in X \),

(28)

where \( H = \max\{1, \sup_k \rho_k \} \).
\[
\leq \frac{\rho_1}{\rho_1 + \rho_2} \times \sup_k \left[ \sum_{j=1}^{\infty} a_{kj} \left( M_j \left( \frac{\Delta_m^t (A_j(x))}{\rho_1} \right) \right) \right]^{p_j} \]
\[
\times \sup_k \left[ \sum_{j=1}^{\infty} a_{kj} \left( M_j \left( \frac{\Delta_m^t (A_j(y))}{\rho_2} \right) \right) \right]^{p_j} \]
\[
= \left\{ \rho_1 : \sup_k \left[ \sum_{j=1}^{\infty} a_{kj} \right]^{p_j} \leq 1 \right\}
\]

\[
A_4 = \left\{ \rho_1 : \sup_k \left[ \sum_{j=1}^{\infty} a_{kj} \right]^{p_j} \leq 1 \right\}
\]

\[
g_\Delta (x + y) = \sum_{j=1}^{m} \left[ x_j + y_j, z_1, z_2, \ldots, z_{n-1} \right] \]
\[
+ \inf \left\{ (\rho_1 + \rho_2)^{p_j} : \rho_1 \in A_1, \rho_2 \in A_2 \right\}
\]
\[
\leq \sum_{j=1}^{m} \left[ x_j, z_1, z_2, \ldots, z_{n-1} \right] \]
\[
+ \inf \left\{ (\rho_1)^{p_j} : \rho_1 \in A_1 \right\}
\]
\[
+ \sum_{j=1}^{m} \left[ y_j, z_1, z_2, \ldots, z_{n-1} \right] \]
\[
+ \inf \left\{ (\rho_2)^{p_j} : \rho_2 \in A_2 \right\}
\]
\[
g_\Delta (x) + g_\Delta (y).
\]

\[
(30)
\]

Let \( \lambda' \rightarrow \lambda \) where \( \lambda', \lambda \in \mathbb{C} \), and let \( g_\Delta (\lambda' x - \lambda x) \rightarrow 0 \) as \( t \rightarrow \infty \). We have to show that \( g_\Delta (\lambda' x - \lambda x) \rightarrow 0 \) as \( t \rightarrow \infty \). We set

\[
A_3
\]
\[
= \left\{ \rho_1 : \sup_k \left[ \sum_{j=1}^{\infty} a_{kj} \right]^{p_j} \leq 1 \right\}
\]
\[
\times \left[ \sup_k \left[ \sum_{j=1}^{\infty} a_{kj} \right]^{p_j} \right]
\]
\[
+ \sup_k \left[ \sum_{j=1}^{\infty} a_{kj} \right]^{p_j} \]

If \( \rho_1 \in A_3 \) and \( \rho_1^j \in A_4 \), then by using non-decreasing and convexity of the Orlicz function \( M_j \) for all \( j \in \mathbb{N} \) we get

\[
(31)
\]
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\[
\times \left[ M_j \left( \frac{\Delta_m' \left( \lambda x'_j - \lambda x_j \right)}{\left| \lambda^t - \lambda \right| \rho_1 + |\lambda| \rho_1^t}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{1/H} \leq \frac{|\lambda^t - \lambda| \rho_1}{|\lambda^t - \lambda| \rho_1 + |\lambda| \rho_1^t}
\]

Then, by the definition of ideal convergent, we have the set \( A_\delta \in I \). If \( n \notin A_\delta \), then we have

\[
\sum_{j=1}^{m_1} \left\| \lambda^t x_j' - \lambda x_j \right\| + \inf \left\{ \left( \frac{|\lambda^t - \lambda| \rho_1}{|\lambda^t - \lambda| \rho_1 + |\lambda| \rho_1^t} \right)^{p_1/H} : \rho_1 \in A_3, \rho_1^t \in A_4 \right\}
\]

Note that \( g_\delta(x') \leq g_\delta(x) + g_\delta(x' - x) \), for all \( t \in \mathbb{N} \). Hence, by our assumption, the right hand of (34) tends to 0 as \( t \to \infty \), and the result follows. This completes the proof of the theorem.

**Theorem 12.** Let \( \mathcal{M} = (M_j') \), \( \mathcal{M}' = (M'_j) \), and \( \mathcal{M}'' = (M''_j) \) be the Musielak-Orlicz functions. Then, the following hold:

(a) \( W[\mathcal{A}, \mathcal{M}, \Delta_m', \Lambda, p, \| \cdots \|_{0}^1] \subseteq W[\mathcal{A}, \mathcal{M}, \Delta_m', \Lambda, p, \| \cdots \|_{0}^1] \), provided \( p = (p_k) \) such that \( G_0 = \inf p_k > 0 \),

(b) \( W[\mathcal{A}, \mathcal{M}, \Delta_m', \Lambda, p, \| \cdots \|_{0}^1] \subseteq W[\mathcal{A}, \mathcal{M}'', \Delta_m''', \Lambda, p, \| \cdots \|_{0}^1] \).

**Proof.** (a) Let \( \epsilon > 0 \) be given. Choose \( \epsilon_1 > 0 \) such that \( \sup_k (\sum_{j=1}^{m_1} a_{kj}) \max \{ \epsilon_1^G, \epsilon_1^G \} < \epsilon \). Using the continuity of the Orlicz function \( M \), choose \( 0 < \delta < 1 \) such that \( 0 < t < \delta \) implies that \( M(t) < \epsilon_1 \).

Let \( x = (x_k) \) be any element in \( W[\mathcal{A}, \mathcal{M}', \Delta_m', \Lambda, p, \| \cdots \|_{0}^1] \) and put

\[
A_\delta = \left\{ k \in \mathbb{N} : \sum_{j=1}^{m_1} a_{kj} \left[ M_j' \left( \left| \frac{\Delta_m' \left( A_j (x_k) \right)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right| \right) \right]^{p_j} \geq \delta^G \right\}.
\]

Then, by the definition of ideal convergent, we have the set \( A_\delta \in I \). If \( n \notin A_\delta \), then we have
\[ \sum_{j=1}^{\infty} a_{kj} \left[ M_j^\prime \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j} \]
\[ < \delta^G \left[ M_j^\prime \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j} \]
\[ < \delta^G \Rightarrow M_j^\prime \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right) < \delta. \]

(36)

Using the continuity of the Orlicz function \( M_j \) for all \( j \) and the relation (36), we have
\[ M_j \left[ M_j^\prime \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right) \right] < \varepsilon_1. \]

(37)

Consequently, we get
\[ \sum_{j=1}^{\infty} a_{kj} \left[ M_j M_j^\prime \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j} \]
\[ < \sup_k \left( \sum_{j=1}^{\infty} a_{kj} \right) \max \left\{ \varepsilon_1, \varepsilon_2 \right\} < \varepsilon \]
\[ \Rightarrow \sum_{j=1}^{\infty} a_{kj} \left[ M_j M_j^\prime \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j} \]
\[ < \varepsilon. \]

(38)

This shows that
\[ \left\{ \begin{array}{l}
 k \in \mathbb{N} : \\
 \sum_{j=1}^{\infty} a_{kj} \left[ M_j M_j^\prime \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j} \\
 \geq \varepsilon
 \end{array} \right\} \subseteq A_\delta \in I. \]

(39)

This proves the assertion.

(b) Let \( x = (x_k) \) be any element in \( W[A, \mathcal{M}, \Lambda, \rho, \| \cdot \|_1] \). Then, by the following inequality, the results follow:
\[ \sum_{j=1}^{\infty} a_{kj} \left[ \left( M_j^\prime + M_j^\prime \right) \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j} \]
\[ \leq D \sum_{j=1}^{\infty} a_{kj} \left[ M_j^\prime \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j} \]
\[ + D \sum_{j=1}^{\infty} a_{kj} \left[ M_j^\prime \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j}. \]

(40)

Theorem 13. The inclusions \( Z[A, \mathcal{M}, \Lambda, \rho, \| \cdot \|_1] \subseteq Z[A, \mathcal{M}, \Lambda, \rho, \| \cdot \|_1] \) are strict for \( s, m \geq 1 \) in general where \( Z = W^s, W^t, W^{p, t} \).

Proof. We will give the proof for \( W[A, \mathcal{M}, \Lambda, \rho, \| \cdot \|_1] \subseteq W[A, \mathcal{M}, \Lambda, \rho, \| \cdot \|_1] \) only. The others can be proved by similar arguments. Let \( x = (x_k) \in W[A, \mathcal{M}, \Lambda, \rho, \| \cdot \|_1] \). Then let \( \varepsilon > 0 \) be given; there exist \( \rho > 0 \) such that
\[ \left\{ \begin{array}{l}
 k \in \mathbb{N} : \sum_{j=1}^{\infty} a_{kj} \left[ \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j} \\
 \geq \varepsilon
 \end{array} \right\} \in I. \]

(41)

Since \( M_j \) for all \( j \in \mathbb{N} \) is non-decreasing and convex, it follows that
\[ \sum_{j=1}^{\infty} a_{kj} M_j \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{2\rho}, z_1, z_2, \ldots, z_{n-1} \right) \]
\[ = \sum_{j=1}^{\infty} a_{kj} M_j \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{2\rho}, z_1, z_2, \ldots, z_{n-1} \right) \]
\[ \times M_j \left( \frac{\Delta^s_m \Lambda_{j+1} \left( x \right) - \Delta^s_m \Lambda_j \left( x \right)}{2\rho}, z_1, z_2, \ldots, z_{n-1} \right) \]
\[ \leq \frac{1}{2} \sum_{j=1}^{\infty} a_{kj} M_j \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right) \]
\[ + \frac{1}{2} \sum_{j=1}^{\infty} a_{kj} M_j \left( \frac{\Delta^s_m \left( \Lambda_j \left( x \right) \right)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right). \]

(42)
and then we have
\[
\left\{ \begin{array}{l}
k \in \mathbb{N}: \sum_{j=1}^{\infty} a_{kj} \\
z_1, z_2, \ldots, z_{n-1} \end{array} \right\} \geq \varepsilon
\]
\[
\leq \left\{ \begin{array}{l}k \in \mathbb{N}: \frac{1}{2} \\
z_1, z_2, \ldots, z_{n-1} \end{array} \right\} \geq \varepsilon
\]
\[
\cup \left\{ \begin{array}{l}k \in \mathbb{N}: \frac{1}{2} \\
z_1, z_2, \ldots, z_{n-1} \end{array} \right\} \geq \varepsilon
\]

for a sufficiently large value of \(j\). Since \(M_j\) for all \(j \in \mathbb{N}\) is non-decreasing, we get
\[
\sup_{k} \sum_{j=1}^{\infty} a_{kj} \left[ M_j \left( \left\| \frac{\Delta^s_m (\Lambda_j (x))}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{q_j}
\leq \sup_{k} \sum_{j=1}^{\infty} a_{kj} \left[ M_j \left( \left\| \frac{\Delta^s_m (\Lambda_j (x))}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_j} < \infty.
\]

Thus, \(x \in W[A, M, \Delta^s_m, \Lambda, p, \ldots, \| \cdot \|_{\infty}]\). This completes the proof of the theorem.

**Theorem 15.** (i) If \(0 < \inf p_k \leq p_k < 1\), then
\[
W[A, M, \Delta^s_m, \Lambda, p, \ldots, \| \cdot \|_{\infty}] \subset W[A, M, \Delta^s_m, \Lambda, p, \ldots, \| \cdot \|_{\infty}].
\]

(ii) If \(1 < p_k \leq \sup p_k < \infty\), then \(W[A, M, \Delta^s_m, \Lambda, p, \ldots, \| \cdot \|_{\infty}] \subset W[A, M, \Delta^s_m, \Lambda, p, \ldots, \| \cdot \|_{\infty}].
\]

Proof. (i) Let \(x = (x_j) \in W[A, M, \Delta^s_m, \Lambda, p, \ldots, \| \cdot \|_{\infty}];\) since \(0 < \inf p_k \leq p_k < 1\), then we have
\[
\sup_{k} \sum_{j=1}^{\infty} a_{kj} \left[ M_j \left( \left\| \frac{\Delta^s_m (\Lambda_j (x))}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_j} < \infty,
\]

and hence \(x \in W[A, M, \Delta^s_m, \Lambda, p, \ldots, \| \cdot \|_{\infty}].\)

(ii) Let \(1 < p_k \leq \sup p_k < \infty\) and \(x = (x_j) \in W[A, M, \Delta^s_m, \Lambda, p, \ldots, \| \cdot \|_{\infty}].\) Then for each \(0 < \varepsilon < 1\) there exists a positive integer \(j_0\) such that
\[
\sup_{k} \sum_{j=1}^{\infty} a_{kj} \left[ M_j \left( \left\| \frac{\Delta^s_m (\Lambda_j (x))}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_j} \leq \varepsilon < 1,
\]

for all \(j \geq j_0\). This implies that
\[
\sup_{k} \sum_{j=1}^{\infty} a_{kj} \left[ M_j \left( \left\| \frac{\Delta^s_m (\Lambda_j (x))}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_j} \leq \varepsilon < 1.
\]

Thus \(x \in W[A, M, \Delta^s_m, \Lambda, p, \ldots, \| \cdot \|_{\infty}]\) and this completes the proof.
Proof. Let \( x = (x_j) \in W[A, \Delta_m^s, \Lambda, p, \| \cdot \|_{\infty}] \) and \( \varepsilon > 0 \) be given. Then, there exist \( \rho > 0 \) such that the set

\[
\begin{align*}
&\left\{ k \in \mathbb{N} : \right. \\
&\left. \sum_{j=1}^{\infty} a_{kj} \left( \frac{\sum_{j=1}^{m} \Delta_m^s(A_j(x)) - 1}{\rho}, z_1, z_2, \ldots, z_{n-1} \right)^{p_j} \geq \varepsilon \right\}
\end{align*}
\]

\( \in I \), for some \( I. \)

By taking \( y_j = \|\Delta_m^s (A_j(x)) - 1\|/\rho, z_1, z_2, \ldots, z_{n-1} \|, \) let \( \varepsilon > 0 \) and choose \( \delta \) with \( 0 < \delta < 1 \) such that \( M_j(t) < \varepsilon \) for all \( j \in \mathbb{N} \); for \( 0 \leq t \leq \delta \), consider that

\[
\sum_{j=1}^{\infty} [M_j(y_j)]^{p_j} = \sum_{j=1}^{\infty} \left[ M_j \left( \frac{y_j}{\delta} \right) \right]^{p_j} + \frac{\sum_{j=1}^{\infty} M_j(y_j)}{\delta}^{p_j},
\]

since \( M_j \) is continuous for all \( n \in \mathbb{N} \).

\[
\sum_{j=1, y_j \in [0, \delta]} [M_j(y_j)]^{p_j} < \varepsilon \quad \text{and for } y_j > \delta, \text{we use the fact that } y_j < y_j/\delta < 1 + y_j/\delta. \text{Since } M = (M_j) \text{ is non-decreasing and convex, it follows that}
\]

\[
M_j \left( \frac{y_j}{\delta} \right) < M_j \left( 1 + \frac{y_j}{\delta} \right) < \frac{1}{2} M_j(2) + \frac{1}{2} M_j \left( \frac{2y_j}{\delta} \right).
\]

Since \( M = (M_j) \) satisfies the \( \Delta_2 \)-condition, then

\[
M_j \left( \frac{y_j}{\delta} \right) < \frac{y_j}{\delta} M_j(2) + \frac{y_j}{2\delta} M_j(2) = \frac{y_j}{\delta} L M_j(2).
\]

Hence

\[
\sum_{j=1, y_j \in [0, \delta]} [M_j(y_j)]^{p_j} < \max \left\{ 1, \sup_j \left( L \delta^{-1} M_j(2) \right)^{p_j} \right\} \times \sum_{j=1, y_j \in [0, \delta]} (y_j)^{p_j},
\]

and then we have

\[
\sum_{j=1}^{\infty} [M_j(y_j)]^{p_j} < \varepsilon + \max \left\{ 1, \sup_j \left( L \delta^{-1} M_j(2) \right)^{p_j} \right\} \sum_{j=1, y_j \in [0, \delta]} (y_j)^{p_j}.
\]

This proves that \( W[A, \Delta_m^s, \Lambda, p, \| \cdot \|_{\infty}] \subseteq W[A, \Lambda, \Delta_m^s, \Lambda, p, \| \cdot \|_{\infty}] \).

**Theorem 17.** Let \( 0 < p_n \leq q_n < 1 \) and \( (q_n/p_n) \) be bounded; then

\[
W[A, \Lambda, \Delta_m^s, \Lambda, q, \| \cdot \|_{\infty}] \subseteq W[A, \Lambda, \Delta_m^s, \Lambda, p, \| \cdot \|_{\infty}] \).
\]

**Proof.** Let \( x = (x_j) \in W[A, \Lambda, \Delta_m^s, \Lambda, q, \| \cdot \|_{\infty}] \) and we put

\[
y_j = \left( M_j \left( \frac{\sum_{j=1}^{m} \Delta_m^s(A_j(x)) - 1}{\rho}, z_1, z_2, \ldots, z_{n-1} \right) \right)^{q_j},
\]

\[
\beta_j = \frac{p_j}{q_j} \quad \forall j \in \mathbb{N}.
\]

Then \( 0 < \beta_j \leq 1 \), for all \( j \in \mathbb{N} \). Let it be such that \( 0 < \beta \leq \beta_j \) for all \( j \in \mathbb{N} \). Define the sequences \((a_j)\) and \((b_j)\) as follows: for \( j \geq 1 \), let \( a_j = y_j \) and \( b_j = 0 \); for \( j < 1 \), let \( a_j = 0 \) and \( b_j = y_j \). Then, clearly, for all \( j \in \mathbb{N} \) we have \( y_j = a_j + b_j \), \( y_j^\beta = a_j^\beta + b_j^\beta \), \( a_j \leq a_j \leq y_j \), and \( b_j \leq b_j \). Therefore, we have

\[
\sum_{j=1}^{\infty} a_{kj} y_j^\beta \leq \sum_{j=1}^{\infty} a_{kj} y_j \leq \sum_{j=1}^{\infty} a_{kj} y_j.
\]

Hence \( x \in W[A, \Lambda, \Delta_m^s, \Lambda, p, \| \cdot \|_{\infty}] \).

**Theorem 18.** For any two sequences \( p = (p_k) \) and \( q = (q_k) \) of positive real numbers and for any two \( n \)-norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) on \( X \), the following holds:

\[
Z[A, \Lambda, \Delta_m^s, \Lambda, p, \| \cdot \|_1] \cap Z[A, \Lambda, \Delta_m^s, \Lambda, q, \| \cdot \|_2] \neq \emptyset.
\]

**Proof.** The proof of the theorem is obvious, because the zero element belongs to each of the sequence spaces involved in the intersection.

**Theorem 19.** The sequence spaces \( W[A, \Lambda, \Delta_m^s, \Lambda, p, \| \cdot \|_1] \), \( W[A, \Lambda, \Delta_m^s, \Lambda, p, \| \cdot \|_2] \), \( W[A, \Lambda, \Delta_m^s, \Lambda, p, \| \cdot \|_\infty] \), and \( W[A, \Lambda, \Delta_m^s, \Lambda, p, \| \cdot \|_\infty] \) are neither solid nor symmetric nor sequence algebras for \( s, m \geq 1 \) in general.

**Proof.** The proof is obtained by using the same techniques of Theorem 15 and Theorems 17, 18, and 19.

**Note.** It is clear from definitions that

\[
W[A, \Lambda, \Delta_m^s, \Lambda, p, \| \cdot \|_0] \subseteq W[A, \Lambda, \Delta_m^s, \Lambda, p, \| \cdot \|_1] \subseteq W[A, \Lambda, \Delta_m^s, \Lambda, p, \| \cdot \|_2] \subseteq W[A, \Lambda, \Delta_m^s, \Lambda, p, \| \cdot \|_\infty].
\]
Theorem 20. The spaces $Z[A, M, \Delta', \Lambda, p, \| \ldots \|]$ and $Z[A, M, p, \| \ldots \|]$ are equivalent as topological spaces, where $Z = W^t, W^d, W^i, \text{ and } W^i_\infty$.

Proof. Consider the mapping

$$T: Z[A, M, \Delta', \Lambda, p, \| \ldots \|] \rightarrow Z[A, M, p, \| \ldots \|],$$

(63)

defined by $T(x) = (\Delta'_{m}(\Lambda, x))$ for each $x \in Z[A, M, \Delta', \Lambda, p, \| \ldots \|]$. Then, clearly $T$ is a linear homeomorphism and the proof follows. $\square$

Remark 21. If we replace the difference operator $\Delta'$ by $\Delta^{(i)}$, then for each $\varepsilon > 0$ we get the following sequence spaces:

$$W[A, M, \Delta^{(i)}, \Lambda, p, \| \ldots \|]^i$$

$$= \left\{ x \in \omega(n - X) : k \in \mathbb{N} : \lambda_k^{-1} \sum_{j \in I_k} \left[ M_j \left( \frac{\Delta^{(i)}_{m} x_j - l}{\rho}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j} \geq \varepsilon \right\} \in I,$$

for some $\rho > 0$ and each $z_1, z_2, \ldots, z_{n-1} \in X$.

$$W[A, M, \Delta^{(i)}, \Lambda, p, \| \ldots \|]_0^i$$

$$= \left\{ x \in \omega(n - X) : \exists K > 0, \sup_{k} \lambda_k^{-1} \sum_{j \in I_k} \left[ M_j \left( \frac{\Delta^{(i)}_{m} x_j}{\rho}, z_1, z_2, \ldots, z_{n-1} \right) \right]^{p_j} < \infty, \right\} \in I.$$

for some $\rho > 0$ and each $z_1, z_2, \ldots, z_{n-1} \in X$.

Corollary 22. The sequence spaces $Z[A, M, \Delta^{(i)}, \Lambda, p, \| \ldots \|]$, where $Z = W^i, W^d_\infty, W^i_\infty$, and $W^i_\infty$ are paranormed spaces (not totally paranormed) with respect to the paranorm $h_\Delta$ defined by

$$h_\Delta(x) = \sum_{j=1}^{m} \left\| x_j, z_1, z_2, \ldots, z_{n-1} \right\| + \inf \left\{ \rho^{p_i/H} : \right\}.$$
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\[ \sup_k \left[ \sum_{j=1}^{\infty} a_{kj} \times M_j \left( \frac{\Delta_m^{(j)}(\Lambda_j(x))}{\rho_j} \right)^{\frac{1}{p_j}} \right]_{z_1, z_2, \ldots, z_{n-1}} \leq 1, \quad \text{for some } \rho > 0, \]

where \( H = \max\{1, \sup_k p_k\} \) and \( Z = W^I, W^0, W^I_0 \), and \( W_\infty \). Also it is clear that the paranorms \( g_\Lambda \) and \( h_\Lambda \) are equivalent.

We state the following theorem in view of Lemma 9.

**Theorem 23.** Let \( X \) be a standard \( n \)-normed space and \( \{e_1, e_2, \ldots, e_n\} \) an orthogonal set in \( X \). Then, the following hold:

(a) \( W[A, A, \Delta_m^{(s)}, \Lambda, p; \| \ldots \|_{\infty} ] = W[A, A, \Delta_m^{(s)}, \Lambda, p; \| \ldots \|_{n-1} ] \),

(b) \( W[A, A, \Delta_m^{(0)}, \Lambda, p; \| \ldots \|_{0} ] = W[A, A, \Delta_m^{(0)}, \Lambda, p; \| \ldots \|_{n-1} ] \),

(c) \( W[A, A, \Delta_m^{(s)}, \Lambda, p; \| \ldots \|_{\infty} ] = W[A, A, \Delta_m^{(s)}, \Lambda, p; \| \ldots \|_{n-1} ] \),

(d) \( W[A, A, \Delta_m^{(s)}, \Lambda, p; \| \ldots \|_{\infty} ] = W[A, A, \Delta_m^{(s)}, \Lambda, p; \| \ldots \|_{n-1} ] \),

where \( \| \ldots \|_{\infty} \) is the derived \((n-1)\)-norm defined with respect to the set \( \{e_1, e_2, \ldots, e_n\} \) and \( \| \ldots \|_{n-1} \) is the standard \((n-1)\)-norm on \( X \).

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**References**


