Research Article

Some Operator Inequalities on Chaotic Order and Monotonicity of Related Operator Function

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We will discuss some operator inequalities on chaotic order about several operators, which are generalization of Furuta inequality and show monotonicity of related Furuta type operator function.

1. Introduction

An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all vectors $x$ in a Hilbert space, and $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible.

Theorem LH (Löwner-Heinz inequality, denoted by (LH) briefly). If $A \geq B \geq 0$ holds, then $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$.

This was originally proved in [1, 2] and then in [3]. Although (LH) asserts that $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$, unfortunately $A^\alpha \geq B^\alpha$ does not always hold for $\alpha > 1$. The following result has been obtained from this point of view.

Theorem F (Furuta inequality). If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $(B^r A^p B^r)^1/q \geq (B^r B^p B^r)^1/q$,

(ii) $(A^r A^p A^r)^1/q \geq (A^r B^p A^r)^1/q$

hold for $p \geq 0$ and $q \geq 1$ with $(1 + r)q \geq p + r$.

The original proof of Theorem F is shown in [4], an elementary one-page proof is in [5], and alternative ones are in [6, 7]. We remark that the domain of the parameters $p, q$, and $r$ in Theorem F is the best possible for the inequalities (i) and (ii) under the assumption $A \geq B \geq 0$; see [8].

We write $A \gg B$ if $\log A \geq \log B$ for $A, B > 0$, which is called the chaotic order.

Theorem A. For $A, B > 0$, the following (i) and (ii) hold:

(i) $A \gg B$ holds if and only if $A^\delta \geq (A^r B^p A^r)^1/(p+r)$ for $p, r \geq 0$;

(ii) $A \gg B$ holds if and only if for any fixed $\delta \geq 0$, $F_{A, B}(p, r) = A^{-\delta/2}(A^r B^p A^r)^{(\delta+r)/(p+r)} A^{-\delta/2}$ is a decreasing function of $p \geq \delta$ and $r \geq 0$.

(i) in Theorem A is shown in [9, 10], an excellent proof in [II], a proof in the case $p = r$ in [12], (ii) in [9, 10], and so forth.

Lemma B (see [11]). Let $A$ be a positive invertible operator, and let $B$ be an invertible operator. For any real number $\lambda$,

$$(BA^*) = BA^{1/2} \left( A^{1/2} B^* A^{1/2} \right)^{\lambda-1} A^{1/2} B^*.$$  (1)

Definition 1. Let $A_n, A_{n-1}, \ldots, A_2, A_1, B \geq 0$, $r_1, r_2, \ldots, r_n \geq 0$, and $p_1, p_2, \ldots, p_n \geq 0$ for a natural number $n$.

Let $C_{A, n}[n]$ be defined by

$$C_{A,B}[n] = A_{n/2} \left( \cdots A_{n/2} A_{n/2} \left( A_{n-1/2} B P_{n} A_{n-1/2} \right) P_{n} A_{n-1/2} \right) \cdots A_{n/2}.$$  (2)
For example,
\[
C_{A_i,B}[2] = A_2^{r_2/2} A_{i1}^{r_1/2} B P_i A_{i2}^{r_2/2},
\]
\[
C_{A_i,B}[4] = A_4^{r_4/2} \left[ A_2^{r_2/2} \left( A_{i1}^{r_1/2} B P_i A_{i2}^{r_2/2} \right) P_i \right] \times A_3^{r_3/2} \right] P_i A_{i4}^{r_4/2}.
\]
(3)

Let \( q[n] \) be defined by
\[
q[n] = [\cdots (p_1 + r_1) p_2 + r_2] p_3 + \cdots + r_n] p_n + r_n.
\]
(4)

For example,
\[
q[1] = p_1 + r_1, \quad q[2] = (p_1 + r_1) p_2 + r_2,
\]
\[
\]
(5)

For the sake of convenience, we define
\[
C_{A_i,B}[0] = B, \quad q[0] = 1,
\]
(6)

and these definitions in (6) may be reasonable by (2) and (4).

Lemma 2. For \( A_1, A_{n-1}, \ldots, A_2, A_1, B \geq 0 \) and any natural number \( n \), we have

(i) \( C_{A_i,B}[n] = A_n^{r_n/2} C_{A_i,B}[n-1] P_n A_n^{r_n/2} \),

(ii) \( q[n] = q[n-1] p_n + r_n \).

Proof. (i) and (ii) can be easily obtained by definitions (2) and (4).

\[\square\]

2. Basic Results Associated with
\( C_{A_i,B}[n] \) and \( q[n] \)

We will give some operator inequalities on chaotic order, and Theorem 5 is further extension of Theorem 3.1 in [13].

Lemma 3. If \( A \gg B \), for \( p \geq 0 \) and \( r \geq 0 \), then \( A \gg \left( A^{r/2} B^p A^{r/2} \right)^{1/(p+r)} \).

Proof. Since \( A \gg B \), we can obtain the following inequality.
\[
A \geq A^{r/2} B^p A^{r/2} \}
(7)

Take the logarithm on both sides of the previous inequality; that is,
\[
\log A \geq \log \left( A^{r/2} B^p A^{r/2} \right)^{1/(p+r)},
\]
therefore we have
\[
A \gg \left( A^{r/2} B^p A^{r/2} \right)^{1/(p+r)}.
\]
(8)

\[\square\]

Theorem 4. If \( A_n \gg A_{n-1} \gg \cdots \gg A_2 \gg A_1 \gg B \) and \( r_1, r_2, \ldots, r_n \geq 0, p_1, p_2, \ldots, p_n \geq 0 \) for a natural number \( n \). Then the following inequality holds:
\[
A_n \gg C_{A_i,B}[n]^{1/q[n]},
\]
(9)

where \( C_{A_i,B}[n] \) and \( q[n] \) are defined in (2) and (4).

Proof. We will show (9) by mathematical induction. In the case \( n = 1 \).

Since \( A_1 \gg B \) implies
\[
A_1 \gg \left( A_1^{r_1/2} B^{p_1} A_1^{r_1/2} \right)^{1/(p_1+r_1)}
\]
(10)

holds for any \( p_1 \geq 0 \) and \( r_1 \geq 0 \) by Lemma 3, whence (9) for \( n = 1 \).

Assume that (9) holds for a natural number \( k (1 \leq k < n) \). We will show that (9) holds \( r_1, r_2, \ldots, r_k, r_{k+1} \geq 0 \) and \( p_1, p_2, \ldots, p_k, p_{k+1} \geq 0 \) for \( k + 1 \).

Put \( D = A_{k+1}, E = A_k, \) and \( F = C_{A_i,B}[k]^{1/q[k]} \), and (9) holds for \( n = k \) implying
\[
D \gg E \gg F > 0.
\]
(11)

Equation (11) yields the following by Lemma 3, for \( r \geq 0 \) and \( p \geq 0 \):
\[
D \gg \left( D^{r/2} E^p D^{r/2} \right)^{1/(p+r)},
\]
(12)

that is,
\[
A_{k+1} \gg \left( A_{k+1}^{r_{k+1}/2} C_{A_i,B}[k]^{p_{k+1}/q[k]} A_{k+1}^{r_{k+1}/2} \right)^{1/(p+r)}.
\]
(13)

Put \( r = r_{k+1}, p = q[k] p_{k+1} \) in (13), then by (ii) of Lemma 2, the exponential power \((p+r)\) of the right hand side of (13) can be written as follows:
\[
\frac{1}{p+r} = \frac{1}{q[k] p_{k+1} + r_{k+1}} = \frac{1}{q[k+1]},
\]
(14)

and we have the following desired (15) by (12) and (13):
\[
A_{k+1} \gg \left( A_{k+1}^{r_{k+1}/2} C_{A_i,B}[k]^{p_{k+1}/q[k]} A_{k+1}^{r_{k+1}/2} \right)^{1/q[k+1]}
\]
(15)

so that (15) shows that (9) holds for \( k + 1 \).
\[\square\]
Theorem 5. If \( A_n \gg A_{n-1} \gg \cdots \gg A_2 \gg A_1 \gg B \) and \( r_1, r_2, \ldots, r_n \geq 0 \) for a natural number \( n \). For any fixed \( \delta \geq 0 \), let \( p_1, p_2, \ldots, p_n \) be satisfied by
\[
\begin{align*}
p_1 & \geq \delta, \\
p_2 & \geq \frac{\delta + r_1}{p_1 + r_1}, \\
& \vdots \\
p_k & \geq \frac{\delta + r_1 + r_2 + \cdots + r_{k-1}}{q[k-1]}, \\
& \vdots \\
p_n & \geq \frac{\delta + r_1 + r_2 + \cdots + r_{n-1}}{q[n-1]}.
\end{align*}
\]

The operator function \( I_k(p_k, r_k) \) for any natural number \( k \) such that \( 1 \leq k \leq n \) is defined by
\[
I_k(p_k, r_k) = A^{-r_{k+1}/2} A^{-r_k/2} A^{-r_{k-1}/2} A^{-r_{k-2}/2} \cdots A^{-r_2/2} A^{-r_1/2} A^{-r_{k+1}/2}.
\]

Then the following inequality holds:
\[
A^{r_{k+1}/2} I_{k-1}(p_{k-1}, r_{k-1}) A^{r_k/2} I_k(p_k, r_k) \geq A_k(p_k, r_k)
\]
for every natural number \( k \) such that \( 1 \leq k \leq n \), where \( A_{k-1} \) and \( q[k] \) are defined in (2) and (4).

Proof. Since \( C_{A,B}[0] = B, q[0] = 1 \) in (6), we may define
\[
I_0(p_0, r_0) = B^\delta \text{ for } p_0 = r_0 = 0.
\]
Because \( A_1 \gg B \), then for any fixed \( \delta \geq 0 \),
\[
B^\delta \geq A^{-r_{k+1}/2} A^{-r_k/2} A^{-r_{k-1}/2} A^{-r_{k-2}/2} \cdots A^{-r_2/2} A^{-r_1/2} A^{-r_{k+1}/2}
\]
for \( p_1 \geq \delta, r_1 \geq 0, \) since \( F_{A,B}(\delta, r_0) \geq F_{A,B}(p_1, r_1) \) holds by (ii) of Theorem A. And (19) can be expressed as
\[
B^\delta = A^{-r_{k+1}/2} I_0(p_0, r_0) A^{r_0/2} I_1(p_1, r_1).
\]

We can apply Theorem 4, and we have the following (21) for any natural number \( k \) such that \( 1 \leq k \leq n \):
\[
A_{k-1} \gg A_k \gg C_{A,B}[k]^{1/q[k]}.
\]

Since \( X \gg Y \) implies that \( X^t \gg Y^t \) holds for any \( t \geq 0 \), (21) ensures
\[
A^{\delta r_1 + r_2 + r_3 + \cdots + r_k} \gg C_{A,B}[k]^{(\delta r_1 + r_2 + r_3 + \cdots + r_k)/q[k]}.
\]

Putting \( A = A^{\delta r_1 + r_2 + r_3 + \cdots + r_k}, B_1 = C_{A,B}[k]^{(\delta r_1 + r_2 + r_3 + \cdots + r_k)/q[k]} \) and applying (19) for \( \delta = 1 \) and \( A \gg B_1 \), we have
\[
B_1 \geq A^{-r_{k+1}/2} A^{r_0/2} I_0(p_0, r_0) A^{r_0/2} I_1(p_1, r_1) A^{-r_{k+1}/2}
\]
holds for \( p_1 \geq 1 \) and \( r \geq 0 \).

Putting \( r_k \geq r(\delta + r_1 + r_2 + \cdots + r_k) \) in (23), then (23) can be rewritten by
\[
B_1 \geq A_k^{-r_{k+1}/2} \left( A_k^{-r_0/2} A_{k-1}^{r_0/2} A_{k-2}^{r_0/2} \cdots A_1^{r_0/2} \right)^{1/r_0} A^{-r_{k+1}/2}.
\]

Putting \( p = (q[k] p_{k+1} + \delta + r_1 + r_2 + \cdots + r_k) \geq 1 \), since \( p_{k+1} \geq (\delta + r_1 + r_2 + \cdots + r_k)/q[k] \) in (16), then we have
\[
A_k^{-r_{k+1}/2} I_k(p_k, r_k) A^{r_{k+1}/2} = B_1 = C_{A,B}[k]^{(\delta r_1 + r_2 + \cdots + r_k)/q[k]} \geq A_k^{-r_{k+1}/2} \frac{(\delta r_1 + r_2 + \cdots + r_k)/q[k]}{q[k]}
\]
for any fixed \( \delta \geq 0, r_1 \geq 0, \) since \( F_{A,B}(\delta, r_0) \geq F_{A,B}(p_1, r_1) \) holds by (ii) of Theorem A. And (19) can be expressed as
\[
B^\delta = A^{-r_{k+1}/2} I_0(p_0, r_0) A^{r_0/2} I_1(p_1, r_1).
\]

We can apply Theorem 4, and we have the following (21) for any natural number \( k \) such that \( 1 \leq k \leq n \):
\[
A_{k-1} \gg A_k \gg C_{A,B}[k]^{(1/q[k])}. \quad (21)
\]

Since \( X \gg Y \) implies that \( X^t \gg Y^t \) holds for any \( t \geq 0 \), (21) ensures
\[
A^{\delta r_1 + r_2 + r_3 + \cdots + r_k} \gg C_{A,B}[k]^{(\delta r_1 + r_2 + r_3 + \cdots + r_k)/q[k]}.
\]

Putting \( A = A^{\delta r_1 + r_2 + r_3 + \cdots + r_k}, B_1 = C_{A,B}[k]^{(\delta r_1 + r_2 + r_3 + \cdots + r_k)/q[k]} \) and applying (19) for \( \delta = 1 \) and \( A \gg B_1 \), we have
\[
B_1 \geq A^{-r_{k+1}/2} A^{r_0/2} I_0(p_0, r_0) A^{r_0/2} I_1(p_1, r_1) A^{-r_{k+1}/2}
\]
holds for \( p_1 \geq 1 \) and \( r \geq 0 \).
Proof. Applying (18) of Theorem 5 for \( k \) such that \( 1 \leq k \leq n \), we have

\[
B^\delta = A_1^{r/2}I_0 \left(p_0, r_0\right) A_1^{-r/2} \\
\geq I_1 \left(p_1, r_1\right) \\
= A_1^{-r/2} \left(A_1^{r/2} B^p A_1^{-r/2}\right)^{(dr, r_1)/(p_1+r_1)} A_1^{-r/2} \\
\geq A_1^{-r/2} I_2 \left(p_2, r_2\right) A_1^{-r/2} \\
= A_1^{-r/2} A_2^{-r/2} \left[A_2^{r/2} \left(A_1^{r/2} B^p A_1^{-r/2}\right)^{p_2} \\
\times A_2^{-r/2} \right]\left((dr, r_2)/(p_2+r_2)\right) \\
\times A_2^{-r/2} A_1^{-r/2}. \tag{27}
\]

3. Monotonicity Property on Operator Functions

We would like to emphasize that the condition of Theorem 7 is stronger than Theorem 5, and moreover when we discuss monotonicity property on operator functions, we can only apply Theorem 7.

Theorem 7. If \( A_n \gg A_{n-1} \gg \cdots \gg A_2 \gg A_1 \gg B \) and \( r_1, r_2, \ldots, r_n \geq 0, p_1, p_2, \ldots, p_n \geq 0 \) for a natural number \( n \). Then the following inequality holds:

\[
A_n^{r_n} \geq C_{A_n, B^n}[r_n/q[n]], \tag{28}
\]

where \( C_{A_n, B^n}[n] \) and \( q[n] \) are defined in (2) and (4).

Proof. We will show (28) by mathematical induction. In the case \( n = 1 \).

Since \( A_1 \gg B \) implies

\[
A_1 \geq A_1^{r/2} B^p A_1^{-r/2}/(p+r), \tag{29}
\]

holds for any, \( p, r \geq 0 \) and \( r_1 \geq 0 \) by (i) of Theorem A, whence (28) for \( n = 1 \).

Assume that (28) holds for a natural number \( k \) (\( 1 \leq k < n \)). We will show (28) for \( r_1, r_2, \ldots, r_{k+1} \geq 0 \) and \( p_1, p_2, \ldots, p_k, p_{k+1} \geq 0 \) for \( k + 1 \).

We can obtain the following inequality from the hypothesis (28) for the case \( n = k \):

\[
A_k^{r_k} \geq C_{A_k, B^n}[r_k/q[k]], \tag{30}
\]

hence we have \( A_{k+1} \gg A_k \gg C_{A_k, B^n}[r_k/q[k]], \) and (i) of Theorem A ensures

\[
A_{k+1}^r \geq \left(A_{k+1}^{r/2} C_{A_k, B^n}[r_k/q[k]] A_{k+1}^{r/2}\right)^{(r/p+r)} \tag{31}
\]

Putting \( r = r_{k+1} \) and \( p = q[k]p_{k+1} \), then we have the following inequality:

\[
A_{k+1}^{r_{k+1}} \geq \left(A_{k+1}^{r_{k+1}/2} C_{A_k, B^n}[r_{k+1}/2] A_{k+1}^{r_{k+1}/2}\right)^{(r_{k+1}/q[k]p_{k+1}+r_{k+1})} = C_{A_{k+1}, B^n}[k+1][r_{k+1}/q[k]+1], \tag{32}
\]

so that (32) shows (28) for \( k + 1 \).

□

Theorem 8. If \( A_n \gg A_{n-1} \gg \cdots \gg A_2 \gg A_1 \gg B \) and \( r_1, r_2, \ldots, r_n \geq 0 \) for a natural number \( n \). For any fixed \( \delta \geq 0 \), let \( p_1, p_2, \ldots, p_n \) be satisfied by (16).

Then

\[
I_n \left(p_n, r_n\right) = A_n^{-r_n} C_{A_n, B^n}[n]\left[(dr, r_2+\cdots+r_n)/q[n]\right] A_n^{-r_n/2} \tag{33}
\]

is a decreasing function of both \( r_n \geq 0 \) and \( p_n \) which satisfies

\[
p_n \geq \delta + r_1 + r_2 + \cdots + r_{n-1}, \tag{34}
\]

where \( C_{A_n, B^n}[n] \) and \( q[n] \) are defined in (2) and (4).

Proof. Since the condition (16) with \( \delta \geq 0 \) suffices (28) in Theorem 7, we have the following inequality by Theorem 7; see (28).

We state the following important inequality (35) for the forthcoming discussion which is the inequality in (16):

\[
q[n] = q[n-1] p_n + r_n \geq \delta + r_1 + r_2 + \cdots + r_{n-1} + r_n \tag{35}
\]

because the inequality in (35) follows by (ii) of Lemma 2, and the inequality follows by

\[
q[n-1] p_n \geq \delta + r_1 + r_2 + \cdots + r_{n-1} \tag{36}
\]

obtained by (34).

(a) Proof of the result that \( I_n(p_n, r_n) \) is a decreasing function of \( p_n \).

Without loss of generality, we can assume that \( p_n > 0 \). We can obtain the following inequality by (28) and by (i) of Lemma 2:

\[
A_n^{r_n} \geq C_{A_n, B^n}[n]^{r_n/q[n]} \left[A_n^{r_n/2} C_{A_n, B^n}[n-1] p_n A_n^{r_n/2}\right]^{r_n/q[n]} = A_n^{r_n/2} C_{A_n, B^n}[n-1] p_n A_n^{r_n/2} \times \left(C_{A_n, B^n}[n-1] p_n A_n^{r_n/2} C_{A_n, B^n}[n-1] p_n A_n^{r_n/2}\right)\left(r_n-q[n]/q[n]\right) \times C_{A_n, B^n}[n-1] p_n A_n^{r_n/2}, \tag{37}
\]
and (37) implies
\[
(\mathbb{C}_n,\mathbb{B}[n^{-1}])^{(q[n]-\omega)/q[n]}(\gamma[n])^{-1/2}/q[n]p_n)
\geq C_{n,\mathbb{B}[n^{-1}]}^{(\delta+r_1+r_2+\cdots+r_n)}\cdot A_{n}^{-\tau'/2}.
\] (39)

Put \(\alpha = \omega/p_n \in [0,1]\) for \(p_n \geq \omega \geq 0\), then we raise each side of (38) to the power \(\alpha = \omega/p_n \in [0,1]\), then
\[
(\mathbb{C}_n,\mathbb{B}[n^{-1}])^{(q[n]-\omega)/q[n]}(\gamma[n])^{-1/2}/q[n]p_n)
\geq C_{n,\mathbb{B}[n^{-1}]}^{(\delta+r_1+r_2+\cdots+r_n)}\cdot A_{n}^{-\tau'/2}.
\] (40)

Whence we have
\[
I_n(p_n, r_n)
= A_{n}^{-\tau'/2}(A_{n}^{r'/2}C_{n,\mathbb{B}[n^{-1}])^{(\delta+r_1+r_2+\cdots+r_n)}\cdot A_{n}^{\tau'/2}
= A_{n}^{-\tau'/2}
\times \left\{ A_{n}^{r'/2}C_{n,\mathbb{B}[n^{-1}])^{(q[n]-\omega)/q[n]}(\gamma[n])^{-1/2}/q[n]p_n)
\times A_{n}^{\tau'/2} \right\}
\geq C_{n,\mathbb{B}[n^{-1}]}^{(\delta+r_1+r_2+\cdots+r_n)}\cdot A_{n}^{-\tau'/2}.
\] (41)

We state the following inequality by (ii) of Lemma 3 and (35):
\[
q[n] - (\delta + r_1 + r_2 + \cdots + r_n)
= q[n - 1]p_n + r_n - (\delta + r_1 + r_2 + \cdots + r_n)
\geq q[n - 1]p_n - (\delta + r_1 + r_2 + \cdots + r_{n-1}) \geq 0.
\] (42)

Then we have
\[
I_n(p_n, r_n)
= A_{n}^{-\tau'/2}(A_{n}^{r'/2}C_{n,\mathbb{B}[n^{-1}])^{(\delta+r_1+r_2+\cdots+r_n)}\cdot A_{n}^{\tau'/2}
= A_{n}^{-\tau'/2}
\times \left\{ A_{n}^{r'/2}C_{n,\mathbb{B}[n^{-1}])^{(q[n]-\omega)/q[n]}(\gamma[n])^{-1/2}/q[n]p_n)
\times A_{n}^{\tau'/2} \right\}
\geq C_{n,\mathbb{B}[n^{-1}]}^{(\delta+r_1+r_2+\cdots+r_n)}\cdot A_{n}^{-\tau'/2}.
\] (43)
\[ \geq C_{A, B}[n-1]^{p_{\mu}/2} \times \{ C_{A, B}[n-1]^{p_{\mu}/2} A_{n}^{r_{n} + \mu} \times C_{A, B}[n-1]^{p_{\mu}/2} \}^{(\delta + r_{1} + r_{2} + \cdots + r_{n} - q[n])/q[n] + \mu} \]
\[ = I_{n}(p_{\mu}, r_{n} + \mu), \]
(43)
and the last inequality holds by LH because (41) and
\[ \frac{\delta + r_{1} + r_{2} + \cdots + r_{n} - q[n]}{q[n] + \mu} \]
\[ = - \frac{q[n] - (\delta + r_{1} + r_{2} + \cdots + r_{n})}{q[n] + \mu} \in [-1, 0], \]
so that \( I_{k}(p_{k}, r_{k}) \) is a decreasing function of \( r_{n} \).

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