Research Article

Remark on Existence and Uniqueness of Solutions for a Coupled System of Multiterm Nonlinear Fractional Differential Equations

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The aim of this paper is to extend the work of Sun et al. (2012) to a more general case for a wider range of function classes of \( f \) and \( g \). Our results include the case of the previous work, which are essential improvement of the work of Sun et al. (2012), especially.

1. Introduction

Fractional calculus can give a more vivid and accurate description of problems in various fields of sciences than the traditional calculus [1–3]. Recently many complicated dynamic phenomena were modeled by fractional order calculus system and have received more and more attention; see [4–16].

In recent work [12], Sun et al. studied the existence and uniqueness of solutions for a coupled system of multiterm nonlinear fractional differential equations with an initial value condition

\[
-\mathcal{D}^\alpha x(t) = f \left( t, y(t), \mathcal{D}^{\beta_1} y(t), \ldots, \mathcal{D}^{\beta_N} y(t) \right),
\]

\[
\mathcal{D}^{\alpha-j} x(0) = 0, \quad i = 1, 2, \ldots, n_1,
\]

\[
-\mathcal{D}^\sigma y(t) = g \left( t, x(t), \mathcal{D}^{\rho_1} x(t), \ldots, \mathcal{D}^{\rho_N} x(t) \right),
\]

\[
\mathcal{D}^{\sigma-j} y(0) = 0, \quad j = 1, 2, \ldots, n_2,
\]

where \( t \in (0,1] \), \( \alpha > \beta_1 > \beta_2 > \cdots > \beta_N > 0 \), \( \sigma > \rho_1 > \rho_2 > \cdots > \rho_N > 0 \), \( n_1 = [\alpha] + 1 \), \( n_2 = [\sigma] + 1 \) for \( \alpha, \sigma \notin \mathbb{N} \) and \( n_1 = \alpha, n_2 = \sigma \) for \( \alpha, \sigma \in \mathbb{N} \), \( \beta_1, \beta_2, \ldots, \beta_N, \rho_1, \rho_2, \ldots, \rho_N < 1 \) for any \( q \in \{1, 2, \ldots, N\} \), \( \mathcal{D} \) is the standard Riemann–Liouville derivative, and \( f, g : [0,1] \times \mathbb{R}^{N+1} \to \mathbb{R} \) are given functions. In order to obtain the existence and uniqueness of solutions of (1), the following growth conditions are introduced in [12].

(H1) There exist two nonnegative functions \( a(t), b(t) \in L^1[0,1] \) such that

\[
\left| f \left( t, x_0, x_1, \ldots, x_N \right) \right| \leq a(t) + c_0 |x_0|^\gamma_0 + c_1 |x_1|^\gamma_1 + \cdots + c_N |x_N|^\gamma_N,
\]

\[
\left| g \left( t, x_0, x_1, \ldots, x_N \right) \right| \leq b(t) + d_0 |x_0|^\theta_0 + d_1 |x_1|^\theta_1 + \cdots + d_N |x_N|^\theta_N,
\]

where \( c_i, d_i \geq 0, 0 < \gamma_i, \theta_i < 1 \) for \( i = 0, 1, 2, \ldots, N \).

(H2) The functions \( f \) and \( g \) satisfy

\[
\left| f \left( t, x_0, x_1, \ldots, x_N \right) \right| \leq c_0 |x_0|^{\gamma_0} + c_1 |x_1|^{\gamma_1} + \cdots + c_N |x_N|^{\gamma_N},
\]

\[
\left| g \left( t, x_0, x_1, \ldots, x_N \right) \right| \leq d_0 |x_0|^{\theta_0} + d_1 |x_1|^{\theta_1} + \cdots + d_N |x_N|^{\theta_N},
\]

where \( c_i, d_i \geq 0, \gamma_i, \theta_i > 1 \) for \( i = 0, 1, 2, \ldots, N \).

However, there are many functions which cannot satisfy conditions (H1) and (H2); for example,

\[
g \left( t, x_0, x_1 \right) = \frac{t}{6.08} + \frac{1}{25.26} \left[ x_0 + e^x \right].
\]

Hence the results of [12] are limited only to a small subset of functions which satisfy (H1) and (H2). This paper thus aims to
extend the work of Sun et al. [12] to a more general case with more general conditions on $f$ and $g$. Our major contributions of this paper include three aspects.

(1) We extend the function classes to more general case; that is, the power growth assumptions (H1) and (H2) are replaced by a very general assumption where the functions $\phi ((|x_0|,|x_1|,\ldots,|x_N|))$ and $\psi ((|x_0|,|x_1|,\ldots,|x_N|))$ are only required to be nondecreasing function classes (see (A1)), which implies that the function classes are extended to more general case and also include the case of [12] as a special case. In mathematics and applied science, this generalization is important and interesting.

(2) In [12], the weight functions considered constants $c_0, c_1, \ldots, c_N$. But in physics, the influence of weight functions for the whole system is important, so in this work, we improve the weight functions to general Lebesgue integral functions $b(t), d(t) \in L^1[0,1]$, which is also an essential improvement.

(3) In this paper, the nonlinearities $f$ and $g$ are allowed to be exponential growth. However, in [12], the nonlinearities $f$ and $g$ are only allowed to be power growth. It is known that in most cases exponential growth is faster than power growth. From this aspect, this is also a major contribution of this paper.

The remaining part of the paper is organized as follows. In Section 2, some preliminary results including definitions, notations, and lemmas are given. Section 3 presents the main results and the proof of the results. In addition, an example is given to illustrate the application of the main results.

2. Preliminaries and Lemmas

Definition 1 (see [1–3]). The fractional integral of order $\alpha > 0$ of a function $x : (a, +\infty) \to R$ is given by

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) \, ds, \quad (5)$$

provided that the right-hand side is pointwisely defined on $(a, +\infty)$.

Definition 2 (see [1–3]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $x : (a, +\infty) \to R$ is given by

$$D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t-s)^{n-\alpha-1} x(s) \, ds, \quad (6)$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of the number $\alpha$, and $t > a$, provided that the right-hand side is defined on $(a, +\infty)$.

Lemma 3 (see [1]). Assume that $x \in L^1[0,1]$ with a fractional derivative of order $\alpha > 0$; then

$$I^\alpha D^\alpha x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad (7)$$

where $c_i \in R, i = 1, 2, \ldots, n, n = [\alpha] + 1$.

Lemma 4 (see [12]). Suppose that $h \in L^1[0,1]$. Then the initial value problem

$$D^\alpha x(t) = h(t), \quad \alpha > 0, \quad t \in [a, b], \quad (8)$$

has a unique solution

$$x(t) = \sum_{j=1}^n \frac{b_j}{\Gamma(\alpha-j+1)} (t-a)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{h(s)}{(t-s)^{1-\alpha}} \, ds, \quad (9)$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $\alpha = n$ for $\alpha \in \mathbb{N}$.

Let $I = [0,1]$ and let $C(I)$ be the space of all continuous functions defined on $I$. We define the space

$$X \times Y = \{ (x, y) \mid (x, y) \in C(I) \times C(I), \quad x = D^{\rho_1} x(t), \quad y = D^{\rho_2} y(t) \} \in C(I), \quad (10)$$

$$\times C(I), \quad j = 1, 2, \ldots, N \}$$

endowed with the norm $\| (x, y) \|_{X \times Y} = \max \{ \| x \|_X, \| y \|_Y \}$, where

$$\| x \|_X = \max_{t \in I} |x(t)| + \sum_{j=1}^N \max_{t \in I} \| D^{\rho_j} x(t) \|, \quad (11)$$

$$\| y \|_Y = \max_{t \in I} |y(t)| + \sum_{j=1}^N \max_{t \in I} \| D^{\rho_j} y(t) \|.$$
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\[
\frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \times g(s, x(s), D^{\rho_1}x(s), \ldots, D^{\rho_N}x(s)) \, ds.
\]

(13)

It is obvious that a fixed point of operator \(T\) is the solution of coupled system (1).

3. Main Result

Theorem 5. Let \(f, g : [0, 1] \times \mathbb{R}^{N+1} \to \mathbb{R}\) be continuous. Assume that

(A1) there exist nonnegative functions \(a, b, c, d \in L^1[0, 1]\) and nonnegative nondecreasing functions \(\phi, \psi\) with respect to each variable \(x_i, i = 0, 1, 2, \ldots, N\), such that

\[
|f(t, x_0, x_1, \ldots, x_N)| \leq a(t) + b(t) \phi \left( |x_0|, |x_1|, \ldots, |x_N| \right),
\]

\[
|g(t, x_0, x_1, \ldots, x_N)| \leq c(t) + d(t) \psi \left( |x_0|, |x_1|, \ldots, |x_N| \right);
\]

(14)

(A2) there exists a constant \(R_0 > \max\{k_1, l_1\}\) such that

\[
\phi(R_0, R_0, \ldots, R_0) \leq \frac{R_0 - k_1}{k_2},
\]

\[
\psi(R_0, R_0, \ldots, R_0) \leq \frac{R_0 - l_1}{l_2},
\]

where

\[
k_1 = \max_{t \in T} \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |a(s)| \, ds + \sum_{j=1}^N \frac{1}{\Gamma(\alpha - \beta_j)} \int_0^t (t-s)^{\alpha-\beta_j-1} |a(s)| \, ds \right),
\]

\[
k_2 = \max_{t \in T} \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |b(s)| \, ds + \sum_{j=1}^N \frac{1}{\Gamma(\alpha - \beta_j)} \int_0^t (t-s)^{\alpha-\beta_j-1} |b(s)| \, ds \right),
\]

\[
l_1 = \max_{t \in T} \left( \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} |c(s)| \, ds + \sum_{j=1}^N \frac{1}{\Gamma(\sigma - \beta_j)} \int_0^t (t-s)^{\sigma-\beta_j-1} |c(s)| \, ds \right),
\]

\[
l_2 = \max_{t \in T} \left( \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} |d(s)| \, ds + \sum_{j=1}^N \frac{1}{\Gamma(\sigma - \beta_j)} \int_0^t (t-s)^{\sigma-\beta_j-1} |d(s)| \, ds \right).
\]

(16)

Then the coupled system (1) has a solution.

Proof. Define a closed ball of Banach space \(X \times Y\)

\[B = \{(x, y) \in X \times Y : \| (x, y) \|_{X \times Y} \leq R_0 \}.
\]

(17)

We will prove that \(T : B \to B\). In fact, for any \((x, y) \in B\), by (A1), we have

\[
\| T_1 x(t) \| 
= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), D^{\beta_1}y(s), \ldots, D^{\beta_N}y(s)) \, ds \right|
\]

\[
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |a(s)| \, ds
\]

\[
+ \frac{\phi(R_0, R_0, \ldots, R_0)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |b(s)| \, ds,
\]

\[
|D^{\rho}T_1 x(t)|
= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), D^{\beta_1}y(s), \ldots, D^{\beta_N}y(s)) \, ds \right|
\]

\[
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |a(s)| \, ds
\]

\[
+ \frac{\phi(R_0, R_0, \ldots, R_0)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |b(s)| \, ds,
\]

(18)

Thus it follows from (18) and (A2) that

\[
\| T_1 x \|_X = \max_{t \in T} \| T_1 x(t) \| + \sum_{j=1}^N \max_{t \in T} |D^{\rho_j}T_1 x(t)|
\]

\[
\leq k_1 + k_2 \phi(R_0, R_0, \ldots, R_0) \leq R_0,
\]

(19)

In the same way, we also have

\[
\| T_2 y \|_Y = \max_{t \in T} \| T_2 y(t) \| + \sum_{j=1}^N \max_{t \in T} |D^{\rho_j}T_2 y(t)|
\]

\[
\leq l_1 + l_2 \psi(R_0, R_0, \ldots, R_0) \leq R_0,
\]

(20)

Consequently, \(\| T_1 x \|_X \leq R_0\) and \(\| T_2 y \|_Y \leq R_0\), and then \(\| T \|_{X \times Y} \leq R_0\) for any \((x, y) \in B\); that is, \(T : B \to B\).

By [12], we know that the operator \(T\) is completely continuous. Therefore, the Schauder fixed point theorem implies that coupled system (1) has a solution in \(B\). The proof is completed.

□
From Theorem 5, we easily obtain the following corollaries.

**Corollary 6.** Let \( f, g : [0, 1] \times \mathbb{R}^{N+1} \rightarrow \mathbb{R} \) be continuous. Assume that

(A) there exist nonnegative functions \( c, d \in L^1[0, 1] \) and nonnegative nondecreasing functions \( \phi, \psi \) with respect to each variable \( x_i, \ i = 0, 1, 2, \ldots, N \), such that

\[
\begin{align*}
|f(t, x_0, x_1, \ldots, x_N)| &\leq b(t) \min\{ |x_0|, |x_1|, \ldots, |x_N| \}, \\
|g(t, x_0, x_1, \ldots, x_N)| &\leq d(t) \min\{ |x_0|, |x_1|, \ldots, |x_N| \}.
\end{align*}
\]

(B) for any \( s > 0 \),

\[
\phi(s, s, \ldots, s) \leq s, \quad \psi(s, s, \ldots, s) \leq s,
\]
and \( \max\{k_1^2, l_1^2\} < 1 \), where

\[
\begin{align*}
|f(t, x_0, x_1, \ldots, x_N)| &\leq a(t) \min\{ |x_0|, |x_1|, \ldots, |x_N| \}, \\
|g(t, x_0, x_1, \ldots, x_N)| &\leq c(t) \min\{ |x_0|, |x_1|, \ldots, |x_N| \}.
\end{align*}
\]

Then the coupled system (1) has a solution.

**Proof.** In fact, let us choose \( R_0 = \max\{k_1, l_1\} \), where

\[
\begin{align*}
&k_1 = \max_{t \in I} \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |a(s)| \, ds \right), \\
&l_1 = \max_{t \in I} \left( \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} |c(s)| \, ds \right),
\end{align*}
\]

and construct a closed ball of Banach space \( X \times Y \)

\[
B = \{(x, y) \in X \times Y : \|(x, y)\|_{X \times Y} \leq R_0 \}. \tag{26}
\]

The rest of proof is similar to Theorem 5.

**Remark 8.** The condition (A1) is weaker than (H1) and (H2). Clearly, \( \phi \) and \( \psi \) are continuous and bounded. This leads to the Corollary 3.1 of [12]. Therefore, Corollary 3.1 of [12] is only a special case of Corollary 7.

In the following, we focus on the uniqueness of the solution of the system (1).

**Theorem 10.** Let \( f, g : [0, 1] \times \mathbb{R}^{N+1} \rightarrow \mathbb{R} \) be continuous. Assume that

(A1) there exist nonnegative functions \( a, c \in L^1[0, 1] \) and nonnegative nondecreasing functions \( \phi, \psi \) with respect to each variable \( x_i, \ i = 0, 1, 2, \ldots, N \), such that

\[
\begin{align*}
|f(t, u_0, u_1, \ldots, u_N) - f(t, v_0, v_1, \ldots, v_N)| &\leq a(t) \min\{ |u_0 - v_0|, |u_1 - v_1|, \ldots, |u_N - v_N| \}, \\
|g(t, u_0, u_1, \ldots, u_N) - g(t, v_0, v_1, \ldots, v_N)| &\leq b(t) \min\{ |u_0 - v_0|, |u_1 - v_1|, \ldots, |u_N - v_N| \},
\end{align*}
\]

and \( \max\{k_1^2, l_1^2\} < 1 \), where

(B1) for any \( s > 0 \),

\[
\phi(s, s, \ldots, s) \leq s, \quad \psi(s, s, \ldots, s) \leq s,
\]

Then the coupled system (1) has a solution.
\[ k_1 = \max_{t \in I} \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a(s) \, ds \right) \]

\[ + \sum_{j=1}^N \frac{1}{\Gamma(\alpha - \rho_j)} \int_0^t (t-s)^{\alpha-\rho_j-1} a(s) \, ds \right), \]

\[ l_1 = \max_{t \in I} \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} c(s) \, ds \right) \]

\[ + \sum_{j=1}^N \frac{1}{\Gamma(\alpha - \beta_j)} \int_0^t (t-s)^{\alpha-\beta_j-1} c(s) \, ds \right). \]

Then coupled system (1) has a unique solution.

**Proof.** We prove that the operator \( T : X \times Y \to X \times Y \) is contraction. To do this, let \((x_1, y_1), (x_2, y_2) \in X \times Y \); we have

\[ |T_1 x_2(t) - T_1 x_1(t)| \]

\[ = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \right. \]

\[ \times f \left( s, y_2(s), \mathcal{D}^\beta y_2(s), \ldots, \mathcal{D}^{\beta_N} y_2(s) \right) \, ds \]

\[ - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \]

\[ \times f \left( s, y_1(s), \mathcal{D}^\beta y_1(s), \ldots, \mathcal{D}^{\beta_N} y_1(s) \right) \, ds \] \]

\[ \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a(s) \]

\[ \times \phi \left( \|y_2(s) - y_1(s)\|, \|\mathcal{D}^\beta y_2(s) - \mathcal{D}^\beta y_1(s)\|, \ldots, \right. \]

\[ \left| \mathcal{D}^{\beta_N} y_2(s) - \mathcal{D}^{\beta_N} y_1(s) \right| \right] \, ds \]

\[ \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a(s) \]

\[ \times \phi \left( \|y_2(s) - y_1(s)\|, \|\mathcal{D}^\beta y_2(s) - \mathcal{D}^\beta y_1(s)\|, \ldots, \right. \]

\[ \left. \|\mathcal{D}^{\beta_N} y_2(s) - \mathcal{D}^{\beta_N} y_1(s)\| \right) \right] \, ds \]

\[ \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a(s) \]

\[ \times \phi \left( \|y_2(s) - y_1(s)\|, \|y_2(s) - y_1(s)\|, \ldots, \right. \]

\[ \left. \|\mathcal{D}^{\beta_N} y_2(s) - \mathcal{D}^{\beta_N} y_1(s)\| \right) \right] \, ds \]

\[ \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a(s) \]

\[ \times \phi \left( \|y_2(s) - y_1(s)\|, \|y_2(s) - y_1(s)\|, \ldots, \right. \]

\[ \left. \|y_2(s) - y_1(s)\| \right) \right] \, ds \]

\[ \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a(s) \]

\[ \times \phi \left( \|y_2(s) - y_1(s)\|, \|y_2(s) - y_1(s)\|, \ldots, \right. \]

\[ \left. \|y_2(s) - y_1(s)\| \right) \right] \, ds \]

\[ \leq k \|y_2 - y_1\|_Y. \]

Thus it follows from (30) and (B2) that

\[ \|T_1 x_2 \| \leq \max_{t \in I} |T_1 x_2(t) - T_1 x_1(t)| \]

\[ + \sum_{j=1}^N \max_{t \in I} |\mathcal{D}^\rho_j \left( T_1 x_2(t) - T_1 x_1(t) \right) | \]

\[ \leq \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a(s) \right) \, ds \]

\[ + \sum_{j=1}^N \frac{1}{\Gamma(\alpha - \rho_j)} \int_0^t (t-s)^{\alpha-\rho_j-1} a(s) \, ds \right) \|y_2 - y_1\|_Y \]

\[ \leq k \|y_2 - y_1\|_Y. \]
Similarly, we can get
\[ |T_1 y_2(t) - T_1 y_1(t)| \leq \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} c(s) ds \|x_2 - x_1\|_X, \]
\[ |\mathcal{D}^\beta T_1 y_2(t) - \mathcal{D}^\beta T_1 y_1(t)| \leq \frac{1}{\Gamma(\sigma - \beta)} \int_0^t (t-s)^{\sigma-\beta-1} c(s) ds \|x_2 - x_1\|_X, \]
\[ \|T_1 y_2 - T_1 y_1\|_Y = \max_{t \in I} |T_1 y_2(t) - T_1 y_1(t)| \]
\[ + \sum_{j=1}^N \max_{t \in I} |\mathcal{D}^\rho (T_1 y_2(t) - T_1 y_1(t))| \]
\[ \leq \left( \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} c(s) ds \right) \|x_2 - x_1\|_X \]
\[ + \sum_{j=1}^N \frac{1}{\Gamma(\sigma - \beta)} \int_0^t (t-s)^{\sigma-\beta-1} c(s) ds \|x_2 - x_1\|_X \]
\[ \leq \left( \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} c(s) ds \right) \|x_2 - x_1\|_X \]
\[ \leq l_1 \|x_2 - x_1\|_X. \]  
(32)

Hence, for the Euclidean distance \( d \) on \( \mathbb{R}^2 \), we get
\[ d(T(x_2, y_2), T(x_1, y_1)) = \sqrt{\|T_1 x_2 - T_1 x_1\|_X^2 + \|T_1 y_2 - T_1 y_1\|_Y^2} \]
\[ \leq k_0^2 \|x_2 - x_1\|_X^2 + l_1^2 \|y_2 - y_1\|_Y^2 \]
\[ \leq \max\{k_0^2, l_1^2\} \|x_2 - x_1\|_X + \|y_2 - y_1\|_Y \]
\[ \leq \max\{k_0^2, l_1^2\} d((x_2, y_2), (x_1, y_1)). \]  
(33)

Thus \( T \) is a contraction since \( \max\{k_0^2, l_1^2\} < 1 \).

By Banach contraction principle, \( T \) has a unique fixed point, which is a solution of the coupled system (1). The proof is completed.

**An Example.** Consider the existence of solutions for the following coupled system of multiterm nonlinear fractional differential equations:
\[-\mathcal{D}^{3.5} x(t) = \frac{t}{6.08} + \frac{1}{25.26} \left[ y(t) + e^{(\mathcal{D}^\rho x(t))} \right],\]
\[-\mathcal{D}^{4.2} y(t) = \frac{10000}{5501} \left[ t^{-1/2} x^{0.2}(t) + t^2 (\mathcal{D}^{1.2} x(t))^{0.5} \right],\]
\[ \mathcal{D}^{4.2-j} y(0) = 0, \quad j = 1, 2, \ldots, 5, \]  
(34)

where \( t \in (0, 1] \).

Let
\[ f(t, x_0, x_1) = \frac{t}{6.08} + \frac{1}{25.26} \left[ x_0 + e^{x_1} \right], \]  
(35)

and choose
\[ a(t) = \frac{t}{6.08}, \quad b(t) = \frac{1}{25.26}, \]  
\[ \phi(x_0, x_1) = x_0 + e^{x_1}, \quad c(t) = 0, \]  
\[ d(t) = \frac{10000}{5501} \left[ t^{-1/2} + r^2 \right], \]  
\[ \psi(x_0, x_1) = x_0^{0.2} + x_1^{0.5}. \]  
(36)

Then
\[ f(t, x_0, x_1) \leq a(t) + b(t) \phi(x_0, x_1), \]  
\[ g(t, x_0, x_1) \leq c(t) + d(t) \psi(x_0, x_1); \]  
(37)

consequently, (A1) holds.

In the following, we check the condition (A1). Since
\[ k_1 = \max\left\{ \frac{1}{\Gamma(3.5)} \int_0^t (t-s)^{2.5} s ds + \frac{1}{\Gamma(3)} \int_0^t (t-s)^2 ds \right\} \]
\[ = 0.01, \]  
\[ k_2 = \max\left\{ \frac{1}{\Gamma(3.5)} \int_0^t (t-s)^{2.5} s ds + \frac{1}{\Gamma(3)} \int_0^t (t-s)^2 ds \right\} \]
\[ = 0.01, \]  
\[ l_1 = 0, \]  
\[ l_2 = \frac{10000}{5501} \]
\[ \times \max\left\{ \frac{1}{\Gamma(4.2)} \int_0^t (t-s)^{3.2} (s^{-1/2} + s^2) ds \right\} \]
\[ + \frac{1}{\Gamma(3.4)} \int_0^t (t-s)^{2.4} (s^{-1/2} + s^3) ds \right\} = 1, \]  
(38)

take \( R_0 = 5 \); we have
\[ \phi(R_0, R_0) = R_0 + e^{R_0} = 5 + e^5 \]
\[ = 153.44 < \frac{R_0 - k_1}{k_2} = \frac{5 - 0.01}{0.01} = 499, \]  
\[ \psi(R_0, R_0) = R_0^{0.2} + R_0^{0.5} = 5^{0.2} + 5^{0.5} \]
\[ = 3.6158 < \frac{R_0 - l_1}{l_2} = 5, \]  
(39)

which implies that (A2) is satisfied. Hence, by Theorem 5, the coupled system of fractional differential equation (34) has a solution.
Remark 11. In the coupled system of fractional differential equation (34), the nonlinear function $f$ involves exponential growth, but the results of [12] are only allowed to be power growth; that is, (34) cannot be solved by using the results of [12]. So the results obtained in this paper give a significant improvement of the previous work in [12].

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