Research Article

Sufficient Conditions for Non-Bazilevič Functions

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The main purpose of this paper is to derive some sufficient conditions for analytic functions to be of non-Bazilevič type.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad (1)$$

which are analytic in the open unit disk:

$$\mathbb{U} := \{z : z \in \mathbb{C}, |z| < 1\}. \quad (2)$$

For $0 \leq \alpha < 1$ and $0 < \mu < 1$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{N}(\mu, \alpha)$ if it satisfies the condition

$$\Re \left(f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \right) > \alpha, \quad (z \in \mathbb{U}). \quad (3)$$

As usual, the class $\mathcal{N}(\mu, \alpha)$ is said to be non-Bazilevič functions of order α (see [1]).

For some recent investigations of non-Bazilevič functions, see, for example the works of [2–6] and the references cited therein.

For two functions f and g , analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$ and write

$$f(z) < g(z), \quad (z \in \mathbb{U}), \quad (4)$$

if there exists a Schwarz function ω , which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0, \quad |\omega(z)| < 1, \quad (z \in \mathbb{U}), \quad (5)$$

such that

$$f(z) = g(\omega(z)), \quad (z \in \mathbb{U}). \quad (6)$$

Indeed, it is known that

$$f(z) < g(z), \quad (7)$$

$$(z \in \mathbb{U}) \implies f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) < g(z), \quad (8)$$

$$(z \in \mathbb{U}) \implies f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

To derive our main results, we need the following lemmas.

Lemma 1 (see [7]). Let $p(z) = 1 + b_1 z + b_2 z^2 + \dots$ be analytic in $\mathbb{U}$ and let h be analytic and starlike (with respect to the origin) univalent in $\mathbb{U}$ with $h(0) = 0$. If

$$z p'(z) < h(z), \quad (9)$$

then

$$p(z) < 1 + \int_0^z \frac{h(t)}{t} dt. \quad (10)$$

Lemma 2 (see [8]). Let q be univalent in $\mathbb{U}$. Also let ϕ be analytic in the domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $\phi(\omega) \neq 0$ when $\omega \in q(\mathbb{U})$. Set

$$Q(z) = z q'(z) \phi(q(z)), \quad h(z) = \theta(q(z)) + Q(z). \quad (11)$$

Suppose that

- (1) $Q(z)$ is starlike univalent in $\mathbb{U}$;
- (2) $\Re(zh'(z)/Q(z)) = \Re((\theta'(q(z))/\phi(q(z))) + (zQ'(z)/Q(z))) > 0$ for $z \in \mathbb{U}$.

If p is analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)), \quad (12)$$

then $p < q$, and q is the best dominant.

Lemma 3 (see [9]). Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that Φ is a mapping from $\mathbb{C}^2 \times \mathbb{U}$ to $\mathbb{C}$ which satisfies $\Phi(ix, y; z) \notin \Omega$ for $z \in \mathbb{U}$ and for all real x, y such that $y \leq -(1+x^2)/2$.

If the function $p(z) = 1 + c_1z + c_2z^2 + \dots$ is analytic in $\mathbb{U}$ and $\Phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then $\Re(p(z)) > 0$.

In this paper, we aim at proving some sufficient conditions for analytic functions to be of non-Bazilevič type.

2. Main Results

Our first main result is given by Theorem 4.

Theorem 4. Suppose that $h(z)$ is starlike in $\mathbb{U}$ with $h(0) = 0$. If

$$\frac{zf''(z)}{f'(z)} + (1+\mu)\left(1 - \frac{zf'(z)}{f(z)}\right) < h(z), \quad (0 < \mu < 1), \quad (13)$$

then

$$f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} < \exp\left(1 + \int_0^z \frac{h(t)}{t} dt\right). \quad (14)$$

Proof. We define the function p by

$$p(z) := f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}, \quad (z \in \mathbb{U}; 0 < \mu < 1). \quad (15)$$

Then p is analytic in $\mathbb{U}$ with $p(0) = 1$. It follows from (15) that

$$z(\log(p(z)))' = \frac{zf''(z)}{f'(z)} + (1+\mu)\left(1 - \frac{zf'(z)}{f(z)}\right), \quad (16)$$

$$(0 < \mu < 1).$$

Combining (13) and (16), we find that

$$z(\log(p(z)))' < h(z). \quad (17)$$

By Lemma 1, we deduce that

$$\log(p(z)) < 1 + \int_0^z \frac{h(t)}{t} dt. \quad (18)$$

From (15) and (18), we readily get the assertion (14) of Theorem 4. $\square$

Theorem 5. If $f \in \mathcal{A}$ satisfies the inequality

$$\left| \left[\frac{zf''(z)}{f'(z)} + (1+\mu)\left(1 - \frac{zf'(z)}{f(z)}\right) \right] \times \left(f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \right)^{-1} \right| < \nu, \quad (0 < \mu, \nu < 1), \quad (19)$$

then $f \in \mathcal{N}(\mu, 1/(1+\nu))$.

Proof. Suppose that the function p is defined by (15). It follows that

$$z\left(\frac{1}{p(z)}\right)' = -\left[\frac{zf''(z)}{f'(z)} + (1+\mu)\left(1 - \frac{zf'(z)}{f(z)}\right) \right] \times \left(f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \right)^{-1}. \quad (20)$$

Combining (19) and (20), we know that

$$z\left(\frac{1}{p(z)}\right)' < \nu z. \quad (21)$$

An application of Lemma 1 to (21) yields

$$p(z) < \frac{1}{1+\nu z} =: q(z). \quad (22)$$

By noting that

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) = \Re\left(\frac{1-\nu z}{1+\nu z}\right) \geq \frac{1-\nu}{1+\nu} > 0, \quad (23)$$

$$(0 < \nu < 1; z \in \mathbb{U}),$$

which implies that the region $q(\mathbb{U})$ is symmetric with respect to the real axis and q is convex univalent in $\mathbb{U}$ therefore, we have

$$\Re(q(z)) \geq q(1) \geq 0, \quad (z \in \mathbb{U}). \quad (24)$$

Combining (15), (22), and (24), we conclude that

$$\Re\left(f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\right) > \frac{1}{1+\nu}, \quad (0 < \nu < 1; z \in \mathbb{U}). \quad (25)$$

This completes the proof of Theorem 5. $\square$

Theorem 6. Suppose that q is convex in $\mathbb{U}$ with $q(0) = 1$. If

$$\Re(\lambda q(z)) > 0, \quad (z \in \mathbb{U}; \lambda \in \mathbb{C}), \quad (26)$$

$$\left[\frac{zf''(z)}{f'(z)} + (1+\mu)\left(1 - \frac{zf'(z)}{f(z)}\right) \right] f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} + \lambda\left(f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\right)^2 < zq'(z) + \lambda q^2(z), \quad (27)$$

then

$$f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} < q(z), \quad (28)$$

and q is the best dominant.

Proof. Suppose that the function p is defined by (15). It follows that

$$\begin{aligned} zp'(z) + \lambda p^2(z) &= \left[\frac{zf''(z)}{f'(z)} + (1+\mu) \left(1 - \frac{zf'(z)}{f(z)} \right) \right] \\ &\quad \times f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \\ &\quad + \lambda \left(f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \right)^2. \end{aligned} \quad (29)$$

We now assume that

$$\theta(\omega) = \lambda\omega^2, \quad \phi(\omega) = 1. \quad (30)$$

Obviously, $\theta(\omega)$ and $\phi(\omega)$ are analytic in the ω plane. By noting that the function

$$Q(z) = zp'(z)\phi(p(z)) = zp'(z) \quad (31)$$

is starlike in $\mathbb{U}$ and

$$\chi(z) = \theta(p(z)) + Q(z) = \lambda p^2(z) + zp'(z), \quad (32)$$

it follows from (26) that

$$\Re \left(\frac{z\chi'(z)}{Q(z)} \right) = \Re \left(2\lambda p(z) + \frac{zQ'(z)}{Q(z)} \right) > 0. \quad (33)$$

Combining (27), (29), and Lemma 2, we get the assertion of Theorem 6. $\square$

Remark 7. By taking suitable $h(z)$ and $q(z)$ in Theorems 4 and 6, respectively, we can get some useful consequences. Here we choose to omit the details.

Theorem 8. If $f \in \mathcal{A}$ satisfies the condition

$$\begin{aligned} &\frac{f^{1+\mu}(z)}{z^\mu f'(z)} \left(f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \right)' \\ &> \begin{cases} \frac{\gamma}{2(\gamma-1)}, & (0 \leq \gamma \leq \frac{1}{2}), \\ \frac{\gamma-1}{2\gamma}, & (\frac{1}{2} \leq \gamma < 1), \end{cases} \end{aligned} \quad (34)$$

then $f \in \mathcal{N}(\mu, \gamma)$.

Proof. Suppose that

$$\psi(z) := \frac{f'(z)(z/f(z))^{1+\mu} - \gamma}{1-\gamma}, \quad (0 \leq \gamma < 1; z \in \mathbb{U}). \quad (35)$$

Then ψ is analytic in $\mathbb{U}$. It follows from (35) that

$$\begin{aligned} \frac{f^{1+\mu}(z)}{z^\mu f'(z)} \left(f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \right)' &= \frac{(1-\gamma)z\psi'(z)}{\gamma + (1-\gamma)\psi(z)} \\ &= \Phi(\psi(z), z\psi'(z); z), \end{aligned} \quad (36)$$

where

$$\Phi(r, s; t) = \frac{(1-\gamma)s}{\gamma + (1-\gamma)r}. \quad (37)$$

For all real x and y satisfying $y \leq -(1+x^2)/2$, we have

$$\begin{aligned} \Re(\Phi(ix, y; z)) &= \frac{(1-\gamma)\gamma y}{\gamma^2 + (1-\gamma)^2 x^2} \\ &\leq -\frac{(1-\gamma)\gamma}{2} \cdot \frac{1+x^2}{\gamma^2 + (1-\gamma)^2 x^2} \\ &\leq \begin{cases} -\frac{(1-\gamma)\gamma}{2} \cdot \frac{1}{(1-\gamma)^2}, & (0 \leq \gamma \leq \frac{1}{2}), \\ -\frac{(1-\gamma)\gamma}{2} \cdot \frac{1}{\gamma^2}, & (\frac{1}{2} \leq \gamma < 1). \end{cases} \end{aligned} \quad (38)$$

We now put

$$\Omega = \left\{ \xi : \Re(\xi) > \begin{cases} \frac{\gamma}{2(\gamma-1)} & (0 \leq \gamma \leq \frac{1}{2}) \\ \frac{\gamma-1}{2\gamma} & (\frac{1}{2} \leq \gamma < 1) \end{cases} \right\}. \quad (39)$$

Then $\Phi(ix, y; z) \notin \Omega$ for all real x, y such that $y \leq -(1+x^2)/2$. Moreover, in view of (34), we know that $\Phi(\psi(z), z\psi'(z); z) \in \Omega$. Thus, by Lemma 3, we deduce that

$$\Re(\psi(z)) > 0, \quad (z \in \mathbb{U}), \quad (40)$$

which shows that the desired assertion of Theorem 8 holds. $\square$

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