Research Article

Some Almost Generalized (ψ, ϕ) -Contractions in G -Metric Spaces

Hassen Aydi,¹ Sana Hadj Amor,² and Erdal Karapinar³

¹ Department of Mathematics, Jubail College of Education, Dammam University, P.O. Box 12020, Industrial Jubail 31961, Saudi Arabia

² Laboratoire Physique Mathématique, Fonctions Spéciales et Applications (MAPSEA) LR11ES35, Ecole Supérieure des Sciences et de la Technologie de Hammam Sousse, Université de Sousse, Rue Lamine el Abbassi, 4011 Hammam Sousse, Tunisia

³ Department of Mathematics, Atilim University, İncek, 06836 Ankara, Turkey

Correspondence should be addressed to Hassen Aydi; hassen.aydi@isima.rnu.tn

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In this paper, we introduce some almost generalized (ψ, ϕ) -contractions in the setting of G -metric spaces. We prove some fixed points results for such contractions. The presented theorems improve and extend some known results in the literature. An example is also presented.

1. Introduction and Preliminaries

In 2006, a new structure of a generalized metric space was introduced by Mustafa and Sims [1] as an appropriate notion of a generalized metric space called a G -metric space. Fixed point theory in this space was initiated in [2]. Particularly, Banach contraction mapping principle was established in this work. Since then the fixed point theory in G -metric spaces has been studied and developed by many authors, see [1–29].

The following definitions and results will be needed in the sequel.

Definition 1 (see [1]). Let X be a nonempty set, $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following properties:

- (G1) $G(x, y, z) = 0$ if $x = y = z$,
- (G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
- (G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
- (G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables),
- (G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specially, a G -metric on X , and the pair (X, G) is called a G -metric space.

Every G -metric on X defines a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X. \quad (1)$$

Example 2. Let (X, d) be a metric space. The function $G : X \times X \times X \rightarrow [0, +\infty)$, defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}, \quad (2)$$

or

$$G(x, y, z) = d(x, y) + d(y, z) + d(z, x), \quad (3)$$

for all $x, y, z \in X$, is a G -metric on X .

Definition 3 (see [1]). Let (X, G) be a G -metric space, and let $\{x_n\}$ be a sequence of points of X ; therefore, we say that x_n is G -convergent to $x \in X$ if $\lim_{n, m \rightarrow +\infty} G(x, x_n, x_m) = 0$; that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \rightarrow x$ or $\lim_{n \rightarrow +\infty} x_n = x$.

Proposition 4 (see [1]). Let (X, G) be a G -metric space. The following are equivalent:

- (1) $\{x_n\}$ is G -convergent to x ,
- (2) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow +\infty$,
- (3) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow +\infty$.

Definition 5 (see [1]). Let (X, G) be a G -metric space. A sequence $\{x_n\}$ is called a G -Cauchy sequence if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$; that is, $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow +\infty$.

Proposition 6 (see [1]). Let (X, G) be a G -metric space. Then the following are equivalent:

- (1) the sequence $\{x_n\}$ is G -Cauchy,
- (2) for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq N$.

Definition 7 (see [1]). A G -metric space (X, G) is called G -complete if every G -Cauchy sequence is G -convergent in (X, G) .

Definition 8. Let (X, G) be a G -metric space. A mapping $F : X \rightarrow X$ is said to be G -continuous if for any G -convergent sequence $\{x_n\}$ to x , then $\{F(x_n)\}$ is G -convergent to $F(x)$.

Now, let $\mathcal{F}$ denote the set of functions $f : [0, +\infty) \rightarrow [0, +\infty)$ such that $f(t) = 0$ if and only if $t = 0$. We denote by Ψ and Φ the subsets of $\mathcal{F}$ such that

$$\begin{aligned}\Psi &= \{\psi \in \mathcal{F} : \psi \text{ is continuous and nondecreasing}\}, \\ \Phi &= \{\phi \in \mathcal{F} : \phi \text{ is lower semicontinuous}\}.\end{aligned}\quad (4)$$

There are a lot of fixed point theorems for different type contractions in the literature. In particular, Berinde [30–32] introduced the concept of an almost contraction in metric spaces and studied many interesting fixed point theorems for a Ćirić strong almost contraction. For other fixed point results on generalized almost contractions, see [33–37]. In this paper, we introduce some almost generalized (ψ, ϕ) -contractions in the setting of G -metric spaces, and we establish some fixed points results for such contractions.

2. Main Results

Let (X, G) be a G -metric space. First, we consider the following expressions:

$$\begin{aligned}M(x, y, z) &= \max \{G(x, Tx, y), G(y, T^2x, Ty), G(Tx, T^2x, Ty), \\ &\quad G(x, Tx, z), G(z, T^2x, Tz), \\ &\quad G(Tx, T^2x, Tz), G(x, y, z)\}, \\ N(x, y, z) &= \min \{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), \\ &\quad G(z, Tx, Tx), G(y, Tz, Tz)\},\end{aligned}\quad (5)$$

for all $x, y, z \in X$.

Our first result for almost generalized (ψ, ϕ) -contractions is the following.

Theorem 9. Let (X, G) be a complete G -metric space. Let $T : X \rightarrow X$ be a self-mapping. Suppose there exist $\psi \in \Psi$, $\phi \in \Phi$, and $L \geq 0$ such that for all $x, y, z \in X$,

$$\begin{aligned}\psi(G(Tx, Ty, Tz)) &\leq \psi(M(x, y, z)) \\ &\quad - \phi(M(x, y, z)) + LN(x, y, z).\end{aligned}\quad (6)$$

Then T has a unique fixed point; say $u \in X$.

Proof. Let $x_0 \in X$, and define a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n, \quad \text{for any } n \in \mathbb{N}. \quad (7)$$

If for some $n \in \mathbb{N}$, $x_{n+1} = x_n$, then $x_n = Tx_n$, and the proof is completed. Thus, we may assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. By (6) we have

$$\begin{aligned}\psi(G(x_n, x_{n+1}, x_{n+1})) &= \psi(G(Tx_{n-1}, Tx_n, Tx_n)) \\ &\leq \psi(M(x_{n-1}, x_n, x_n)) \\ &\quad - \phi(M(x_{n-1}, x_n, x_n)) + LN(x_{n-1}, x_n, x_n),\end{aligned}\quad (8)$$

where

$$\begin{aligned}M(x_{n-1}, x_n, x_n) &= \max \{G(x_{n-1}, Tx_{n-1}, x_n), G(x_n, T^2x_{n-1}, Tx_n), \\ &\quad G(Tx_{n-1}, T^2x_{n-1}, Tx_n), G(x_{n-1}, Tx_{n-1}, x_n), \\ &\quad G(x_n, T^2x_{n-1}, Tx_n), \\ &\quad G(Tx_{n-1}, T^2x_{n-1}, Tx_n), G(x_{n-1}, x_n, x_n)\} \\ &= \max \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\ &\quad G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n), \\ &\quad G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), \\ &\quad G(x_{n-1}, x_n, x_n)\} \\ &= \max \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}, \\ N(x_{n-1}, x_n, x_n) &= \min \{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_n, Tx_n), \\ &\quad G(x_n, Tx_n, Tx_n), G(x_n, Tx_{n-1}, Tx_{n-1}), \\ &\quad G(x_n, Tx_n, Tx_n)\}\end{aligned}$$

$$\begin{aligned}
&= \min \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\
&\quad G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n), \\
&\quad G(x_n, x_{n+1}, x_{n+1})\} \\
&= \min \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), 0\} = 0.
\end{aligned} \tag{9}$$

From (8) and (9), we get

$$\begin{aligned}
&\psi(G(x_n, x_{n+1}, x_{n+1})) \\
&\leq \psi(\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}) \\
&\quad - \phi(\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}).
\end{aligned} \tag{10}$$

If

$$\begin{aligned}
&\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} \\
&= G(x_n, x_{n+1}, x_{n+1}),
\end{aligned} \tag{11}$$

then by (10) we have

$$\begin{aligned}
&\psi(G(x_n, x_{n+1}, x_{n+1})) \\
&\leq \psi(G(x_n, x_{n+1}, x_{n+1})) - \phi(G(x_n, x_{n+1}, x_{n+1})).
\end{aligned} \tag{12}$$

Thus $\phi(G(x_n, x_{n+1}, x_{n+1})) = 0$, and hence $G(x_n, x_{n+1}, x_{n+1}) = 0$. Therefore, $x_n = x_{n+1}$ which is a contradiction. So,

$$\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} = G(x_{n-1}, x_n, x_n). \tag{13}$$

Therefore, we get

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n) \quad \forall n \in \mathbb{N}, \tag{14}$$

and (10) becomes

$$\begin{aligned}
&\psi(G(x_n, x_{n+1}, x_{n+1})) \\
&\leq \psi(G(x_{n-1}, x_n, x_n)) - \phi(G(x_{n-1}, x_n, x_n))
\end{aligned} \tag{15}$$

$\forall n \in \mathbb{N}.$

Thus, by (14), the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ is monotone nonincreasing. It follows that $G(x_n, x_{n+1}, x_{n+1}) \rightarrow \alpha$ as $n \rightarrow +\infty$ for some $\alpha \geq 0$. Next we claim that $\alpha = 0$. On taking limit as $n \rightarrow +\infty$ in (15), we obtain

$$\psi(\alpha) \leq \psi(\alpha) - \phi(\alpha). \tag{16}$$

Hence $\phi(\alpha) = 0$ and we get $\alpha = 0$. Hence

$$\lim_{n \rightarrow +\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{17}$$

Next, we show that $\{x_n\}$ is a G-Cauchy sequence. On contrary, assume that $\{x_n\}$ is not a G-Cauchy sequence. Then, there is

an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}, \{x_{m(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) \geq k$ such that

$$\begin{aligned}
&G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geq \varepsilon, \\
&G(x_{m(k)-1}, x_{n(k)}, x_{n(k)}) < \varepsilon.
\end{aligned} \tag{18}$$

Using (18) and (G5), we have

$$\begin{aligned}
&\varepsilon \leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\
&\leq G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, x_{n(k)}, x_{n(k)}) \\
&< \varepsilon + 2G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}).
\end{aligned} \tag{19}$$

Taking limit as $k \rightarrow +\infty$ and using (17), we have

$$\lim_{k \rightarrow +\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{20}$$

By (G5), we get

$$\begin{aligned}
&G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\
&\leq G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) \\
&\quad + G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) \\
&\leq G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) \\
&\quad + 2G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) \\
&\leq G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) \\
&\quad + G(x_{m(k)}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) \\
&\quad + 2G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) \\
&\leq 3G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\
&\quad + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + 2G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}).
\end{aligned} \tag{21}$$

On taking limit as $k \rightarrow +\infty$ in the above inequality and using (17) and (20), we obtain

$$\lim_{k \rightarrow +\infty} G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{22}$$

Again, by (G5), we have

$$\begin{aligned}
&G(x_{n(k)}, x_{m(k)+2}, x_{n(k)+1}) \\
&\leq G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+2}) \\
&\leq G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{m(k)+2}, x_{m(k)+1}, x_{m(k)+1}) \\
&\quad + G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) \\
&\leq G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + 2G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}) \\
&\quad + G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}) \\
&\quad + G(x_{m(k)+2}, x_{n(k)+1}, x_{n(k)+1})
\end{aligned}$$

$$\begin{aligned}
&\leq G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + 2G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}) \\
&\quad + G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}) + G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) \\
&\quad + G(x_{n(k)}, x_{n(k)+1}, x_{m(k)+2}) \\
&\leq G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + 2G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}) \\
&\quad + G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}) \\
&\quad + 2G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)}, x_{n(k)+1}, x_{m(k)+2}). \quad (23)
\end{aligned}$$

On taking limit as $k \rightarrow +\infty$ in the above inequality and using (17) and (22), we obtain

$$\lim_{k \rightarrow +\infty} G(x_{n(k)}, x_{m(k)+2}, x_{n(k)+1}) = \varepsilon. \quad (24)$$

We have also, by (G5),

$$\begin{aligned}
&G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) \\
&\leq G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}), \\
&G(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}) \\
&\leq G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}). \quad (25)
\end{aligned}$$

On taking limit as $k \rightarrow +\infty$ in the above inequality and using (17) and (G3), we obtain

$$\begin{aligned}
&\lim_{k \rightarrow +\infty} G(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}) \\
&= \lim_{k \rightarrow +\infty} G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}). \quad (26)
\end{aligned}$$

We have also, by (G3) and (G5),

$$\begin{aligned}
&G(x_{n(k)}, x_{m(k)+1}, x_{m(k)+1}) \\
&\leq G(x_{n(k)}, x_{m(k)+1}, x_{n(k)+1}) \\
&\leq G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}), \\
&G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq G(x_{m(k)}, x_{n(k)}, x_{m(k)+1}). \quad (27)
\end{aligned}$$

Now, on taking limit as $k \rightarrow +\infty$ in (27), and using (17), (20), and (26), we obtain

$$\begin{aligned}
&\lim_{k \rightarrow +\infty} G(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}) \\
&= \lim_{k \rightarrow +\infty} G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) = \varepsilon. \quad (28)
\end{aligned}$$

Furthermore, by (G3) and (G5), we get

$$\begin{aligned}
G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) &\leq G(x_{m(k)+1}, x_{m(k)+2}, x_{n(k)+1}) \\
&\leq G(x_{m(k)+2}, x_{m(k)+1}, x_{m(k)+1}) \\
&\quad + G(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1}) \\
&\leq 2G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}) \\
&\quad + G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) \\
&\quad + G(x_{n(k)}, x_{m(k)+1}, x_{m(k)+1}) \\
&\leq 2G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}) \\
&\quad + 2G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) \\
&\quad + G(x_{n(k)}, x_{m(k)+1}, x_{m(k)+1}). \quad (29)
\end{aligned}$$

On taking limit as $k \rightarrow +\infty$ in the above inequality, and using (17), (20), (22), and (28), we obtain

$$\lim_{k \rightarrow +\infty} G(x_{m(k)+1}, x_{m(k)+2}, x_{n(k)+1}) = \varepsilon. \quad (30)$$

Now, we have

$$\begin{aligned}
&\psi(G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1})) \\
&= \psi(G(Tx_{m(k)}, Tx_{n(k)}, Tx_{n(k)})) \\
&\leq \psi(M(x_{m(k)}, x_{n(k)}, x_{n(k)})) \\
&\quad - \phi(M(x_{m(k)}, x_{n(k)}, x_{n(k)})) \\
&\quad + LN(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\
&\leq \psi(\max\{G(x_{m(k)}, Tx_{m(k)}, x_{n(k)}), \\
&\quad G(x_{n(k)}, T^2x_{m(k)}, Tx_{n(k)}), \\
&\quad G(Tx_{m(k)}, T^2x_{m(k)}, Tx_{n(k)}), \\
&\quad G(x_{m(k)}, Tx_{m(k)}, x_{n(k)}), \\
&\quad G(x_{n(k)}, T^2x_{m(k)}, Tx_{n(k)}), \\
&\quad G(Tx_{m(k)}, T^2x_{m(k)}, Tx_{n(k)}), \\
&\quad G(x_{m(k)}, x_{n(k)}, x_{n(k)})\})
\end{aligned}$$

$$\begin{aligned}
 & -\phi(\max\{G(x_{m(k)}, Tx_{m(k)}, x_{n(k)}), \\
 & \quad G(x_{n(k)}, T^2x_{m(k)}, Tx_{n(k)}), \\
 & \quad G(Tx_{m(k)}, T^2x_{m(k)}, Tx_{n(k)}), \\
 & \quad G(x_{m(k)}, Tx_{m(k)}, x_{n(k)}), \\
 & \quad G(x_{n(k)}, T^2x_{m(k)}, Tx_{n(k)}), \\
 & \quad G(Tx_{m(k)}, T^2x_{m(k)}, Tx_{n(k)}), \\
 & \quad G(x_{m(k)}, x_{n(k)}, x_{n(k)})\}) \\
 & + L(\min\{G(x_{m(k)}, Tx_{m(k)}, Tx_{m(k)}), \\
 & \quad G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)}), \\
 & \quad G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)}), \\
 & \quad G(x_{n(k)}, Tx_{m(k)}, Tx_{m(k)}), \\
 & \quad G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)})\}) \\
 & \leq \psi(\max\{G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}), \\
 & \quad G(x_{n(k)}, x_{m(k)+2}, x_{n(k)+1}), \\
 & \quad G(x_{m(k)+1}, x_{m(k)+2}, x_{n(k)+1}), \\
 & \quad G(x_{m(k)}, x_{n(k)}, x_{n(k)})\}) \\
 & -\phi(\max\{G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}), \\
 & \quad G(x_{n(k)}, x_{m(k)+2}, x_{n(k)+1}), \\
 & \quad G(x_{m(k)+1}, x_{m(k)+2}, x_{n(k)+1}), \\
 & \quad G(x_{m(k)}, x_{n(k)}, x_{n(k)})\}) \\
 & + L(\min\{G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}), \\
 & \quad G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}), \\
 & \quad G(x_{n(k)}, x_{m(k)+1}, x_{m(k)+1})\}).
 \end{aligned} \tag{31}$$

Letting $k \rightarrow +\infty$, and using (17), (20), (22), (24), (28), and (30), and the properties of ϕ and ψ , we have

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon). \tag{32}$$

Thus $\phi(\varepsilon) = 0$ and hence $\varepsilon = 0$, a contradiction. Thus $\{x_n\}$ is a G -Cauchy sequence in X .

Now, since (X, G) is G -complete, there are $x \in X$ such that $\{x_n\}$ is G -convergent to x ; that is

$$\lim_{n \rightarrow +\infty} G(x_n, x_n, x) = \lim_{n \rightarrow +\infty} G(x_n, x, x) = 0. \tag{33}$$

By (6), we get

$$\begin{aligned}
 & \psi(G(x_{n+1}, x_{n+1}, Tx)) \\
 & = \psi(G(Tx_n, Tx_n, Tx)) \\
 & \leq \psi(M(x_n, x_n, x)) - \phi(M(x_n, x_n, x)) + LN(x_n, x_n, x),
 \end{aligned} \tag{34}$$

where

$$\begin{aligned}
 M(x_n, x_n, x) & = \max\{G(x_n, Tx_n, x_n), G(x_n, T^2x_n, Tx_n), \\
 & \quad G(Tx_n, T^2x_n, Tx_n), G(x_n, Tx_n, x), \\
 & \quad G(x, T^2x_n, Tx), \\
 & \quad G(Tx_n, T^2x_n, Tx), G(x_n, x_n, x)\} \\
 & = \max\{G(x_n, x_{n+1}, x_n), G(x_n, x_{n+2}, x_{n+1}), \\
 & \quad G(x_{n+1}, x_{n+2}, x_{n+1}), G(x_n, x_{n+1}, x), \\
 & \quad G(x, x_{n+2}, Tx), G(x_{n+1}, x_{n+2}, Tx), \\
 & \quad G(x_n, x_n, x)\}, \\
 N(x_n, x_n, x) & = \min\{G(x_n, Tx_n, Tx_n), G(x_n, Tx_n, Tx_n), \\
 & \quad G(x, Tx, Tx), G(x, Tx_n, Tx_n), \\
 & \quad G(x_n, Tx, Tx)\} \\
 & = \min\{G(x_n, x_{n+1}, x_{n+1}), G(x, Tx, Tx), \\
 & \quad G(x, x_{n+1}, x_{n+1}), G(x_n, Tx, Tx)\}.
 \end{aligned} \tag{35}$$

Letting $n \rightarrow +\infty$ in (35), we get

$$\begin{aligned}
 \lim_{n \rightarrow +\infty} M(x_n, x_n, x) & = G(x, x, Tx), \\
 \lim_{n \rightarrow +\infty} N(x_n, x_n, x) & = 0.
 \end{aligned} \tag{36}$$

On letting $n \rightarrow +\infty$ in (34), and using the properties of ψ and ϕ and (36), we obtain

$$\psi(G(x, x, Tx)) \leq \psi(G(x, x, Tx)) - \phi(G(x, x, Tx)). \tag{37}$$

Therefore $G(x, x, Tx) = 0$ and hence $x = Tx$. Thus x is a fixed point of T .

Now our purpose is to check that such point is unique. Suppose that there are two fixed points of T ; say $x, y \in X$ such that $x \neq y$. By (6), we have

$$\begin{aligned}
 & \psi(G(x, x, y)) \\
 & = \psi(G(Tx, Tx, Ty)) \\
 & \leq \psi(M(x, x, y)) - \phi(M(x, x, y)) + LN(x, x, y),
 \end{aligned} \tag{38}$$

where

$$\begin{aligned}
 M(x, x, y) &= \max \{G(x, Tx, x), G(x, T^2x, Tx), \\
 &\quad G(Tx, T^2x, Tx), G(x, Tx, y), \\
 &\quad G(y, T^2x, Ty), \\
 &\quad G(Tx, T^2x, Ty), G(x, x, y)\} \\
 &= \max \{G(x, x, y), G(y, x, y)\}, \\
 N(x, x, y) &= \min \{G(x, Tx, Tx), G(x, Tx, Tx), \\
 &\quad G(y, Ty, Ty), G(y, Tx, Tx), \\
 &\quad G(x, Ty, Ty)\} = 0.
 \end{aligned} \quad (39)$$

Similarly, we can prove that

$$\begin{aligned}
 \psi(G(x, y, y)) &= \psi(G(Tx, Ty, Ty)) \\
 &\leq \psi(M(x, y, y)) - \phi(M(x, y, y)) + LN(x, y, y),
 \end{aligned} \quad (40)$$

where

$$\begin{aligned}
 M(x, y, y) &= \max \{G(x, x, y), G(y, x, y)\}, \\
 N(x, y, y) &= 0.
 \end{aligned} \quad (41)$$

If

$$\max \{G(x, x, y), G(y, x, y)\} = G(x, x, y). \quad (42)$$

By (38) and (39), we get

$$\psi(G(x, x, y)) \leq \psi(G(x, x, y)) - \phi(G(x, x, y)), \quad (43)$$

a contradiction. Then

$$\max \{G(x, x, y), G(y, x, y)\} = G(y, x, y). \quad (44)$$

By (40) and (41), we get

$$\psi(G(x, y, y)) \leq \psi(G(x, y, y)) - \phi(G(x, y, y)), \quad (45)$$

a contradiction. Thus, $x = y$, and hence the fixed point of T is unique. $\square$

As consequence of Theorem 9, we present the following corollaries.

Corollary 10. Let (X, G) be a complete G -metric space. Let $T : X \rightarrow X$ be a self mapping. Suppose there exist $k \in [0, 1)$ and $L \geq 0$ such that for all $x, y, z \in X$

$$G(Tx, Ty, Tz) \leq kM(x, y, z) + LN(x, y, z). \quad (46)$$

Then T has a unique fixed point; say $u \in X$.

Proof. It suffices to take $\psi(t) = t$ and $\phi(t) = (1 - k)t$ in Theorem 9. $\square$

Corollary 11. Let (X, G) be a complete G -metric space. Let $T : X \rightarrow X$ be a self mapping. Suppose there exist $k \in [0, 1)$ and $L \geq 0$ such that for all $x, y, z \in X$

$$G(Tx, Ty, Tz) \leq kG(x, y, z) + LN(x, y, z). \quad (47)$$

Then T has a unique fixed point; say $u \in X$.

Proof. Note that $G(x, y, z) \leq M(x, y, z)$ in Corollary 10. $\square$

Remark 12. Corollary 11 is a generalization of Mustafa's result [2].

Our second main result is given as follows.

Theorem 13. Let (X, G) be a complete G -metric space. Let $T : X \rightarrow X$ be a self mapping. Suppose there exist $\psi \in \Psi$, $\phi \in \Phi$ and $L \geq 0$ such that for all $x, y \in X$:

$$\begin{aligned}
 \psi(G(Tx, Ty, T^2x)) &\leq \psi(M^*(x, y, x)) - \phi(M^*(x, y, x)) + LN^*(x, y, x),
 \end{aligned} \quad (48)$$

where

$$\begin{aligned}
 M^*(x, y, x) &= \max \{G(x, Tx, y), G(y, T^2x, Ty), G(Tx, T^2x, Ty), \\
 &\quad G(x, Tx, Tx), G(Tx, T^2x, T^2x), G(x, y, Tx)\}, \\
 N^*(x, y, x) &= \min \{G(x, Tx, Tx), G(y, Ty, Ty), G(y, Tx, Tx)\}
 \end{aligned} \quad (49)$$

for all $x, y \in X$. Then T has a unique fixed point; say $u \in X$.

Proof. Let x_0 be an arbitrary point in X and define a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n, \quad \text{for any } n \in \mathbb{N}. \quad (50)$$

If for some $n \in \mathbb{N}$, $x_{n+1} = x_n$, then $x_n = Tx_n$ and the proof is completed. Thus, we may assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. By (48), we have

$$\begin{aligned}
 \psi(G(x_n, x_{n+1}, x_{n+1})) &= \psi(G(Tx_{n-1}, Tx_n, T^2x_{n-1})) \\
 &\leq \psi(M^*(x_{n-1}, x_n, x_{n-1})) - \phi(M^*(x_{n-1}, x_n, x_{n-1})) \\
 &\quad + LN^*(x_{n-1}, x_n, x_{n-1}),
 \end{aligned} \quad (51)$$

where

$$\begin{aligned}
 & M^*(x_{n-1}, x_n, x_{n-1}) \\
 &= \max \{G(x_{n-1}, Tx_{n-1}, x_n), G(x_n, T^2x_{n-1}, Tx_n), \\
 &\quad G(Tx_{n-1}, T^2x_{n-1}, Tx_n), G(x_{n-1}, Tx_{n-1}, Tx_n), \\
 &\quad G(Tx_{n-1}, T^2x_{n-1}, T^2x_{n-1}), G(x_{n-1}, x_n, Tx_{n-1})\} \\
 &= \max \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\
 &\quad G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n), \\
 &\quad G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n)\} \\
 &= \max \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}, \\
 & N^*(x_{n-1}, x_n, x_{n-1}) \\
 &= \min \{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_n, Tx_n), \\
 &\quad G(x_n, Tx_{n-1}, Tx_{n-1})\} \\
 &= \min \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n)\} \\
 &= 0.
 \end{aligned} \tag{52}$$

From (51) and (52), we get

$$\begin{aligned}
 & \psi(G(x_n, x_{n+1}, x_{n+1})) \\
 &\leq \psi(\max \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}) \\
 &\quad - \phi(\max \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}).
 \end{aligned} \tag{53}$$

If

$$\begin{aligned}
 & \max \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} \\
 &= G(x_n, x_{n+1}, x_{n+1}),
 \end{aligned} \tag{54}$$

then by (53) we have

$$\begin{aligned}
 & \psi(G(x_n, x_{n+1}, x_{n+1})) \\
 &\leq \psi(G(x_n, x_{n+1}, x_{n+1})) - \phi(G(x_n, x_{n+1}, x_{n+1})).
 \end{aligned} \tag{55}$$

Thus $\psi(G(x_n, x_{n+1}, x_{n+1})) = 0$ and hence $G(x_n, x_{n+1}, x_{n+1}) = 0$. Therefore, $x_n = x_{n+1}$, which is a contradiction. So,

$$\max \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} = G(x_{n-1}, x_n, x_n). \tag{56}$$

Therefore, we get

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n) \quad \forall n \in \mathbb{N}, \tag{57}$$

and (53) becomes

$$\begin{aligned}
 & \psi(G(x_n, x_{n+1}, x_{n+1})) \leq \psi(G(x_{n-1}, x_n, x_n)) \\
 &\quad - \phi(G(x_{n-1}, x_n, x_n)) \\
 &\quad \forall n \in \mathbb{N}.
 \end{aligned} \tag{58}$$

Thus, by (57), the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ is monotone nonincreasing. It follows that $G(x_n, x_{n+1}, x_{n+1}) \rightarrow g$ as $n \rightarrow +\infty$ for some $g \geq 0$. Next we claim that $g = 0$. On taking limit as $n \rightarrow +\infty$ in (58), we obtain

$$\psi(g) \leq \psi(g) - \phi(g). \tag{59}$$

Hence $\phi(g) = 0$ and we get $g = 0$. Then

$$\lim_{n \rightarrow +\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{60}$$

Next, we show that $\{x_n\}$ is a G-Cauchy sequence. On contrary, assume that $\{x_n\}$ is not a G-Cauchy sequence. Then, there is an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}, \{x_{m(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) \geq k$ such that

$$\begin{aligned}
 & G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geq \varepsilon, \\
 & G(x_{m(k)-1}, x_{n(k)}, x_{n(k)}) < \varepsilon.
 \end{aligned} \tag{61}$$

Using (61) and (G5), we have

$$\begin{aligned}
 & \varepsilon \leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\
 &\leq G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, x_{n(k)}, x_{n(k)}) \\
 &< \varepsilon + 2G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}).
 \end{aligned} \tag{62}$$

Taking limit as $k \rightarrow +\infty$ and using (60), we have

$$\lim_{k \rightarrow +\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{63}$$

By (G5), and using (60) and (63), we obtain, as stated in the proof of Theorem 9,

$$\lim_{k \rightarrow +\infty} G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{64}$$

Again, by (G5), and using (60) and (64), similarly as in the proof of Theorem 9, we obtain

$$\lim_{k \rightarrow +\infty} G(x_{n(k)}, x_{m(k)+2}, x_{n(k)+1}) = \varepsilon. \tag{65}$$

Furthermore, by (G3) and (G5), we get

$$\begin{aligned}
 & G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) \\
 &\leq G(x_{m(k)+1}, x_{m(k)+2}, x_{n(k)+1}) \\
 &\leq G(x_{m(k)+2}, x_{m(k)+1}, x_{m(k)+1}) \\
 &\quad + G(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1}) \\
 &\leq 2G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}) \\
 &\quad + G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{m(k)+1}, x_{m(k)+1}) \\
 &\leq 2G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}) \\
 &\quad + 2G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) \\
 &\quad + G(x_{n(k)}, x_{m(k)+1}, x_{m(k)+1}).
 \end{aligned} \tag{66}$$

By the proof of Theorem 9, we have stated that

$$\lim_{k \rightarrow +\infty} G(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}) = \varepsilon. \quad (67)$$

So, on taking limit as $k \rightarrow +\infty$ in (66), and using (60), (64), and (67), we obtain

$$\lim_{k \rightarrow +\infty} G(x_{m(k)+1}, x_{m(k)+2}, x_{n(k)+1}) = \varepsilon. \quad (68)$$

Furthermore, by (G3) and (G5), we get

$$\begin{aligned} & G(x_{n(k)-1}, x_{n(k)}, x_{m(k)+1}) \\ & \leq G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)}, x_{m(k)+1}) \\ & \leq G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) \\ & \quad + G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ & \leq G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + 2G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) \\ & \quad + G(x_{m(k)}, x_{n(k)}, x_{n(k)}), \\ & G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ & \leq G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{n(k)}, x_{n(k)}) \\ & \leq G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) \\ & \quad + G(x_{n(k)-1}, x_{n(k)}, x_{m(k)+1}). \end{aligned} \quad (69)$$

On taking limit as $k \rightarrow +\infty$ in above inequalities, and using (60) and (63), we have

$$\lim_{k \rightarrow +\infty} G(x_{n(k)-1}, x_{n(k)}, x_{m(k)+1}) = \varepsilon. \quad (70)$$

Also, by (G3) and (G5), we get

$$\begin{aligned} & G(x_{m(k)+1}, x_{n(k)}, x_{n(k)}) \\ & \leq G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ & \leq 2G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)}, x_{n(k)}, x_{n(k)}), \\ & G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ & \leq G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{n(k)}, x_{n(k)}). \end{aligned} \quad (71)$$

On taking limit as $k \rightarrow +\infty$ in above inequalities, and using (60), (63), and (70), we obtain

$$\lim_{k \rightarrow +\infty} G(x_{m(k)+1}, x_{n(k)}, x_{n(k)}) = \varepsilon. \quad (72)$$

Now, we have

$$\begin{aligned} & \psi(G(x_{n(k)}, x_{m(k)+2}, x_{n(k)+1})) \\ & = \psi(G(Tx_{n(k)-1}, Tx_{m(k)+1}, T^2x_{n(k)-1})) \end{aligned}$$

$$\begin{aligned} & \leq \psi(M^*(x_{n(k)-1}, x_{m(k)+1}, x_{n(k)-1})) \\ & \quad - \phi(M^*(x_{n(k)-1}, x_{m(k)+1}, x_{n(k)-1})) \\ & \quad + LN^*(x_{n(k)-1}, x_{m(k)+1}, x_{n(k)-1}) \\ & \leq \psi(\max\{G(x_{n(k)-1}, Tx_{n(k)-1}, x_{m(k)+1}), \\ & \quad G(x_{m(k)+1}, T^2x_{n(k)-1}, Tx_{m(k)+1}), \\ & \quad G(Tx_{n(k)-1}, T^2x_{n(k)-1}, Tx_{m(k)+1}), \\ & \quad G(x_{n(k)-1}, Tx_{n(k)-1}, Tx_{n(k)-1}), \\ & \quad G(Tx_{n(k)-1}, T^2x_{n(k)-1}, T^2x_{n(k)-1}), \\ & \quad G(x_{n(k)-1}, x_{m(k)+1}, Tx_{n(k)-1})\}) \\ & \quad - \phi(\max\{G(x_{n(k)-1}, Tx_{n(k)-1}, x_{m(k)+1}), \\ & \quad G(x_{m(k)+1}, T^2x_{n(k)-1}, Tx_{m(k)+1}), \\ & \quad G(Tx_{n(k)-1}, T^2x_{n(k)-1}, Tx_{m(k)+1}), \\ & \quad G(x_{n(k)-1}, Tx_{n(k)-1}, Tx_{n(k)-1}), \\ & \quad G(Tx_{n(k)-1}, T^2x_{n(k)-1}, T^2x_{n(k)-1}), \\ & \quad G(x_{n(k)-1}, x_{m(k)+1}, Tx_{n(k)-1})\}) \\ & \quad + L \min\{G(x_{n(k)-1}, Tx_{n(k)-1}, Tx_{n(k)-1}), \\ & \quad G(x_{m(k)+1}, Tx_{m(k)+1}, Tx_{m(k)+1}), \\ & \quad G(x_{m(k)+1}, Tx_{n(k)-1}, Tx_{n(k)-1})\} \\ & \leq \psi(\max\{G(x_{n(k)-1}, x_{n(k)}, x_{m(k)+1}), \\ & \quad G(x_{m(k)+1}, x_{n(k)+1}, x_{m(k)+2}), \\ & \quad G(x_{n(k)}, x_{n(k)+1}, x_{m(k)+2}), \\ & \quad G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}), \\ & \quad G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1})\}) \\ & \quad - \phi(\max\{G(x_{n(k)-1}, x_{n(k)}, x_{m(k)+1}), \\ & \quad G(x_{m(k)+1}, x_{n(k)+1}, x_{m(k)+2}), \\ & \quad G(x_{n(k)}, x_{n(k)+1}, x_{m(k)+2}), \\ & \quad G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}), \\ & \quad G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1})\}) \\ & \quad + L \min\{G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}), \\ & \quad G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}), \\ & \quad G(x_{m(k)+1}, x_{n(k)}, x_{n(k)})\}. \end{aligned} \quad (73)$$

Letting $k \rightarrow +\infty$, and using (60), (65), (68), (70), and (72), and the properties of ϕ and ψ , we have

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon). \quad (74)$$

Thus $\phi(\varepsilon) = 0$ and hence $\varepsilon = 0$, a contradiction. Thus $\{x_n\}$ is a G -Cauchy sequence in X .

Now, since (X, G) is G -complete, there is $x \in X$ such that $\{x_n\}$ is G -convergent to x ; that is

$$\lim_{n \rightarrow +\infty} G(x_n, x_n, x) = \lim_{n \rightarrow +\infty} G(x_n, x, x) = 0. \quad (75)$$

By (48), we get

$$\begin{aligned} \psi(G(x_n, Tx, x_{n+1})) &= \psi(G(Tx_{n-1}, Tx, T^2x_{n-1})) \\ &\leq \psi(M^*(x_{n-1}, x, x_{n-1})) \\ &\quad - \phi(M^*(x_{n-1}, x, x_{n-1})) \\ &\quad + LN^*(x_{n-1}, x, x_{n-1}), \end{aligned} \quad (76)$$

where

$$\begin{aligned} M^*(x_{n-1}, x, x_{n-1}) &= \max \{G(x_{n-1}, Tx_{n-1}, x), G(x, T^2x_{n-1}, Tx), \\ &\quad G(Tx_{n-1}, T^2x_{n-1}, Tx), \\ &\quad G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), \\ &\quad G(Tx_{n-1}, T^2x_{n-1}, T^2x_{n-1}), \\ &\quad G(x_{n-1}, x, Tx_{n-1})\} \\ &= \max \{G(x_{n-1}, x_n, x), G(x, x_{n+1}, Tx), \\ &\quad G(x_n, x_{n+1}, Tx), G(x_{n-1}, x_n, x_n), \\ &\quad G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x, x_n)\}, \quad (77) \\ N^*(x_{n-1}, x, x_{n-1}) &= \min \{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), \\ &\quad G(x, Tx, Tx), G(x, Tx_{n-1}, Tx_{n-1})\} \\ &= \min \{G(x_{n-1}, x_n, x_n), \\ &\quad G(x, Tx, Tx), G(x, x_n, x_n)\}. \end{aligned}$$

Letting $n \rightarrow +\infty$ in (77), we get

$$\begin{aligned} \lim_{n \rightarrow +\infty} M^*(x_{n-1}, x, x_{n-1}) &= G(x, x, Tx), \\ \lim_{n \rightarrow +\infty} N^*(x_{n-1}, x, x_{n-1}) &= 0. \end{aligned} \quad (78)$$

On letting $n \rightarrow +\infty$ in (76), and using the properties of ψ and ϕ and (78), we obtain

$$\psi(G(x, x, Tx)) \leq \psi(G(x, x, Tx)) - \phi(G(x, x, Tx)). \quad (79)$$

Therefore $G(x, x, Tx) = 0$ and hence $x = Tx$. Thus x is a fixed point of T .

Now our purpose is to check that such point is unique. Suppose that there are two fixed points of T ; say $x, y \in X$ such that $x \neq y$. By (48), we have

$$\begin{aligned} \psi(G(x, y, x)) &= \psi(G(Tx, Ty, T^2x)) \\ &\leq \psi(M^*(x, y, x)) - \phi(M^*(x, y, x)) \\ &\quad + LN^*(x, y, x), \end{aligned} \quad (80)$$

where

$$\begin{aligned} M^*(x, y, x) &= \max \{G(x, Tx, y), G(y, T^2x, Ty), G(Tx, T^2x, Ty), \\ &\quad G(x, Tx, Tx), G(Tx, T^2x, T^2x), G(x, y, Tx)\} \\ &= \max \{G(x, x, y), G(y, x, y)\}, \\ N^*(x, y, x) &= \min \{G(x, Tx, Tx), G(y, Ty, Ty), G(y, Tx, Tx)\} = 0. \end{aligned} \quad (81)$$

Similarly, we can prove that

$$\begin{aligned} \psi(G(y, x, y)) &= \psi(G(Ty, Tx, T^2y)) \\ &\leq \psi(M^*(y, x, y)) - \phi(M^*(y, x, y)) + LN^*(y, x, y), \end{aligned} \quad (82)$$

where

$$\begin{aligned} M^*(y, x, y) &= \max \{G(x, x, y), G(y, x, y)\}, \\ N^*(y, x, y) &= 0. \end{aligned} \quad (83)$$

If

$$\max \{G(x, x, y), G(y, x, y)\} = G(x, x, y). \quad (84)$$

By (80) and (81), we get

$$\psi(G(x, x, y)) \leq \psi(G(x, x, y)) - \phi(G(x, x, y)), \quad (85)$$

a contradiction. Then

$$\max \{G(x, x, y), G(y, x, y)\} = G(y, x, y). \quad (86)$$

By (82) and (83), we get

$$\psi(G(x, y, y)) \leq \psi(G(x, y, y)) - \phi(G(x, y, y)), \quad (87)$$

a contradiction. Thus, $x = y$, and hence the fixed point of T is unique. $\square$

Finally, we provide the following example.

Example 14. Let $X = [0, 1]$ and $G : X \times X \times X \rightarrow \mathbb{R}$ be defined by

$$G(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise.} \end{cases} \quad (88)$$

Then (X, G) is a G -complete G -metric space. Let $T : X \rightarrow X$ be defined by

$$Tx = \begin{cases} \frac{1}{4}x & \text{if } 0 \leq x < \frac{1}{3}, \\ \frac{1}{8}x^4 & \text{if } \frac{1}{3} \leq x \leq 1. \end{cases} \quad (89)$$

Take $\psi(t) = (1/2)t$ and $\phi(t) = (1/4)t$ for all $t \in [0, +\infty[$. We examine the following cases.

(i) Let $0 \leq x, y < 1/3$. Then

$$\begin{aligned} G(Tx, T^2x, Ty) &= \max\left\{\frac{1}{4}x, \frac{1}{16}x, \frac{1}{4}y\right\} = \frac{1}{4} \max\left\{x, \frac{1}{4}x, y\right\} \\ &\leq \frac{1}{4} \max\{x, y\} = \psi(M^*(x, y, x)) - \phi(M^*(x, y, x)), \end{aligned} \quad (90)$$

where $M^*(x, y, x) = \max\{x, y\}$.

(ii) Let $1/3 \leq x, y < 1$. Then

$$\begin{aligned} G(Tx, T^2x, Ty) &= \max\left\{\frac{1}{8}x^4, \frac{1}{64}x^{16}, \frac{1}{8}y^4\right\} \leq \frac{1}{4} \max\left\{x, \frac{1}{8}x^4, y\right\} \\ &\leq \frac{1}{4} \max\{x, y\} = \psi(M^*(x, y, x)) - \phi(M^*(x, y, x)), \end{aligned} \quad (91)$$

where $M^*(x, y, z) = \max\{x, y\}$.

(iii) Let $0 \leq x < 1/3 \leq y < 1$. Then

$$\begin{aligned} G(Tx, T^2x, Ty) &= \max\left\{\frac{1}{4}x, \frac{1}{16}x, \frac{1}{8}y^4\right\} \leq \frac{1}{4} \max\left\{x, \frac{1}{4}x, y\right\} \\ &\leq \frac{1}{4}y = \psi(M^*(x, y, x)) - \phi(M^*(x, y, x)), \end{aligned} \quad (92)$$

where $M^*(x, y, x) = y$.

(iv) Let $0 \leq y < 1/3 \leq x < 1$. Then

$$\begin{aligned} G(Tx, T^2x, Ty) &= \max\left\{\frac{1}{8}x^4, \frac{1}{64}x^{16}, \frac{1}{4}y\right\} \leq \frac{1}{4} \max\left\{x, \frac{1}{8}x^4, y\right\} \\ &\leq \frac{1}{4}x = \psi(M^*(x, y, x)) - \phi(M^*(x, y, x)), \end{aligned} \quad (93)$$

where $M^*(x, y, x) = x$. Then

$$\begin{aligned} G(Tx, T^2x, Ty) &\leq \psi(M^*(x, y, x)) - \phi(M^*(x, y, x)) \\ &\quad + LN^*(x, y, x) \end{aligned} \quad (94)$$

for all $L \geq 0$. Then the conditions of Theorem 13, hold and T has a unique fixed point. Notice that $u = 0$ is the desired fixed point of T .

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