Research Article

Some Properties of the $q$-Extension of the $p$-Adic Gamma Function

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We study the $q$-extension of the $p$-adic gamma function $\Gamma_{p,q}$. We give a new identity for the $q$-extension of the $p$-adic gamma function $\Gamma_{p,q}$ in the case $p=2$. Also, we derive some properties and new representations of the $q$-extension of the $p$-adic gamma function $\Gamma_{p,q}$ in general case.

1. Introduction

Let $p$ be a prime number and let $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of $p$-adic integers, the field of $p$-adic numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. It is well known that the analogous of the classical gamma function $\Gamma$ in $p$-adic context depends on modifying the factorial function $n!$ [1]. The factorial function $(n!)_p$ in $\mathbb{Q}_p$ is defined as

$$ (n!)_p = \prod_{j<n, (p,j)=1} j. $$

The $p$-adic gamma function $\Gamma_p$ is defined by Morita [2] as the continuous extension to $\mathbb{Z}_p$ of the function $n \to (-1)^n (n!)_p$. That is, $\Gamma_p(x)$ is defined by the formula

$$ \Gamma_p(x) = \lim_{n \to x} (-1)^n \prod_{j<n, (p,j)=1} j $$

for $x \in \mathbb{Z}_p$, where $n$ approaches $x$ through positive integers. The $p$-adic gamma function $\Gamma_p(x)$ had been studied by Diamond [3], Barsky [4], and others. The relationship between some special functions and the $p$-adic gamma function $\Gamma_p(x)$ were investigated by Gross and Koblitz [5], Cohen and Friedman [6], and Shapiro [7].

The $q$-extension of the $p$-adic gamma function $\Gamma_{p,q}(x)$ is defined by Koblitz as follows.

**Definition 1** (see [8]). Let $q \in \mathbb{C}_p$, $|q - 1|_p < 1$, $q \neq 1$. The $q$-extension of the $p$-adic gamma function $\Gamma_{p,q}(x)$ is defined by formula

$$ \Gamma_{p,q}(x) = \lim_{n \to x} (-1)^n \prod_{j<n, (p,j)=1} \frac{1 - q^j}{1 - q} $$

for $x \in \mathbb{Z}_p$, where $n$ approaches $x$ through positive integers. We recall that $\lim_{q \to 1} \Gamma_{p,q} = \Gamma_p$.

The $q$-extension of the $p$-adic gamma function $\Gamma_{p,q}(x)$ was studied by Koblitz [8, 9], Nakazato [10], Kim et al. [11], and Kim [12].

2. Main Results

In the present work, we give a new identity for the $q$-extension of the $p$-adic gamma function $\Gamma_{p,q}(x)$ in special case $p = 2$. Also, we derive some properties and representations for the $q$-extension of the $p$-adic gamma function $\Gamma_{p,q}(x)$.

**Theorem 2.** If $p = 2$, then for all $x \in \mathbb{Z}_2$

$$ \Gamma_{2,q}(x) \Gamma_{2,q}(1-x) = (-1)^{\sigma(x)} \lim_{n \to x} \prod_{j<\mathbb{Z}_2, (2,j)=1} q^j $$

(4)
where $\sigma_1$ is defined by the formula

$$\sigma_1 \left( \sum_{j=0}^{\infty} a_j 2^j \right) = a_1. \quad (5)$$

Proof. Let $p = 2$ and $n \in \mathbb{N}$. From Proposition 3 in [12] we know that

$$\Gamma_{2,q} (n+1) \Gamma_{2,q} (-n) = (-1)^{n+1-\left[\frac{n}{2}\right]} \prod_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{j}. \quad (6)$$

Here, $\left[ \cdot \right]$ is the greatest integer function. Taking $n-1$ in place of $n$, the relation becomes

$$\Gamma_{2,q} (n) \Gamma_{2,q} (1-n) = (-1)^{n-\left[\frac{n}{2}\right]} \prod_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{j}. \quad (7)$$

Now, let $n = a_0 + a_1 2 + a_2 2^2 + \cdots$ in base 2. If $a_0 \neq 0$, then $a_1 = 1$ in base 2 and

$$\left[ \frac{n-1}{2} \right] = \left[ \left( a_0 - 1 + a_1 2 + a_2 2^2 + \cdots \right) \right]_{2} = \left[ a_1 + a_2 2 + \cdots \right] \equiv a_1 \pmod{2}. \quad (8)$$

Thus, we get

$$(-1)^n - \left[\frac{n}{2}\right] = (-1)^n - \left[\frac{n-1}{2}\right] = (-1)^{n-1} \cdot a_0 \equiv (-1)^{n-1} \cdot a_1 \equiv (-1)^{\sigma_1} \cdot a_1. \quad (9)$$

If $a_0 = 0$, then

$$\left[ \frac{n-1}{2} \right] = \left[ \left( -1 + a_1 2 + a_2 2^2 + \cdots \right) \right]_{2} = \left[ \left( 1 + (a_1 - 1) 2 + a_2 2^2 + \cdots \right) \right]_{2} \equiv a_1 - 1 \pmod{2}. \quad (10)$$

Hence,

$$(-1)^n - \left[\frac{n}{2}\right] = (-1)^n - \left[\frac{n-1}{2}\right] = (-1)^{n-1} \cdot a_0 \equiv (-1)^{n-1} \cdot a_1 \equiv (-1)^{\sigma_1} \cdot a_1. \quad (11)$$

Thus, we have

$$\Gamma_{2,q} (n) \Gamma_{2,q} (1-n) = (-1)^{\sigma_1 (n)} \prod_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{j}. \quad (12)$$

and thus, we obtain

$$\Gamma_{2,q} (x) \Gamma_{2,q} (1-x) = (-1)^{\sigma_1 (x)} \lim_{n \to x} \prod_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{j}. \quad (13)$$

We recall that the $q$-factorial $[n; q]!$ is defined in [13] by the formula

$$[n; q]! = [n; q] [n-1; q] \cdots [2; q] [1; q] \quad (14)$$

for $n \geq 1$, where

$$[x; q] = \frac{1 - q^x}{1 - q}. \quad (15)$$

Note that for $n = 0$, we can define $[0; q]! = 1$.

We use the following theorem to prove our results.

**Theorem 3** (see [12]). Let $n \in \mathbb{N}$. Then,

$$\Gamma_{p,q} (n+1) = (-1)^{n+1} \frac{[n; q]!}{[p; q]^{[n/p]} \cdot \left[ \left[ \frac{n}{p} \right] \right] \Gamma_{p,q} (p^n)}. \quad (16)$$

where $\left[ \cdot \right]$ is the greatest integer function. In particular,

$$[p^n - 1; q]! = (-1)^p \frac{[p; q]^{p^n-1}}{[p^n - 1; q] \Gamma_{p,q} (p^n)}. \quad (17)$$

**Theorem 4.** Let $n \in \mathbb{N}$ and let $s_n$ be the sum of the digits of $n = \sum_{j=0}^{\infty} a_j p^j$ ($a_j \neq 0$) in base $p$. Then

$$n! = \prod_{j=1}^{\left\lfloor \frac{n}{p} \right\rfloor} (\frac{n}{p})^{[s_j]} \Gamma_{p,q} (p^n) + 1. \quad (18)$$

By taking $[n/p^0], [n/p^1], \ldots, [n/p^n]$ instead of $n$, respectively, we get the relations

$$\left[ \left[ \frac{n}{p^i} \right] \right] \cdot [q]! = (-1)^{[n/p^i]} \left[ \left[ \frac{n}{p^i} \right] \right] [p; q]^{[n/p^i]} \times \left[ \left[ \frac{n}{p^i} \right] \right] \cdot [q]! \Gamma_{p,q} \left( \left[ \frac{n}{p^i} \right] + 1 \right). \quad (19)$$

$$\left[ \left[ \frac{n}{p^i} \right] \right] \cdot [q]! = (-1)^{[n/p^i]} \left[ \left[ \frac{n}{p^i} \right] \right] [p; q]^{[n/p^i]} \times \left[ \left[ \frac{n}{p^i} \right] \right] \cdot [q]! \Gamma_{p,q} \left( \left[ \frac{n}{p^i} \right] + 1 \right). \quad (20)$$

$$\vdots$$

$$\left[ \left[ \frac{n}{p^n} \right] \right] \cdot [q]! = (-1)^{[n/p^n]} \left[ \left[ \frac{n}{p^n} \right] \right] [p; q]^{[n/p^n]} \times \left[ \left[ \frac{n}{p^n} \right] \right] \cdot [q]! \Gamma_{p,q} \left( \left[ \frac{n}{p^n} \right] + 1 \right). \quad (21)$$
By multiplying of the equalities above, we can easily obtain
\[
\left[\frac{n}{p^j}\right]! = (-1)^{n/p^j}(\left[\frac{n}{p^j}\right] + 1)
\times \prod_{j=0}^{s-1} \left[\frac{n/p^{j+1}}{[n/p^j]}\right] ! \prod_{j=0}^{s-1} \Gamma_{p,q} \left[\frac{n}{p^j}\right] + 1 
\times \prod_{j=0}^{s} \Gamma_{p,q} \left(\frac{n}{p^j}\right) + 1
\]

Therefore, we get the relation (a)
\[
\left[\frac{n}{p^j}\right]! = (-1)^{n-1-s}\left(-\left[\frac{n}{p^j}\right] + \left[\frac{n}{p^j}\right] + 1\right)
\times \prod_{j=0}^{s-1} \left[\frac{n/p^{j+1}}{[n/p^j]}\right] ! \prod_{j=0}^{s-1} \Gamma_{p,q} \left[\frac{n}{p^j}\right] + 1 
\times \prod_{j=0}^{s} \Gamma_{p,q} \left(\frac{n}{p^j}\right) + 1
\]

Therefore, we get the relation (b)
\[
\left[\frac{n}{p^j}\right]! = (-1)^{n-1-s}\left(-\left[\frac{n}{p^j}\right] + \left[\frac{n}{p^j}\right] + 1\right)
\times \prod_{j=0}^{s-1} \left[\frac{n/p^{j+1}}{[n/p^j]}\right] ! \prod_{j=0}^{s-1} \Gamma_{p,q} \left[\frac{n}{p^j}\right] + 1 
\times \prod_{j=0}^{s} \Gamma_{p,q} \left(\frac{n}{p^j}\right) + 1
\]

**Theorem 5.** Let \( n \in \mathbb{N} \) and let \( n = \sum_{j=0}^{s} a_j p^j \) (\( a_s \neq 0 \)). Then
\[
[p^n - 1; q]! = (-1)^p\left(-\left[\frac{n}{p^j}\right]!\right)\Gamma_{p,q}(p^j)
\times \prod_{j=0}^{n-1} \left[\frac{p^j - 1; q}{p^j - 1; q}\right] ! \prod_{j=0}^{n} \Gamma_{p,q} \left(p^j\right).
\]

**Proof.** From Theorem 3 it follows that
\[
\left[\frac{n}{p^j}\right]! = (-1)^{n/p^j-1}\left[\frac{n}{p^j}\right] ! \prod_{j=0}^{n-1} \Gamma_{p,q} \left[\frac{n}{p^j}\right] + 1
\times \prod_{j=0}^{s} \Gamma_{p,q} \left(\frac{n}{p^j}\right) + 1
\]

Taking of \( 0, 1, \ldots, n \) instead of \( j \), respectively, we have the equalities
\[
[p^0 - 1; q]! = 1 = (-1)^0\Gamma_{p,q}(p^0),
\]
\[
[p^1 - 1; q]! = (-1)^1\left[\frac{n}{p^j}\right] ! \prod_{j=0}^{n-1} \Gamma_{p,q} \left[\frac{n}{p^j}\right] + 1
\times \prod_{j=0}^{s} \Gamma_{p,q} \left(\frac{n}{p^j}\right) + 1
\]

By multiplying of the equalities above, we can easily obtain
\[
[p^n - 1; q]! = (-1)^n\left[\frac{n}{p^j}\right] ! \prod_{j=0}^{n-1} \Gamma_{p,q} \left[\frac{n}{p^j}\right] + 1
\times \prod_{j=0}^{s} \Gamma_{p,q} \left(\frac{n}{p^j}\right) + 1
\]

Thus,
\[
[p^n - 1; q]! = (-1)^n\left(-\left[\frac{n}{p^j}\right]!\right)\Gamma_{p,q}(p^j)
\times \prod_{j=0}^{n-1} \left[\frac{p^j - 1; q}{p^j - 1; q}\right] ! \prod_{j=0}^{n} \Gamma_{p,q} \left(p^j\right).
\]

**Lemma 6.** Let \( n \in \mathbb{Z}^+ \), \( n = \sum_{j=0}^{s} a_j p^j \) (\( a_s \neq 0 \)), and let \( p \) be a prime number. Then, for \( j = 0, 1, \ldots, s \)
\[
\left[\frac{n}{p^j}\right]! = \prod_{k=1}^{n/p^j} \frac{1}{1 - q^k}
\]

\( 0 \leq k \leq s \).
Proof. For \( j = 0 \)
\[
\frac{[n;q]!}{[p;q]^n [n;q;p]!} = \frac{[1;q] [2;q] \cdots [n,q]}{[p;q]^n [1;q] [2;q;p] \cdots [n,q;p]}
\]
\[
= \left( \frac{1 - q - q^2 \cdots 1 - q^n}{1 - q} \right)
\times \left( \left( \frac{1 - q^p}{1 - q} \right)^{n - 1} \frac{1 - q^{np}}{1 - q^p} \right)
\]
\[
= \left( \frac{1 - q - q^2 \cdots 1 - q^n}{1 - q} \right)
\times \left( \frac{1 - q^p}{1 - q} \left( 1 - q^{2p} \right) \cdots \left( 1 - q^{np} \right) \right)^{n-1}
\]
\[
= \left( \frac{1 - q}{1 - q^p} \right) \left( 1 - q^{2p} \right) \cdots \left( 1 - q^{np} \right).
\]
(29)

For \( 1 \leq j \leq s \) it follows that
\[
\frac{[n/p^j];q!}{[p;q]^{[n/p^j]} [n/p^j];q;p!} = \frac{[1;q] [2;q] \cdots [n/p^j];q}{[p;q]^{[n/p^j]} [1;q] [2;q;p] \cdots [n/p^j];q;p}\]
\[
= \left( \frac{1 - q - q^2 \cdots 1 - q^{[n/p^j]}}{1 - q} \right)
\times \left( \left( \frac{1 - q^p}{1 - q} \right)^{[n/p^j] - 1} \frac{1 - q^{[n/p^j]p}}{1 - q^p} \right)
\]
\[
= \left( \frac{1 - q}{1 - q^p} \right) \left( 1 - q^{2p} \right) \cdots \left( 1 - q^{[n/p^j]p} \right).
\]
(30)

Then, we obtain
\[
\frac{[n/p^j];q!}{[p;q]^{[n/p^j]} [n/p^j];q;p!} = \prod_{k=1}^{[n/p^j]} \frac{1 - q^{kp}}{1 - q^{kp}}.
\]
(31)

**Theorem 7.** Let \( n \in \mathbb{N} \) and let \( s_n \) be the sum of the digits of \( n = \sum_{j=0}^{\ell} a_j p^j \) (\( a_j \neq 0 \)) in base \( p \). Then
\[
[n;q]! = (-1)^{(n-s_n)(p-1)} + \prod_{k=1}^{[n/p^j]} \left( 1 - q^{kp} \right)
\]
\[
\prod_{k=1}^{[n/p^j]} \left( 1 - q^{kp} \right) \prod_{j=0}^{s_n} \Gamma_{p,q} \left( \frac{n}{p^j} \right) + 1).
\]
(32)

**Proof.** This theorem can be proved by using Theorem 4 and Lemma 6.

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**References**


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