Research Article

A Generalization of Poly-Cauchy Numbers and Their Properties

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In Komatsu's work (2013), the concept of poly-Cauchy numbers is introduced as an analogue of that of poly-Bernoulli numbers. Both numbers are extensions of classical Cauchy numbers and Bernoulli numbers, respectively. There are several generalizations of poly-Cauchy numbers, including poly-Cauchy numbers with a q parameter and shifted poly-Cauchy numbers. In this paper, we give a further generalization of poly-Cauchy numbers and investigate several arithmetical properties. We also give the corresponding generalized poly-Bernoulli numbers so that both numbers have some relations.

1. Introduction

Let $n \geq 0$, $k \geq 1$ be integers. Poly-Cauchy numbers of the first kind $c_n^{(k)}$ are defined by

$$c_n^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k \text{ times}} (x_1 x_2 \cdots x_k) (x_1 x_2 \cdots x_k - 1) \cdots (x_1 x_2 \cdots x_k - n + 1) dx_1 dx_2 \cdots dx_k \quad (1)$$

[1]. The concept of poly-Cauchy numbers is a generalization of that of the classical Cauchy numbers $c_n = c_n^{(1)}$ defined by

$$c_n = \int_0^1 x(x-1) \cdots (x-n+1) dx \quad (2)$$

(see, e.g., [2, 3]). The generating function of poly-Cauchy numbers ([1], Theorem 2) is given by

$$\text{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}, \quad (3)$$

where

$$\text{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k} \quad (4)$$

is the k th polylogarithm factorial function. An explicit formula for $c_n^{(k)}$ ([1], Theorem 1) is given by

$$c_n^{(k)} = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^{n-m}}{(m+1)^k} \quad (n \geq 0, k \geq 1), \quad (5)$$

where $\begin{bmatrix} n \\ m \end{bmatrix}$ are the (unsigned) Stirling numbers of the first kind, arising as coefficients of the rising factorial

$$x(x+1) \cdots (x+n-1) = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} x^m \quad (6)$$

(see, e.g., [4]). See ([5], A224094–A224101) for the sequences arising from poly-Cauchy numbers.

The concept of poly-Cauchy numbers is an analogue of that of poly-Bernoulli numbers $B_n^{(k)}$ [6] defined by

$$\frac{\text{Li}_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}, \quad (7)$$

where

$$\text{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \quad (8)$$

is the k th polylogarithm function. When $k = 1$, $B_n = B_n^{(1)}$ is the classical Bernoulli number with $B_1^{(1)} = 1/2$, defined by the generating function

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \quad (9)$$

An explicit formula for $B_n^{(k)}$ ([6], Theorem 1) is given by

$$B_n^{(k)} = (-1)^n \sum_{m=0}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \frac{(-1)^m m!}{(m+1)^k} \quad (n \geq 0, k \geq 1), \quad (10)$$

where $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$ are the Stirling numbers of the second kind, determined by

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \frac{1}{m!} \sum_{j=0}^m (-1)^j \binom{m}{j} (m-j)^n \quad (11)$$

(see, e.g., [4]).

There are some kinds of generalizations of poly-Cauchy numbers. One is the *poly-Cauchy number with a q parameter* $c_{n,q}^{(k)}$ [7] defined by

$$c_{n,q}^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_k (x_1 \cdots x_k) (x_1 \cdots x_k - q) \cdots (x_1 \cdots x_k - (n-1)q) dx_1 \cdots dx_k. \quad (12)$$

Another is the *shifted poly-Cauchy number* $c_{n,a}^{(k)}$ [8] defined by

$$c_{n,a}^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_k (x_1 \cdots x_k)^a (x_1 \cdots x_k - 1) \cdots (x_1 \cdots x_k - (n-1)) dx_1 \cdots dx_k. \quad (13)$$

Notice that $c_{n,a}^{(k)}$ can be expressed as

$$c_{n,a}^{(k)} = \sum_{m=0}^n \left[\begin{matrix} n \\ m \end{matrix} \right] \frac{(-1)^{n-m}}{(m+a)^k}. \quad (14)$$

For example, if $n = 5$ and $a = 3$, then

$$\begin{aligned} c_5^{(k)} &= \frac{24}{2^k} - \frac{50}{3^k} + \frac{35}{4^k} - \frac{10}{5^k} + \frac{1}{6^k}, \\ c_{5,3}^{(k)} &= \frac{24}{4^k} - \frac{50}{5^k} + \frac{35}{6^k} - \frac{10}{7^k} + \frac{1}{8^k}. \end{aligned} \quad (15)$$

Therefore, such numbers are *shifted* from the original poly-Cauchy numbers. Remember that the Hurwitz zeta function $\zeta(s, q) = \sum_{n=0}^{\infty} 1/(q+n)^s$ is a generalization of the famous Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ since $\zeta(s) = \zeta(s, 1)$.

In this paper, we give a further generalization of poly-Cauchy numbers, including both kinds of generalizations, and show several combinatorial and characteristic properties. We also give the corresponding poly-Bernoulli numbers so that both numbers have some relations.

2. Definitions and Basic Properties

Let $n \geq 0$, $k \geq 1$ be integers, and let a, q and $l_1, \dots, l_k$ be nonzero real numbers. For simplicity, we write $L = (l_1, \dots, l_k)$ and $\ell = l_1 \cdots l_k$. Define $c_{n,a,q,L}^{(k)}$ by

$$c_{n,a,q,L}^{(k)} = \underbrace{\int_0^{l_1} \cdots \int_0^{l_k}}_k (x_1 \cdots x_k)^a (x_1 \cdots x_k - q) \cdots (x_1 \cdots x_k - (n-1)q) dx_1 \cdots dx_k. \quad (16)$$

Then, $c_{n,a,q,L}^{(k)}$ can be expressed in terms of the Stirling numbers of the first kind $\left[\begin{matrix} n \\ m \end{matrix} \right]$.

Theorem 1. Let a be a positive real number. Then,

$$c_{n,a,q,L}^{(k)} = \sum_{m=0}^n \left[\begin{matrix} n \\ m \end{matrix} \right] \frac{(-q)^{n-m} \ell^{m+a}}{(m+a)^k} \quad (n \geq 0, k \geq 1). \quad (17)$$

Remark 2. If $a = \ell = 1$, then $c_{n,1,q,(1,\dots,1)}^{(k)} = c_{n,q}^{(k)}$ is the poly-Cauchy number with a q parameter ([7], Theorem 1). If $q = \ell = 1$, then $c_{n,a,1,(1,\dots,1)}^{(k)} = c_{n,a}^{(k)}$ is the shifted poly-Cauchy number ([8], Theorem 2).

Proof. By

$$x(x-1) \cdots (x-n+1) = \sum_{m=0}^n \left[\begin{matrix} n \\ m \end{matrix} \right] (-1)^{n-m} x^m, \quad (18)$$

we have

$$\begin{aligned} c_{n,a,q,L}^{(k)} &= \int_0^{l_1} \cdots \int_0^{l_k} \sum_{m=0}^n \left[\begin{matrix} n \\ m \end{matrix} \right] (-1)^{n-m} \\ &\quad \times (x_1 \cdots x_k)^{m+a-1} q^{n-m} dx_1 \cdots dx_k \\ &= \sum_{m=0}^n \left[\begin{matrix} n \\ m \end{matrix} \right] \frac{(-q)^{n-m} \ell^{m+a}}{(m+a)^k}. \end{aligned} \quad (19)$$

□

For an integer k and a positive real number a , define the extended polylogarithm factorial function $\text{Lif}_k(z; a)$ by

$$\text{Lif}_k(z; a) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+a)^k} \quad (20)$$

[8]. When $a = 1$, $\text{Lif}_k(z; 1) = \text{Lif}_k(z)$ is the *polylogarithm factorial function* [1]. The generating function of the number $c_{n,a,q,L}^{(k)}$ ($q \neq 0$) is given by using the extended polylogarithm factorial function $\text{Lif}_k(a; z)$.

Theorem 3. One has

$$\ell^a \text{Lif}_k \left(\frac{\ell \ln(1+qx)}{q}; a \right) = \sum_{n=0}^{\infty} c_{n,a,q,L}^{(k)} \frac{x^n}{n!}. \quad (21)$$

Remark 4. If $a = \ell = 1$, then Theorem 3 is reduced to Theorem 2 in [7]. If $q = \ell = 1$, then Theorem 3 is reduced to Theorem 3 in [8].

Proof. Since

$$\frac{(\ln(1+x))^m}{m!} = (-1)^m \sum_{n=m}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-x)^n}{n!}, \quad (22)$$

by Theorem 1 we have

$$\begin{aligned} \sum_{n=0}^{\infty} c_{n,a,q,L}^{(k)} \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-q)^{n-m} \ell^{m+a}}{(m+a)^k} \frac{x^n}{n!} \\ &= \ell^a \sum_{m=0}^{\infty} \frac{(-q)^{-m} \ell^m}{(m+a)^k} \sum_{n=m}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-qx)^n}{n!} \\ &= \ell^a \sum_{m=0}^{\infty} \frac{(-q)^{-m} \ell^m}{(m+a)^k} (-1)^m \frac{(\ln(1+qx))^m}{m!} \\ &= \ell^a \sum_{m=0}^{\infty} \frac{1}{m!(m+a)^k} \left(\frac{\ell \ln(1+qx)}{q} \right)^m \\ &= \ell^a \text{Lif}_k \left(\frac{\ell \ln(1+qx)}{q}; a \right). \end{aligned} \quad (23)$$

□

The generating function of the number $c_{n,a,q,L}^{(k)}$ can be written in the form of iterated integrals.

Corollary 5. Let a and q be real numbers with $a > 0$ and $q \neq 0$. For $k = 1$, one has

$$\begin{aligned} &\left(\frac{\ell q}{\ln(1+qx)} \right)^a \int_0^x \left(\frac{\ln(1+qx)}{q} \right)^{a-1} (1+qx)^{\ell/q-1} dx \\ &= \sum_{n=0}^{\infty} c_{n,a,q,L}^{(1)} \frac{x^n}{n!}. \end{aligned} \quad (24)$$

For $k > 1$, one has

$$\begin{aligned} &\left(\frac{\ell q}{\ln(1+qx)} \right) \\ &\underbrace{\int_0^x \frac{q}{(1+qx) \ln(1+qx)} \int_0^x \cdots \frac{q}{(1+qx) \ln(1+qx)} \int_0^x}_{k} \\ &\left(\frac{\ln(1+qx)}{q} \right)^{a-1} (1+qx)^{\ell/q-1} \underbrace{dx \cdots dx}_k = \sum_{n=0}^{\infty} c_{n,a,q,L}^{(k)} \frac{x^n}{n!}. \end{aligned} \quad (25)$$

Remark 6. If $a = \ell = 1$, then Corollary 5 is reduced to Corollary 1 in [7]. If $q = \ell = 1$, then Corollary 5 is reduced to Corollary 1 in [8].

Proof. For $k = 1$,

$$\begin{aligned} \text{Lif}_1(z; a) &= \sum_{m=0}^{\infty} \frac{z^m}{m! (m+a)} = \frac{1}{z^a} \sum_{m=0}^{\infty} \frac{z^{m+a}}{m! (m+a)} \\ &= \frac{1}{z^a} \int_0^z \sum_{m=0}^{\infty} \frac{z^{m+a-1}}{m!} = \frac{1}{z^a} \int_0^z z^{a-1} e^z dz \\ &= \frac{1}{z^a} \left((-1)^a (a-1)! \right. \\ &\quad \left. + e^z \sum_{i=0}^{a-1} (-1)^i \frac{(a-1)!}{(a-i-1)!} z^{a-i-1} \right). \end{aligned} \quad (26)$$

Note that the last equation holds only if a is an integer. For $k > 1$, we have

$$\begin{aligned} \text{Lif}_k(z; a) &= \frac{1}{z^a} \sum_{m=0}^{\infty} \frac{z^{m+a}}{m! (m+a)^k} \\ &= \frac{1}{z^a} \int_0^z \sum_{m=0}^{\infty} \frac{z^{m+a-1}}{m! (m+a)^{k-1}} dz \\ &= \frac{1}{z^a} \int_0^z z^{a-1} \text{Lif}_{k-1}(z; a) dz. \end{aligned} \quad (27)$$

Hence,

$$\text{Lif}_k(z; a) = \frac{1}{z^a} \underbrace{\int_0^z \frac{1}{z} \int_0^z \cdots \frac{1}{z} \int_0^z \frac{1}{z} \int_0^z}_{k} z^{a-1} e^z \underbrace{dz \cdots dz}_k. \quad (28)$$

Putting $z = \ell \ln(1+qx)/q$ and multiplying by ℓ^a , we get the result. □

3. Poly-Cauchy Numbers of the Second Kind

In [1], the concept of poly-Cauchy numbers of the second kind is also introduced. The poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ are defined by

$$\hat{c}_n^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (-x_1 x_2 \cdots x_k) (-x_1 x_2 \cdots x_k - 1) \cdots \quad (29)$$

$$(-x_1 x_2 \cdots x_k - n + 1) dx_1 dx_2 \cdots dx_k,$$

and the generating function is given by

$$\text{Lif}_k(-\ln(1+x)) = \sum_{n=0}^{\infty} \hat{c}_n^{(k)} \frac{x^n}{n!}. \quad (30)$$

Then, the poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ can also be expressed in terms of the Stirling numbers of the first kind ([1], Theorem 4). See ([5], A219247, A224102–A224107, A224109) for the sequences arising from poly-Cauchy numbers of the second kind.

Proposition 7. One has

$$\hat{c}_n^{(k)} = (-1)^n \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{(m+1)^k}. \quad (31)$$

Let a be a positive real number. Similar to generalized poly-Cauchy numbers of the first kind $c_{n,a,q,L}^{(k)}$, define the poly-Cauchy numbers of the second kind $\hat{c}_{n,a,q,L}^{(k)}$ ($n \geq 0, k \geq 1$) by

$$\hat{c}_{n,a,q,L}^{(k)} = (-1)^{a-1} \int_0^{l_1} \cdots \int_0^{l_k} (-x_1 \cdots x_k)^a (-x_1 \cdots x_k - q) \cdots (-x_1 \cdots x_k - (n-1)q) dx_1 \cdots dx_k. \quad (32)$$

Then, similar to Theorem 1, $\hat{c}_{n,a,q,L}^{(k)}$ can also be expressed in terms of the Stirling numbers of the first kind $[n_m]$.

Theorem 8. One has

$$\hat{c}_{n,a,q,L}^{(k)} = (-1)^n \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{q^{n-m} \ell^{m+a}}{(m+a)^k} \quad (n \geq 0, k \geq 1). \quad (33)$$

Theorem 9. The generating function of the number $\hat{c}_{n,a,q,L}^{(k)}$ is given by

$$\ell^a \text{Lif}_k \left(-\frac{\ell \ln(1+qx)}{q}; a \right) = \sum_{m=0}^{\infty} \hat{c}_{n,a,q,L}^{(k)} \frac{x^m}{m!}, \quad (34)$$

where

$$\text{Lif}_k(z; a) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+a)^k}. \quad (35)$$

Remark 10. If $a = \ell = 1$, then Theorem 8 is reduced to Theorem 3 in [7] and Theorem 9 is reduced to Theorem 4 in [7]. If $q = \ell = 1$, then Theorem 8 is reduced to Theorem 5 in [8] and Theorem 9 is reduced to Theorem 6 in [8].

The generating function of the number $\hat{c}_{n,a,q,L}^{(k)}$ can be written in the form of iterated integrals.

Corollary 11. Let a be a positive real number. For $k = 1$, one has

$$\begin{aligned} & \left(\frac{\ell q}{\ln(1+qx)} \right)^a \int_0^x \left(\frac{\ln(1+qx)}{q} \right)^{a-1} (1+qx)^{-\ell/q-1} dx \\ &= \sum_{n=0}^{\infty} \hat{c}_{n,a,q,L}^{(1)} \frac{x^n}{n!}. \end{aligned} \quad (36)$$

For $k > 1$, one has

$$\begin{aligned} & \left(\frac{\ell q}{\ln(1+qx)} \right) \int_0^x \frac{q}{(1+qx) \ln(1+qx)} \int_0^x \cdots \int_0^x \frac{q}{(1+qx) \ln(1+qx)} \int_0^x \\ & \left(\frac{\ln(1+qx)}{q} \right)^{a-1} (1+qx)^{-\ell/q-1} \underbrace{dx \cdots dx}_k = \sum_{n=0}^{\infty} \hat{c}_{n,a,q,L}^{(k)} \frac{x^n}{n!}. \end{aligned} \quad (37)$$

Remark 12. When $a = q = k = \ell = 1$ in the first identity, we have the generating function of the classical Cauchy numbers of the second kind:

$$\frac{x}{(1+x) \ln(1+x)} = \sum_{n=0}^{\infty} \hat{c}_n \frac{x^n}{n!}. \quad (38)$$

In addition, there are relations between both kinds of poly-Cauchy numbers if $q = 1$. For simplicity, we write $c_{n,a,L}^{(k)} = c_{n,a,1,L}^{(k)}$ and $\hat{c}_{n,a,L}^{(k)} = \hat{c}_{n,a,1,L}^{(k)}$.

Theorem 13. Let k be an integer and a a positive real number. For $n \geq 1$, one has

$$\begin{aligned} (-1)^n \frac{c_{n,a,L}^{(k)}}{n!} &= \sum_{m=1}^n \binom{n-1}{m-1} \frac{\hat{c}_{m,a,L}^{(k)}}{m!}, \\ (-1)^n \frac{\hat{c}_{n,a,L}^{(k)}}{n!} &= \sum_{m=1}^n \binom{n-1}{m-1} \frac{c_{m,a,L}^{(k)}}{m!}. \end{aligned} \quad (39)$$

Remark 14. If $a = \ell = 1$, then Theorem 13 is reduced to Theorem 7 in [1].

Proof. We will prove the second identity. The first one is proved similarly and omitted. By using the identity (see, e.g., [4], Chapter 6)

$$\frac{(-1)^i}{n!} \begin{bmatrix} n \\ i \end{bmatrix} = \sum_{m=1}^n \frac{(-1)^m}{m!} \binom{n-1}{m-1} \begin{bmatrix} m \\ i \end{bmatrix} \quad (40)$$

and Theorems 1 and 8, we have

$$\begin{aligned} \text{RHS} &= \sum_{m=1}^n \binom{n-1}{m-1} \frac{1}{m!} \\ &\times \sum_{i=1}^m \begin{bmatrix} m \\ i \end{bmatrix} \frac{(-1)^{m-i} \ell^{i+a}}{(i+a)^k} \\ &= \sum_{i=1}^n \frac{(-1)^i \ell^i}{(i+a)^k} \sum_{m=i}^n \frac{(-1)^m}{m!} \binom{n-1}{m-1} \begin{bmatrix} m \\ i \end{bmatrix} \\ &= \sum_{i=1}^n \frac{(-1)^i \ell^{i+a}}{(i+a)^k} \frac{(-1)^i}{n!} \begin{bmatrix} n \\ i \end{bmatrix} = \text{LHS}. \end{aligned} \quad (41)$$

□

4. Some Expressions of Poly-Cauchy Numbers with Negative Indices

It is known that poly-Bernoulli numbers satisfy the duality theorem $B_n^{(-k)} = B_k^{(-n)}$ for $n, k \geq 0$ ([6], Theorem 2) because of the symmetric formula

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}. \quad (42)$$

However, the corresponding duality theorem does not hold for poly-Cauchy numbers for any real number a , by the following results.

Proposition 15. Suppose that $\ell = 1$. Then, for nonnegative integers n and k and a real number $a \neq 0$, one has

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{n,a,q,L}^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = e^{ay} (1+qx)^{e^y/q}, \quad (43)$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \hat{c}_{n,a,q,L}^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^{ay}}{(1+qx)^{e^y/q}}.$$

Remark 16. If $a = \ell = 1$, then Proposition 15 is reduced to Proposition 1 in [7]. If $q = \ell = 1$, then Proposition 15 is reduced to Proposition 3 in [8].

Proof. We will prove the first identity. The second identity is proved similarly. By Theorem 3, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{n,a,q,L}^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} \\ &= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} c_{n,a,q,L}^{(-k)} \frac{x^n}{n!} \right) \frac{y^k}{k!} \\ &= \sum_{k=0}^{\infty} \ell^a \sum_{m=0}^{\infty} \frac{(m+a)^k}{m!} \left(\frac{\ell \ln(1+qx)}{q} \right)^m \frac{y^k}{k!} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\ln(1+qx)}{q} \right)^m \sum_{k=0}^{\infty} \frac{((m+a)y)^k}{k!} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\ln(1+qx)}{q} \right)^m e^{(m+a)y} \\ &= e^{ay} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{e^y}{q} \ln(1+qx) \right)^m = e^{ay} (1+qx)^{e^y/q}. \end{aligned} \quad (44)$$

□

By using Proposition 15, we have explicit expressions of poly-Cauchy numbers with negative indices. For simplicity, we write $c_{n,a,q}^{(-k)} = c_{n,a,q,L}^{(-k)}$ and $\hat{c}_{n,a,q}^{(-k)} = \hat{c}_{n,a,q,L}^{(-k)}$ if $\ell = 1$.

Theorem 17. For nonnegative integers n, k , and a real number $a \neq 0$, one has

$$\begin{aligned} c_{n,a,q}^{(-k)} &= \sum_{i=0}^k \sum_{j=0}^i \sum_{\lambda=0}^n \sum_{\nu=0}^{\lambda} j! \binom{k}{i} \begin{Bmatrix} i \\ j \end{Bmatrix} \binom{n}{\lambda} \begin{bmatrix} n-\lambda \\ j \end{bmatrix} \\ &\quad \times \begin{bmatrix} \lambda \\ \nu \end{bmatrix} a^{k-i} (-q)^{n-j-\nu}, \\ \hat{c}_{n,a,q}^{(-k)} &= \sum_{i=0}^k \sum_{j=0}^i \sum_{\lambda=0}^n \sum_{\nu=0}^{\lambda} (-1)^n j! \binom{k}{i} \begin{Bmatrix} i \\ j \end{Bmatrix} \binom{n}{\lambda} \\ &\quad \times \begin{bmatrix} n-\lambda \\ j \end{bmatrix} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} a^{k-i} q^{n-j-\nu}. \end{aligned} \quad (45)$$

Remark 18. If $a = q = 1$, by

$$\sum_{i=0}^k \binom{k}{i} \begin{Bmatrix} i \\ j \end{Bmatrix} = \begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix} \quad (46)$$

[4], the above identities become

$$\begin{aligned} c_{n,1,1}^{(-k)} &= c_n^{(-k)} \\ &= \sum_{j=0}^k (-1)^{n+j} j! \\ &\quad \times \left(\begin{bmatrix} n \\ j \end{bmatrix} - n \begin{bmatrix} n-1 \\ j \end{bmatrix} \right) \begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix}, \\ \hat{c}_{n,1,1}^{(-k)} &= \hat{c}_n^{(-k)} = \sum_{j=0}^k (-1)^n j! \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} \begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix}. \end{aligned} \quad (47)$$

Proof. By Proposition 15 together with

$$\begin{aligned} \frac{(e^y - 1)^j}{j!} &= \sum_{k=j}^{\infty} \begin{Bmatrix} k \\ j \end{Bmatrix} \frac{y^k}{k!}, \\ \frac{(-\ln(1+x))^j}{j!} &= \sum_{n=j}^{\infty} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(-x)^n}{n!} \end{aligned} \quad (48)$$

[4], we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{n,a,q}^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} \\ &= \left((1+qx)^{1/q} \right)^{e^y-1} (1+qx)^{1/q} e^{ay} \\ &= \exp \left((e^y - 1) (\ln(1+qx))^{1/q} \right) \\ &\quad \times (1+qx)^{1/q} e^{ay} \\ &= \sum_{j=0}^{\infty} \frac{j!}{q^j} \frac{(e^y - 1)^j}{j!} \frac{(\ln(1+qx))^j}{j!} \\ &\quad \times (1+qx)^{1/q} e^{ay} \\ &= \sum_{j=0}^{\infty} (-1)^j \frac{j!}{q^j} e^{ay} \sum_{k=j}^{\infty} \begin{Bmatrix} k \\ j \end{Bmatrix} \frac{y^k}{k!} (1+qx)^{1/q} \\ &\quad \times \sum_{n=j}^{\infty} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(-qx)^n}{n!}. \end{aligned} \quad (49)$$

Since

$$\begin{aligned}
 e^{ay} \sum_{k=j}^{\infty} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \frac{y^k}{k!} &= \sum_{l=0}^{\infty} \frac{(ay)^l}{l!} \sum_{k=j}^{\infty} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \frac{y^k}{k!} \\
 &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^k \frac{a^{k-i}}{(k-i)!} \left\{ \begin{matrix} i \\ j \end{matrix} \right\} \frac{1}{i!} \right) y^k \\
 &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^k \binom{k}{i} \left\{ \begin{matrix} i \\ j \end{matrix} \right\} a^{k-i} \right) \frac{y^k}{k!}, \\
 (1+qx)^{1/q} \sum_{n=j}^{\infty} \left[\begin{matrix} n \\ j \end{matrix} \right] \frac{(-qx)^n}{n!} \\
 &= \sum_{l=0}^{\infty} \left(\frac{1}{q} \right)^l (qx)^l \sum_{n=j}^{\infty} \left[\begin{matrix} n \\ j \end{matrix} \right] \frac{(-qx)^n}{n!} \\
 &= \sum_{l=0}^{\infty} (-1)^l \sum_{\nu=0}^l \left[\begin{matrix} l \\ \nu \end{matrix} \right] \left(-\frac{1}{q} \right)^{\nu} \frac{(qx)^l}{l!} \\
 &\quad \times \sum_{n=j}^{\infty} \left[\begin{matrix} n \\ j \end{matrix} \right] \frac{(-qx)^n}{n!} \\
 &= \sum_{l=0}^{\infty} \sum_{\nu=0}^l \left[\begin{matrix} l \\ \nu \end{matrix} \right] (-q)^{l-\nu} \frac{x^l}{l!} \sum_{n=0}^{\infty} \left[\begin{matrix} n \\ j \end{matrix} \right] \frac{(-qx)^n}{n!} \\
 &= \sum_{n=0}^{\infty} \sum_{\lambda=0}^n \sum_{\nu=0}^{\lambda} \left[\begin{matrix} \lambda \\ \nu \end{matrix} \right] \frac{(-q)^{\lambda-\nu}}{\lambda!} \left[\begin{matrix} n-\lambda \\ j \end{matrix} \right] \frac{(-q)^{n-\lambda}}{(n-\lambda)!} x^n \\
 &= \sum_{n=0}^{\infty} \sum_{\lambda=0}^n \binom{n}{\lambda} \left[\begin{matrix} n-\lambda \\ j \end{matrix} \right] \sum_{\nu=0}^{\lambda} \left[\begin{matrix} \lambda \\ \nu \end{matrix} \right] (-q)^{n-\nu} \frac{x^n}{n!}, \tag{50}
 \end{aligned}$$

we obtain

$$\begin{aligned}
 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{n,a,q}^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} \\
 &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^k \frac{(-1)^j j!}{q^j} \binom{k}{i} \left\{ \begin{matrix} i \\ j \end{matrix} \right\} a^{k-i} \\
 &\quad \times \sum_{\lambda=0}^n \binom{n}{\lambda} \left[\begin{matrix} n-\lambda \\ j \end{matrix} \right] \sum_{\nu=0}^{\lambda} \left[\begin{matrix} \lambda \\ \nu \end{matrix} \right] (-q)^{n-\nu} \frac{x^n}{n!} \frac{y^k}{k!} \\
 &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^k \sum_{\lambda=0}^n \sum_{\nu=0}^{\lambda} j! \binom{k}{i} \left\{ \begin{matrix} i \\ j \end{matrix} \right\} \\
 &\quad \times \binom{n}{\lambda} \left[\begin{matrix} n-\lambda \\ j \end{matrix} \right] \left[\begin{matrix} \lambda \\ \nu \end{matrix} \right] a^{k-i} \\
 &\quad \times (-q)^{n-j-\nu} \frac{x^n}{n!} \frac{y^k}{k!}. \tag{51}
 \end{aligned}$$

Similarly, by

$$\begin{aligned}
 (1+qx)^{-1/q} \sum_{n=j}^{\infty} \left[\begin{matrix} n \\ j \end{matrix} \right] \frac{(-qx)^n}{n!} \\
 &= \sum_{l=0}^{\infty} (-1)^l \sum_{\nu=0}^l \left[\begin{matrix} l \\ \nu \end{matrix} \right] \left(\frac{1}{q} \right)^{\nu} \frac{(qx)^l}{l!} \\
 &\quad \times \sum_{n=j}^{\infty} \left[\begin{matrix} n \\ j \end{matrix} \right] \frac{(-qx)^n}{n!} \\
 &= \sum_{l=0}^{\infty} \sum_{\nu=0}^l \left[\begin{matrix} l \\ \nu \end{matrix} \right] q^{l-\nu} (-1)^l \frac{x^l}{l!} \sum_{n=0}^{\infty} \left[\begin{matrix} n \\ j \end{matrix} \right] \frac{(-qx)^n}{n!} \\
 &= \sum_{n=0}^{\infty} \sum_{\lambda=0}^n \sum_{\nu=0}^{\lambda} \left[\begin{matrix} \lambda \\ \nu \end{matrix} \right] q^{\lambda-\nu} \frac{(-1)^{\lambda}}{\lambda!} \\
 &\quad \times \left[\begin{matrix} n-\lambda \\ j \end{matrix} \right] \frac{(-q)^{n-\lambda}}{(n-\lambda)!} x^n \\
 &= \sum_{n=0}^{\infty} (-1)^n \sum_{\lambda=0}^n \binom{n}{\lambda} \left[\begin{matrix} n-\lambda \\ j \end{matrix} \right] \sum_{\nu=0}^{\lambda} \left[\begin{matrix} \lambda \\ \nu \end{matrix} \right] q^{n-\nu} \frac{x^n}{n!}, \tag{52}
 \end{aligned}$$

we get

$$\begin{aligned}
 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{n,a,q}^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} \\
 &= (1+qx)^{-e^y/q} e^{ay} \\
 &= \exp \left((e^y - 1) (\ln(1+qx))^{-1/q} \right) \\
 &\quad \times (1+qx)^{-1/q} e^{ay} \\
 &= \sum_{j=0}^{\infty} \frac{j!}{q^j} \frac{(e^y - 1)^j}{j!} \frac{(-\ln(1+qx))^j}{j!} \\
 &\quad \times (1+qx)^{-1/q} e^{ay} \\
 &= \sum_{j=0}^{\infty} \frac{j!}{q^j} e^{ay} \sum_{k=j}^{\infty} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \frac{y^k}{k!} (1+qx)^{-1/q} \\
 &\quad \times \sum_{n=j}^{\infty} \left[\begin{matrix} n \\ j \end{matrix} \right] \frac{(-qx)^n}{n!} \\
 &= \sum_{j=0}^{\infty} \frac{j!}{q^j} \sum_{k=0}^{\infty} \left(\sum_{i=0}^k \binom{k}{i} \left\{ \begin{matrix} i \\ j \end{matrix} \right\} a^{k-i} \right) \frac{y^k}{k!}
 \end{aligned}$$

$$\begin{aligned}
 & \times \sum_{n=0}^{\infty} (-1)^n \sum_{\lambda=0}^n \binom{n}{\lambda} \begin{bmatrix} n-\lambda \\ j \end{bmatrix} \sum_{\nu=0}^{\lambda} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} q^{n-\nu} \frac{x^n}{n!} \\
 & = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^k \sum_{j=0}^i \sum_{\lambda=0}^n \sum_{\nu=0}^{\lambda} (-1)^n j! \\
 & \quad \times \begin{pmatrix} k \\ i \end{pmatrix} \begin{Bmatrix} i \\ j \end{Bmatrix} \begin{pmatrix} n \\ \lambda \end{pmatrix} \begin{bmatrix} n-\lambda \\ j \end{bmatrix} \\
 & \quad \times \begin{bmatrix} \lambda \\ \nu \end{bmatrix} a^{k-i} q^{n-j-\nu} \frac{x^n}{n!} \frac{y^k}{k!}.
 \end{aligned} \tag{53}$$

□

5. Poly-Bernoulli Numbers Corresponding to Poly-Cauchy Numbers

In this section, we will consider the corresponding generalized poly-Bernoulli numbers to the generalized poly-Cauchy numbers discussed in the previous sections. Let k be an integer and a a positive real number. An explicit form of poly-Bernoulli number $B_n^{(k)}$ is given by

$$B_n^{(k)} = \sum_{m=0}^n \begin{Bmatrix} n \\ m \end{Bmatrix} \frac{(-1)^{n-m} m!}{(m+1)^k} \tag{54}$$

([6], Theorem 1). In ([1], Theorem 8), one expression of $B_n^{(k)}$ in terms of poly-Cauchy numbers $c_n^{(k)}$ is given.

Proposition 19. *One has*

$$B_n^{(k)} = \sum_{l=1}^n \sum_{m=1}^n m! \begin{Bmatrix} n \\ m \end{Bmatrix} \begin{Bmatrix} m-1 \\ l-1 \end{Bmatrix} c_l^{(k)} \quad (n \geq 1). \tag{55}$$

On the contrary, in ([9], Theorem 2.2), one expression of $c_n^{(k)}$ in terms of $B_n^{(k)}$ is given.

Proposition 20. *One has*

$$c_n^{(k)} = \sum_{l=1}^n \sum_{m=1}^n \frac{(-1)^{n-m}}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} B_l^{(k)} \quad (n \geq 1). \tag{56}$$

As a counterpart of a generalized poly-Cauchy number, we will define a generalized poly-Bernoulli number $B_{n,a,L}^{(k)}$ by

$$\frac{\ell^{a-1} \text{Li}_k(\ell(1-e^{-t}); a-1)}{1-e^{-t}} = \sum_{n=0}^{\infty} B_{n,a,L}^{(k)} \frac{t^n}{n!}, \tag{57}$$

where $\text{Li}_k(z; a)$ is the generalized polylogarithm function defined by

$$\text{Li}_k(z; a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^k}, \tag{58}$$

so that $\text{Li}_k(z; 0) = \text{Li}_k(z)$.

Then, $B_{n,a,L}^{(k)}$ can be expressed explicitly in terms of the Stirling numbers of the second kind. Note that $B_{n,1,(1,\dots,1)}^{(k)} = B_n^{(k)}$.

Proposition 21. *One has*

$$B_{n,a,L}^{(k)} = \sum_{m=0}^n \begin{Bmatrix} n \\ m \end{Bmatrix} \frac{(-1)^{n-m} m! \ell^{m+a}}{(m+a)^k} \quad (n \geq 0). \tag{59}$$

Proof. By

$$\frac{(e^t - 1)^m}{m!} = \sum_{n=m}^{\infty} \begin{Bmatrix} n \\ m \end{Bmatrix} \frac{t^n}{n!}, \tag{60}$$

we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} B_{n,a,L}^{(k)} \frac{t^n}{n!} &= \frac{\ell^a}{\ell(1-e^{-t})} \sum_{m=1}^{\infty} \frac{(\ell(1-e^{-t}))^m}{(m+a-1)^k} \\
 &= \ell^a \sum_{m=0}^{\infty} \frac{(\ell(1-e^{-t}))^m}{(m+a)^k} \\
 &= \ell^a \sum_{m=0}^{\infty} \frac{(-\ell)^m m!}{(m+a)^k} \sum_{n=m}^{\infty} \begin{Bmatrix} n \\ m \end{Bmatrix} \frac{(-t)^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \begin{Bmatrix} n \\ m \end{Bmatrix} \frac{(-1)^{n-m} m! \ell^{m+a}}{(m+a)^k} \right) \frac{t^n}{n!}.
 \end{aligned} \tag{61}$$

Comparing the coefficients on both sides, we get the result. □

For simplicity, we write $c_{n,a,L}^{(k)} = c_{n,a,1,L}^{(k)}$ and $\hat{c}_{n,a,L}^{(k)} = \hat{c}_{n,a,1,L}^{(k)}$. If $a = \ell = 1$, then our results below are reduced to those previous ones.

Theorem 22. *For $n \geq 0$, one has*

$$\begin{aligned}
 B_{n,a,L}^{(k)} &= \sum_{j=1}^n \sum_{m=1}^n m! \begin{Bmatrix} n \\ m \end{Bmatrix} \begin{Bmatrix} m-1 \\ j-1 \end{Bmatrix} c_{j,a,L}^{(k)}, \\
 c_{n,a,L}^{(k)} &= \sum_{j=1}^n \sum_{m=1}^n \frac{(-1)^{n-m}}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} B_{j,a,L}^{(k)}.
 \end{aligned} \tag{62}$$

Proof. For the first identity,

$$\begin{aligned}
 \text{RHS} &= \sum_{j=1}^n \sum_{m=j}^n m! \begin{Bmatrix} n \\ m \end{Bmatrix} \begin{Bmatrix} m-1 \\ j-1 \end{Bmatrix} \\
 &\quad \times \sum_{i=0}^j \begin{bmatrix} j \\ i \end{bmatrix} \frac{(-1)^{j-i} \ell^{i+a}}{(i+a)^k} \\
 &= \sum_{i=1}^n \frac{(-1)^i \ell^{i+a}}{(i+a)^k} \\
 &\quad \times \sum_{j=i}^n \sum_{m=j}^n m! \begin{Bmatrix} n \\ m \end{Bmatrix} \begin{Bmatrix} m-1 \\ j-1 \end{Bmatrix} (-1)^j \begin{bmatrix} j \\ i \end{bmatrix} \\
 &= \sum_{i=1}^n \frac{(-1)^i \ell^{i+a}}{(i+a)^k} \sum_{m=i}^n m! \begin{Bmatrix} n \\ m \end{Bmatrix}
 \end{aligned}$$

$$\begin{aligned}
& \times \sum_{j=i}^m (-1)^j \begin{Bmatrix} m-1 \\ j-1 \end{Bmatrix} \begin{Bmatrix} j \\ i \end{Bmatrix} \\
& = \sum_{i=1}^n \frac{(-1)^i \ell^{i+a}}{(i+a)^k} \sum_{m=i}^n m! \begin{Bmatrix} n \\ m \end{Bmatrix} (-1)^m \binom{m-1}{i-1} \\
& = \sum_{i=1}^n \frac{(-1)^i \ell^{i+a}}{(i+a)^k} (-1)^n i! \begin{Bmatrix} n \\ i \end{Bmatrix} = \text{LHS}.
\end{aligned} \tag{63}$$

For the second identity,

$$\begin{aligned}
\text{RHS} &= \sum_{j=1}^n \sum_{m=1}^n \frac{(-1)^{n-m}}{m!} \begin{Bmatrix} n \\ m \end{Bmatrix} \begin{Bmatrix} m \\ j \end{Bmatrix} \\
& \times \sum_{i=0}^j \begin{Bmatrix} j \\ i \end{Bmatrix} \frac{(-1)^{j-i} i! \ell^{i+a}}{(i+a)^k} \\
&= \sum_{m=1}^n \frac{(-1)^{n-m}}{m!} \begin{Bmatrix} n \\ m \end{Bmatrix} \\
& \times \sum_{j=0}^n \begin{Bmatrix} m \\ j \end{Bmatrix} \sum_{i=0}^j \begin{Bmatrix} j \\ i \end{Bmatrix} \frac{(-1)^{j-i} i! \ell^{i+a}}{(i+a)^k} \\
&= \sum_{m=1}^n \frac{(-1)^{n-m}}{m!} \begin{Bmatrix} n \\ m \end{Bmatrix} \\
& \times \sum_{i=0}^n \frac{(-1)^i i! \ell^{i+a}}{(i+a)^k} \sum_{j=i}^n (-1)^j \begin{Bmatrix} m \\ j \end{Bmatrix} \begin{Bmatrix} j \\ i \end{Bmatrix} \\
&= \sum_{m=0}^n \frac{(-1)^{n-m}}{m!} \begin{Bmatrix} n \\ m \end{Bmatrix} \\
& \times \frac{(-1)^m m! \ell^{m+a}}{(m+a)^k} (-1)^m \\
&= \sum_{m=0}^n \begin{Bmatrix} n \\ m \end{Bmatrix} \frac{(-1)^{n-m} \ell^{m+a}}{(m+a)^k} = \text{LHS}.
\end{aligned} \tag{64}$$

Note that $\begin{Bmatrix} m \\ 0 \end{Bmatrix} = 0$ ($m \geq 1$) and $\begin{Bmatrix} m \\ j \end{Bmatrix} = 0$ ($j > m$), and

$$\sum_{j=i}^m (-1)^{m-j} \begin{Bmatrix} m \\ j \end{Bmatrix} \begin{Bmatrix} j \\ i \end{Bmatrix} = \begin{cases} 1 & (i = m); \\ 0 & (i \neq m). \end{cases} \tag{65}$$

□

Similarly, concerning

$$\hat{c}_{n,a,L}^{(k)} = (-1)^n \sum_{m=0}^n \begin{Bmatrix} n \\ m \end{Bmatrix} \frac{\ell^{m+a}}{(m+a)^k} \quad (n \geq 0) \tag{66}$$

as a generalization of poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$, we have the following.

Theorem 23. One has

$$\begin{aligned}
B_{n,a,L}^{(k)} &= (-1)^n \sum_{j=1}^n \sum_{m=1}^n m! \begin{Bmatrix} n \\ m \end{Bmatrix} \begin{Bmatrix} m \\ j \end{Bmatrix} \hat{c}_{j,a,L}^{(k)}, \\
\hat{c}_{n,a,L}^{(k)} &= (-1)^n \sum_{j=1}^n \sum_{m=1}^n \frac{1}{m!} \begin{Bmatrix} n \\ m \end{Bmatrix} \begin{Bmatrix} m \\ j \end{Bmatrix} B_{j,a,L}^{(k)}.
\end{aligned} \tag{67}$$

Remark 24. If $a = \ell = 1$, these results are reduced to the identities in Theorems 3.2 and 3.1 in [9], respectively.

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