Research Article

Hyponormal Toeplitz Operators on the Dirichlet Spaces

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We completely characterize the hyponormality of bounded Toeplitz operators with Sobolev symbols on the Dirichlet space and the harmonic Dirichlet space.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and dA be the normalized Lebesgue area measure on $\mathbb{D}$. $L^\infty(\mathbb{D}, dA)$ and $L^2(\mathbb{D}, dA)$ denote the essential bounded measurable function space and the space of square integral functions on $\mathbb{D}$ with respect to dA , respectively. The Bergman space L_a^2 consists of all analytic functions in $L^2(\mathbb{D}, dA)$. The Sobolev space $W^{1,2}(\mathbb{D})$ is the space of functions $f : \mathbb{D} \rightarrow \mathbb{C}$ with the following norm:

$$\|f\| = \left[\left| \int_{\mathbb{D}} f dA \right|^2 + \int_{\mathbb{D}} \left| \frac{\partial f}{\partial z} \right|^2 + \left| \frac{\partial f}{\partial \bar{z}} \right|^2 dA \right]^{1/2} < \infty. \quad (1)$$

$W^{1,2}(\mathbb{D})$ is a Hilbert space with the inner product

$$\begin{aligned} \langle f, g \rangle = & \int_{\mathbb{D}} f dA \int_{\mathbb{D}} \bar{g} dA + \left\langle \frac{\partial f}{\partial z}, \frac{\partial g}{\partial z} \right\rangle_{L^2(\mathbb{D}, dA)} \\ & + \left\langle \frac{\partial f}{\partial \bar{z}}, \frac{\partial g}{\partial \bar{z}} \right\rangle_{L^2(\mathbb{D}, dA)}. \end{aligned} \quad (2)$$

The Dirichlet space $\mathcal{D}$ consists of all analytic functions h in $W^{1,2}(\mathbb{D})$ with $h(0) = 0$. The Sobolev space $W^{1,\infty}(\mathbb{D})$ is defined by

$$W^{1,\infty}(\mathbb{D}) = \left\{ u \in W^{1,2}(\mathbb{D}) : u, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \bar{z}} \in L^\infty(\mathbb{D}, dA) \right\}, \quad (3)$$

with the norm

$$\|u\|_{1,\infty} = \max \left\{ \|u\|_\infty, \left\| \frac{\partial u}{\partial z} \right\|_\infty, \left\| \frac{\partial u}{\partial \bar{z}} \right\|_\infty \right\}. \quad (4)$$

Let P be the orthogonal projection of $W^{1,2}(\mathbb{D})$ onto $\mathcal{D}$. P is an integral operator represented by

$$P(f)(z) = \int_{\mathbb{D}} \frac{\partial f}{\partial w} \overline{\left(\frac{\partial K_z}{\partial w} \right)} dA, \quad (5)$$

where $K_z(w) = \sum_{k=1}^{\infty} (\bar{z}^k w^k / k)$ is the reproducing kernel of $\mathcal{D}$. For $u \in W^{1,\infty}(\mathbb{D})$, the Toeplitz operator T_u with symbol u is defined by

$$T_u h = P(uh) \quad h \in \mathcal{D}. \quad (6)$$

T_u is a bounded operator for $u \in W^{1,\infty}(\mathbb{D})$ on $\mathcal{D}$.

Yu gave a decomposition of the Sobolev space $W^{1,2}(\mathbb{D})$ in [1]. Let $\mathcal{P}_0$ be the set of all the following polynomials:

$$\sum_{j \geq -l} \sum_{l \geq 0} a_{l+j,l} z^{l+j} \bar{z}^l, \quad (7)$$

where j and l run over a finite subset of $\mathbb{Z}$ and $\sum_{l \geq 0} a_{l+j,l} = 0$. Let $\mathcal{A}_0$ denote the closure of $\mathcal{P}_0$ in $W^{1,2}(\mathbb{D})$, and let $\mathcal{A}$ denote $\mathcal{A}_0 + \mathbb{C}$. Since the set of all polynomials in z and $\bar{z}$ is dense in $W^{1,2}(\mathbb{D})$, there is the following decomposition:

$$W^{1,2}(\mathbb{D}) = \mathcal{A} \oplus \mathcal{D} \oplus \overline{\mathcal{D}}. \quad (8)$$

Since $W^{1,\infty}(\mathbb{D}) \subseteq W^{1,2}(\mathbb{D})$ and by the above decomposition, it follows that, if $u \in W^{1,\infty}(\mathbb{D})$, then $u = u_0 + f + \bar{g}$, where $u_0 \in \mathcal{A}$, $f, g \in H(\mathbb{D})$ (the space of the analytic functions on $\mathbb{D}$) with $f(0) = g(0) = 0$.

For the space $\mathcal{A}_0$, there is the following proposition.

Proposition 1 (see [1]). *Let $\phi \in W^{1,\infty}(\mathbb{D})$. Then $\phi\mathcal{A}_0 \subset \mathcal{A}_0$.*

A bounded linear operator A on a Hilbert space is called hyponormal if $A^*A - AA^*$ is a positive operator. There is an extensive literature on hyponormal Toeplitz operators on $H^2(\mathbb{T})$ (the Hardy space on $\mathbb{T}$) [2–4]. The corresponding problems for the Toeplitz operators on the Bergman space have been characterized in [5–9]. In the case of the Dirichlet space and the harmonic Dirichlet space, Lu and Yu proved that there are no nonconstant hyponormal Toeplitz operators with certain symbols [10]. In this paper, we completely characterize the Toeplitz operators T_u with $u \in W^{1,\infty}(\mathbb{D})$ on Dirichlet space $\mathcal{D}$ and harmonic Dirichlet space $\mathcal{D}_h$.

2. Case on the Dirichlet Space

In this section, the hyponormality of T_u with $u \in W^{1,\infty}(\mathbb{D})$ on $\mathcal{D}$ will be discussed.

Theorem 2. *Let $u = u_0 + c + f + \bar{g} \in W^{1,\infty}(\mathbb{D})$ with $u_0 + c \in \mathcal{A}$, $f, g \in H(\mathbb{D})$, and $f(0) = g(0) = 0$. Then T_u is hyponormal on $\mathcal{D}$ if and only if $u \in \mathcal{A}$.*

Proof. By Proposition 1, we only need to prove the necessity with $u(z) = f(z) + \bar{g}(z) = \sum_{k=1}^{\infty} f_k z^k + \sum_{k=1}^{\infty} \bar{g}_k \bar{z}^k$.

Let $h(z) = \sum_{k=1}^{\infty} h_k z^k \in \mathcal{D}$. Simple calculations imply that

$$\begin{aligned} T_{f+\bar{g}}h(z) &= T_f h(z) + T_{\bar{g}} h(z) \\ &= f(z)h(z) + P(\bar{g}h)(z) \\ &= f(z)h(z) + \langle \bar{g}h, K_z \rangle \\ &= \sum_{m=2}^{\infty} \left(\sum_{k=1}^{m-1} f_k h_{m-k} \right) z^m + \sum_{l=2}^{\infty} h_l \left(\sum_{k=1}^{l-1} \bar{g}_k z^{l-k} \right) \\ &= \sum_{m=2}^{\infty} \left(\sum_{k=1}^{m-1} f_k h_{m-k} \right) z^m + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \bar{g}_k h_{k+m} z^m. \end{aligned} \quad (9)$$

Furthermore,

$$\begin{aligned} \frac{\partial(T_{f+\bar{g}}h)}{\partial z} &= \sum_{m=1}^{\infty} (m+1) \left(\sum_{k=1}^m f_k h_{m-k+1} \right) z^m \\ &\quad + \sum_{m=0}^{\infty} (m+1) \left(\sum_{k=1}^{\infty} \bar{g}_k h_{k+m+1} \right) z^m \end{aligned}$$

$$\begin{aligned} &= \sum_{k=1}^{\infty} \bar{g}_k h_{k+1} + \sum_{m=1}^{\infty} (m+1) \\ &\quad \times \left(\sum_{k=1}^m f_k h_{m-k+1} + \sum_{k=1}^{\infty} \bar{g}_k h_{k+m+1} \right) z^m. \end{aligned} \quad (10)$$

Therefore

$$\begin{aligned} \|T_{f+\bar{g}}h\|^2 &= \left| \sum_{k=1}^{\infty} \bar{g}_k h_{k+1} \right|^2 \\ &\quad + \sum_{m=1}^{\infty} \left| \sum_{k=1}^m f_k h_{m-k+1} + \sum_{k=1}^{\infty} \bar{g}_k h_{k+m+1} \right|^2. \end{aligned} \quad (11)$$

Similarly, we have

$$\begin{aligned} T_{f+\bar{g}}^*h(z) &= \langle T_{f+\bar{g}}^*h, K_z \rangle = \langle h, T_{f+\bar{g}}K_z \rangle \\ &= \sum_{m=2}^{\infty} \frac{1}{m} \left(\sum_{l=1}^{m-1} l g_{m-l} h_l \right) z^m \\ &\quad + \sum_{m=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{m+k}{m} \bar{f}_k h_{k+m} \right) z^m, \\ \frac{\partial(T_{f+\bar{g}}^*h)}{\partial z} &= \sum_{k=1}^{\infty} (k+1) \bar{f}_k h_{k+1} \\ &\quad + \sum_{m=1}^{\infty} \left[\sum_{k=1}^{\infty} (m+k+1) \bar{f}_k h_{m+k+1} \right. \\ &\quad \left. + \sum_{l=1}^m l g_{m-l+1} h_l \right] z^m, \end{aligned} \quad (12)$$

$$\begin{aligned} \|T_{f+\bar{g}}^*h\|^2 &= \left| \sum_{k=1}^{\infty} (k+1) \bar{f}_k h_{k+1} \right|^2 \\ &\quad + \sum_{m=1}^{\infty} \frac{1}{m+1} \left| \sum_{k=1}^{\infty} (m+k+1) \bar{f}_k h_{m+k+1} \right. \\ &\quad \left. + \sum_{l=1}^m l g_{m-l+1} h_l \right|^2. \end{aligned}$$

Denote $e_i(z) = (1/i)z^i$ for $i \geq 1$. Since T_u is hyponormal, we have

$$\|T_u e_i\|^2 - \|T_u^* e_i\|^2 \geq 0 \quad \text{for } i \geq 1. \quad (13)$$

For $i \geq 2$, $\|T_u e_i\|^2 - \|T_u^* e_i\|^2 \geq 0$ implies that

$$\begin{aligned} \frac{1}{i^2} \left(\sum_{k=1}^{i-1} |g_k|^2 + \sum_{l=1}^{\infty} |f_l|^2 \right) &\geq \sum_{k=1}^{i-1} \frac{1}{i-k} |f_k|^2 + \sum_{l=1}^{\infty} \frac{1}{i+l} |g_l|^2 \\ &\geq \frac{1}{i} |f_1|^2. \end{aligned} \quad (14)$$

Hence

$$\left(\sum_{i=2}^N \frac{1}{i^2} \right) \left(\sum_{l=1}^{\infty} |g_l|^2 + \sum_{l=1}^{\infty} |f_l|^2 \right) \geq \left(\sum_{i=2}^N \frac{1}{i} \right) |f_1|^2 \quad \text{for } N \geq 2. \quad (15)$$

Letting $N \rightarrow \infty$, since $\sum_{i=2}^N (1/i^2)$ and $(\sum_{l=1}^{\infty} |g_l|^2 + \sum_{l=1}^{\infty} |f_l|^2)$ are convergent and $\sum_{i=2}^N (1/i)$ is divergent, we get $f_1 = 0$. Similarly, by choosing i , we get $f_l = 0$ for $l \geq 1$. Note that $\|T_u e_1\|^2 - \|T_u^* e_1\|^2 \geq 0$ implies that $\sum_{l=1}^{\infty} |f_l|^2 \geq \sum_{l=1}^{\infty} (1/(l+1)) |g_l|^2$. Thus $g_l = 0$ for $l \geq 1$ and the proof is finished. $\square$

The following corollary generalizes Theorems 1 and 2 in [10]. Denote

$$\Omega = \{u : u = f + \bar{g}, f, g \in H^{\infty}(\mathbb{D}), |f|^2 dA \text{ is a } \mathcal{D}\text{-Carleson measure}\}, \quad (16)$$

where $H^{\infty}(\mathbb{D})$ is the space of the bounded analytic functions on $\mathbb{D}$.

Corollary 3. *Let $u \in \Omega$. Then T_u is hyponormal on $\mathcal{D}$ if and only if u is a constant function.*

3. Case on the Harmonic Dirichlet Space

In this section, we will characterize the hyponormality of T_u with $u \in W^{1,\infty}(\mathbb{D})$ on $\mathcal{D}_h$.

The harmonic Dirichlet space $\mathcal{D}_h$ consists of all harmonic functions in $W^{1,2}(\mathbb{D})$. It is a closed subspace of $W^{1,2}(\mathbb{D})$, and hence it is a Hilbert space with the following reproducing kernel:

$$R_z(w) = \overline{K_z(w)} + K_z(w) + 1 = \ln \frac{1}{1-z\bar{w}} + \ln \frac{1}{1-\bar{z}w} + 1. \quad (17)$$

Let Q be the orthogonal projection of $W^{1,2}(\mathbb{D})$ onto $\mathcal{D}_h$. Q is an integral operator represented by

$$\begin{aligned} Q(f)(z) &= \langle f, R_z \rangle \\ &= \int_{\mathbb{D}} \frac{\partial f}{\partial w} \left(\overline{\frac{\partial K_z}{\partial w}} \right) dA(w) \\ &\quad + \int_{\mathbb{D}} \frac{\partial f}{\partial \bar{w}} \left(\overline{\frac{\partial K_z}{\partial \bar{w}}} \right) dA(w) + \int_{\mathbb{D}} f dA. \end{aligned} \quad (18)$$

For $u \in W^{1,\infty}(\mathbb{D})$, the Toeplitz operator $\widetilde{T}_u$ with symbol u is defined by

$$\widetilde{T}_u h = Q(uh) \quad h \in \mathcal{D}_h. \quad (19)$$

$\widetilde{T}_u$ is a bounded operator for $u \in W^{1,\infty}(\mathbb{D})$ on $\mathcal{D}_h$ (see [11]).

Theorem 4. *Let $u = u_0 + c + f + \bar{g} \in W^{1,\infty}(\mathbb{D})$ with $u_0 + c \in \mathcal{A}$, $f, g \in H(\mathbb{D})$, and $f(0) = g(0) = 0$. Then $\widetilde{T}_u$ is hyponormal on $\mathcal{D}_h$ if and only if $u \in \mathcal{A}$.*

Proof. By Proposition 1, we only need to prove the necessity with $u(z) = f(z) + \bar{g}(z) = \sum_{k=1}^{\infty} f_k z^k + \sum_{k=1}^{\infty} \bar{g}_k \bar{z}^k$.

Let $h(z) = a_0 + \sum_{k=1}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k \bar{z}^k = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k \bar{z}^k \in \mathcal{D}_h$. Since $\widetilde{T}_u$ is hyponormal on $\mathcal{D}_h$, we have $\|\widetilde{T}_{f+\bar{g}} h\|^2 - \|\widetilde{T}_{f+\bar{g}}^* h\|^2 \geq 0$. Note that

$$\begin{aligned} \widetilde{T}_{f+\bar{g}} h(z) &= Q[(f+\bar{g})h](z) = \langle (f+\bar{g})h, R_z \rangle \\ &= \int_{\mathbb{D}} \frac{\partial [(f+\bar{g})h]}{\partial w} \frac{z}{1-z\bar{w}} dA \\ &\quad + \int_{\mathbb{D}} \frac{\partial [(f+\bar{g})h]}{\partial \bar{w}} \frac{\bar{z}}{1-\bar{z}w} dA + \int_{\mathbb{D}} (f+\bar{g})h dA \\ &= \sum_{k=1}^{\infty} \frac{1}{k+1} (b_k f_k + a_k \bar{g}_k) \\ &\quad + \left[a_0 f_1 + \sum_{k=1}^{\infty} (b_k f_{k+1} + a_{k+1} \bar{g}_k) \right] z \\ &\quad + \sum_{m=2}^{\infty} \left[\sum_{k=1}^{m-1} a_{m-k} f_k + a_0 f_m + \sum_{k=1}^{\infty} (b_k f_{k+m} + a_{k+m} \bar{g}_k) \right] z^m \\ &\quad + \left[a_0 \bar{g}_1 + \sum_{k=1}^{\infty} (a_k \bar{g}_{k+1} + b_{k+1} f_k) \right] \bar{z} \\ &\quad + \sum_{m=2}^{\infty} \left[\sum_{k=1}^{m-1} b_{m-k} \bar{g}_k + a_0 \bar{g}_m + \sum_{k=1}^{\infty} (a_k \bar{g}_{k+m} + b_{k+m} f_k) \right] \bar{z}^m. \end{aligned} \quad (20)$$

Thus

$$\begin{aligned} \|\widetilde{T}_{f+\bar{g}} h\|^2 &= \left| \int_{\mathbb{D}} \widetilde{T}_{f+\bar{g}} h dA \right|^2 \\ &\quad + \int_{\mathbb{D}} \left| \frac{\partial \widetilde{T}_{f+\bar{g}} h}{\partial z} \right|^2 dA + \int_{\mathbb{D}} \left| \frac{\partial \widetilde{T}_{f+\bar{g}} h}{\partial \bar{z}} \right|^2 dA \\ &= \left| \sum_{k=1}^{\infty} \frac{1}{k+1} (b_k f_k + a_k \bar{g}_k) \right|^2 \\ &\quad + \left| a_0 f_1 + \sum_{k=1}^{\infty} (b_k f_{k+1} + a_{k+1} \bar{g}_k) \right|^2 \\ &\quad + \sum_{m=2}^{\infty} \left| \sum_{k=1}^{m-1} a_{m-k} f_k + a_0 f_m \right. \\ &\quad \quad \left. + \sum_{k=1}^{\infty} (b_k f_{k+m} + a_{k+m} \bar{g}_k) \right|^2 \\ &\quad + \left| a_0 \bar{g}_1 + \sum_{k=1}^{\infty} (a_k \bar{g}_{k+1} + b_{k+1} f_k) \right|^2 \end{aligned}$$

$$\begin{aligned}
& + \sum_{m=2}^{\infty} \left| \sum_{k=1}^{m-1} b_{m-k} \bar{g}_k + a_0 \bar{g}_m \right. \\
& \quad \left. + \sum_{k=1}^{\infty} (a_k \bar{g}_{k+m} + b_{k+m} f_k) \right|^2.
\end{aligned} \tag{21}$$

Similarly, we have

$$\begin{aligned}
\tilde{T}_{f+\bar{g}}^* h(z) &= \langle \tilde{T}_{f+\bar{g}}^* h, R_z \rangle = \langle h, \tilde{T}_{f+\bar{g}} R_z \rangle = \langle h, (f + \bar{g}) R_z \rangle \\
&= \sum_{k=1}^{\infty} k (a_k \bar{f}_k + b_k g_k) \\
&\quad + \left\{ \frac{a_0}{2} g_1 + \sum_{k=1}^{\infty} [k b_k g_{k+1} + (k+1) a_{k+1} \bar{f}_k] \right\} z \\
&\quad + \sum_{m=2}^{\infty} \frac{1}{m} \left\{ \sum_{k=1}^{m-1} k a_k g_{m-k} + \frac{a_0}{m+1} g_m \right. \\
&\quad \quad \left. + \sum_{k=1}^{\infty} [(m+k) a_{k+m} \bar{f}_k + k b_k g_{k+m}] \right\} z^m \\
&\quad + \left\{ \frac{a_0}{2} \bar{f}_1 + \sum_{k=1}^{\infty} [k a_k \bar{f}_{k+1} + (k+1) b_{k+1} g_k] \right\} \bar{z} \\
&\quad + \sum_{m=2}^{\infty} \frac{1}{m} \left\{ \sum_{k=1}^{m-1} k b_k \bar{f}_{m-k} + \frac{a_0}{m+1} \bar{f}_m \right. \\
&\quad \quad \left. + \sum_{k=1}^{\infty} [(m+k) b_{k+m} g_k + k a_k \bar{f}_{k+m}] \right\} \bar{z}^m
\end{aligned}$$

$$\begin{aligned}
\|\tilde{T}_{f+\bar{g}}^* h\|^2 &= \left| \int_{\mathbb{D}} \tilde{T}_{f+\bar{g}}^* h dA \right|^2 + \int_{\mathbb{D}} \left| \frac{\partial (\tilde{T}_{f+\bar{g}}^* h)}{\partial z} dA \right|^2 \\
&\quad + \int_{\mathbb{D}} \left| \frac{\partial (\tilde{T}_{f+\bar{g}}^* h)}{\partial \bar{z}} \right|^2 dA = \left| \sum_{k=1}^{\infty} k (a_k \bar{f}_k + b_k g_k) \right|^2 \\
&\quad + \left| \frac{a_0}{2} g_1 + \sum_{k=1}^{\infty} [k b_k g_{k+1} + (k+1) a_{k+1} \bar{f}_k] \right|^2 \\
&\quad + \sum_{m=2}^{\infty} \frac{1}{m} \left| \sum_{k=1}^{m-1} k a_k g_{m-k} + \frac{a_0}{m+1} g_m \right. \\
&\quad \quad \left. + \sum_{k=1}^{\infty} [(m+k) a_{k+m} \bar{f}_k + k b_k g_{k+m}] \right|^2 \\
&\quad + \left| \frac{a_0}{2} \bar{f}_1 + \sum_{k=1}^{\infty} [k a_k \bar{f}_{k+1} + (k+1) b_{k+1} g_k] \right|^2
\end{aligned}$$

$$\begin{aligned}
& + \sum_{m=2}^{\infty} \frac{1}{m} \left| \sum_{k=1}^{m-1} k b_k \bar{f}_{m-k} + \frac{a_0}{m+1} \bar{f}_m \right. \\
& \quad \left. + \sum_{k=1}^{\infty} [(m+k) b_{k+m} g_k + k a_k \bar{f}_{k+m}] \right|^2.
\end{aligned} \tag{22}$$

For $i \geq 2$, let $a_i = 1/i$ and $a_j = 0$ for $j \neq i$. It follows that

$$\begin{aligned}
& \frac{1}{i^2} \left(\sum_{k=1}^{\infty} |f_k|^2 + \sum_{k=1}^{i-1} |g_k|^2 + \left| \frac{1}{i+1} g_i \right|^2 + \sum_{k=i+1}^{\infty} |g_k|^2 \right) \\
& \geq \sum_{k=1}^{\infty} \frac{1}{i+k} |g_k|^2 + |f_i|^2 + \sum_{k=1}^{i-1} \frac{1}{k} |f_{i-k}|^2 + \sum_{k=1}^{\infty} \frac{1}{k} |f_{i+k}|^2.
\end{aligned} \tag{23}$$

Therefore,

$$\frac{1}{i^2} \left(\sum_{k=1}^{\infty} |f_k|^2 + \sum_{k=1}^{\infty} |g_k|^2 \right) \geq \sum_{k=1}^{\infty} \frac{1}{i+k} (|f_k|^2 + |g_k|^2). \tag{24}$$

For every $k \geq 1$, we have

$$\left(\sum_{i=2}^N \frac{1}{i^2} \right) \left(\sum_{l=1}^{\infty} |f_l|^2 + \sum_{l=1}^{\infty} |g_l|^2 \right) \geq \left(\sum_{i=2}^N \frac{1}{i+k} \right) (|f_k|^2 + |g_k|^2), \tag{25}$$

where $N \geq 2$. Letting $N \rightarrow \infty$, Since $\sum_{i=2}^N (1/i^2)$ and $(\sum_{l=1}^{\infty} |f_l|^2 + \sum_{l=1}^{\infty} |g_l|^2)$ are convergent and $\sum_{i=2}^N (1/(i+k))$ ($k \geq 1$ is fixed) is divergent, we get $|f_k| = |g_k| = 0$ for $k \geq 1$. The proof is finished. $\square$

The following corollary generalizes Theorem 3 in [10].

Corollary 5. Suppose that $u = f + \bar{g} \in W^{1,\infty}(\mathbb{D})$ with $f, g \in H(\mathbb{D})$. Then T_u is hyponormal on $\mathcal{D}_h$ if and only if u is a constant function.

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