Research Article

Stability of a Functional Equation Deriving from Quadratic and Additive Functions in Non-Archimedean Normed Spaces

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We obtain the general solution of the generalized mixed additive and quadratic functional equation $f(x + my) + f(x - my) = 2f(x) - 2m^2 f(y) + m^2 f(2y)$, m is even; $f(x + y) + f(x - y) - 2(m^2 - 1)f(y) + (m^2 - 1)f(2y)$, m is odd, for a positive integer m . We establish the Hyers-Ulam stability for these functional equations in non-Archimedean normed spaces when m is an even positive integer or $m = 3$.

1. Introduction

The basic problem of the stability of functional equations was formulated by Ulam in 1940 in the following form. Suppose that a mapping f satisfies the additive functional equation $f(x + y) = f(x) + f(y)$ only approximately. Then does there exist an additive function which approximates f ? (See also [1].) In 1941, Hyers [2] gave the following answer to this question for Banach spaces. The result of Hyers was generalized in 1950 by Aoki [3] for approximately additive mappings and in 1978 by Rassias [4] for approximately linear mappings, by considering the unbounded Cauchy differences. A further generalization was obtained by Găvruta [5] in 1994, by replacing the Cauchy differences by a control function φ satisfying a very simple condition of convergence.

The Hyers-Ulam stability problem for the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad (1)$$

was first proved by Skof for a function $f: \mathcal{X} \rightarrow \mathcal{Y}$, where $\mathcal{X}$ is a normed space and $\mathcal{Y}$ is a Banach space [6]. One year later, Cholewa [7] demonstrated that Skof's theorem is still true if relevant domain is replaced by an abelian group. After that, in [8], Czerwik proved the Hyers-Ulam stability of the quadratic functional equation (1) as a special case. In [9], it was shown

that a mapping $f: \mathcal{X} \rightarrow \mathcal{Y}$ is quadratic if and only if $f(kx + y) + f(kx - y) = 2k^2 f(x) + 2f(y)$ for all $x, y \in \mathcal{X}$. Also, f is quadratic if and only if $2f((kx + ky)/2) + 2f((kx - ky)/2) = k^2 f(x) + k^2 f(y)$ for all $x, y \in \mathcal{X}$ [10]. Cădariu and Radu investigated the stability of the Cauchy functional equation [11] and for the quadratic functional equation [12]. Stability problems of miscellaneous functional equations have been investigated by several authors during the last decades (see, e.g., [13–15]).

In [16], Eskandani et al. determined the general solution of the following mixed additive and quadratic functional equation:

$$f(x + 2y) + f(x - 2y) + 8f(y) = 2f(x) + 4f(2y). \quad (2)$$

They studied the Hyers-Ulam stability of (2) in non-Archimedean Banach modules over a unital Banach algebra. In [17], Najati and Moghimi established the general solution of the mixed type additive and quadratic functional equation

$$\begin{aligned} f(2x + y) + f(2x - y) \\ = f(x + y) + f(x - y) + 2f(2x) - 2f(x) \end{aligned} \quad (3)$$

and investigated the stability of this equation in quasi-Banach spaces. The stability of (3) in random normed spaces is proved in [18].

In this paper, we consider the following functional equations:

$$f(x + my) + f(x - my) = 2f(x) - 2m^2 f(y) + m^2 f(2y), \quad (4)$$

where m is an even positive integer and

$$\begin{aligned} f(x + 3y) + f(x - 3y) \\ = f(x + y) + f(x - y) - 16f(y) + 8f(2y). \end{aligned} \quad (5)$$

Indeed, (4) and (5) are different from (2) and (3). It is easily verified that the function $f(x) = \alpha x^2 + \beta x$ is a solution of the functional equations (4) and (5). We show that these functional equations are mixed additive and quadratic mappings. We also prove the Hyers-Ulam stability problem for these equations. As a corollary, the hyperstability of (4) and (5) under some conditions in non-Archimedean normed spaces is shown as well.

2. General Solution of (4) and (5)

To achieve our aim in this paper, we need the following lemma which is a fundamental tool.

Lemma 1. Let $\mathcal{X}$ and $\mathcal{Y}$ be real vector spaces.

- (i) If an odd mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies the functional equation (4), then f is additive.
- (ii) If an odd mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies the functional equation (5), then f is additive.
- (iii) If an even mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies the functional equation (4), then f is quadratic.
- (iv) If an even mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies the functional equation (5), then f is quadratic.

Proof. (i) Letting $x = 0$ in (4), we get $f(2y) = 2f(y)$ for all $y \in \mathcal{X}$. This equality implies that

$$f(x + my) + f(x - my) = 2f(x) \quad (6)$$

for all $x, y \in \mathcal{X}$. Replacing y by y/m in (6), we have

$$f(x + y) + f(x - y) = 2f(x) \quad (7)$$

for all $x, y \in \mathcal{X}$. Substituting x, y by y, x in (7), respectively, we obtain

$$f(x + y) - f(x - y) = 2f(y) \quad (8)$$

for all $x, y \in \mathcal{X}$. The equalities (7) and (8) show that

$$f(x + y) = f(x) + f(y) \quad (x, y \in \mathcal{X}). \quad (9)$$

(ii) Suppose that f satisfies (5). Similar to the part (i), by the oddness of f , we have $f(2x) = 2f(x)$ for all $x \in \mathcal{X}$. Thus

$$f(x + 3y) + f(x - 3y) = f(x + y) + f(x - y) \quad (10)$$

for all $x, y \in \mathcal{X}$. We substitute x by $x + y$ in (10) and then x by $x - y$ in (10); we get

$$f(x + 4y) + f(x - 2y) = f(x + 2y) + f(x), \quad (11)$$

$$f(x + 2y) + f(x - 4y) = f(x) + f(x - 2y) \quad (12)$$

for all $x, y \in \mathcal{X}$. Then, by adding (11) to (12), we lead to

$$f(x + 4y) + f(x - 4y) = 2f(x) \quad (13)$$

for all $x, y \in \mathcal{X}$. Similar to the part (i), we can show that f is additive.

(iii) By the assumption, the equality

$$f(x + my) + f(x - my) = 2f(x) - 2m^2 f(y) + m^2 f(2y) \quad (14)$$

holds for a fixed even positive integer m . Putting $x = y = 0$ in (14), we get $f(0) = 0$. Once more, by letting $x = 0$ in (14), we have

$$2f(my) = -2m^2 f(y) + m^2 f(2y) \quad (15)$$

for all $y \in \mathcal{X}$. Interchanging x, y into mx, x in (14), respectively, we deduce that

$$f(2mx) = 2f(mx) - 2m^2 f(x) + m^2 f(2x) \quad (16)$$

for all $x \in \mathcal{X}$. Plugging (15) into (16), we have $f(2mx) = 4f(mx)$ for all $x \in \mathcal{X}$ and thus $f(2x) = 4f(x)$ for all $x \in \mathcal{X}$. Using the last equality and (14), we have

$$f(x + my) + f(x - my) = 2f(x) + 2m^2 f(y) \quad (17)$$

for all $x, y \in \mathcal{X}$. Setting $x = 0$ in (17), we obtain $f(my) = m^2 f(y)$ for all $y \in \mathcal{X}$. Applying this equality and putting x by mx in (17), we get

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad (18)$$

for all $x, y \in \mathcal{X}$. This shows that f is a quadratic mapping.

(iv) Suppose that f satisfies (5). Replacing x by $x + y$ and $x - y$ in (5), respectively, we have

$$\begin{aligned} f(x + 4y) + f(x - 2y) \\ = f(x + 2y) + f(x) - 16f(y) + 8f(2y), \end{aligned} \quad (19)$$

$$\begin{aligned} f(x + 2y) + f(x - 4y) \\ = f(x) + f(x - 2y) - 16f(y) + 8f(2y) \end{aligned} \quad (20)$$

for all $x, y \in \mathcal{X}$. The equalities (19) and (20) imply that

$$f(x + 4y) + f(x - 4y) = 2f(x) - 32f(y) + 16f(2y) \quad (21)$$

for all $x, y \in \mathcal{X}$. Now, the above equality is a special case of the part (iii) when $m = 4$. $\square$

In the following theorem, we solve (4) in which m is an even positive integer and

$$\begin{aligned} f(x+my) + f(x-my) \\ = f(x+y) + f(x-y) \\ - 2(m^2-1)f(y) + (m^2-1)f(2y), \end{aligned} \quad (22)$$

where m is an odd positive integer.

Theorem 2. Let $\mathcal{X}$ and $\mathcal{Y}$ be real vector spaces. Then a mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies the functional equation (2) if and only if it satisfies

$$\begin{aligned} f(x+my) + f(x-my) \\ = \begin{cases} 2f(x) - 2m^2f(y) + m^2f(2y), & m \text{ is even,} \\ f(x+y) + f(x-y) \\ - 2(m^2-1)f(y) + (m^2-1)f(2y), & m \text{ is odd} \end{cases} \end{aligned} \quad (23)$$

for all $m \geq k$, where k is a fixed positive integer with $k \geq 3$.

Proof. Suppose that $f : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies the functional equation (2). Putting $x = y = 0$ in (2), we get $f(0) = 0$. Replacing x by $x+y$ and $x-y$ in (2), respectively, we have

$$\begin{aligned} f(x+3y) + f(x-3y) \\ = f(x+y) + f(x-y) - 16f(y) + 8f(2y). \end{aligned} \quad (24)$$

Similar to the above, we get

$$f(x+4y) + f(x-4y) = 2f(x) - 32f(y) + 16f(2y). \quad (25)$$

Using the above method, we can deduce that

$$\begin{aligned} f(x+my) + f(x-my) \\ = \begin{cases} 2f(x) - 2p_m f(y) + p_m f(2y), & m \text{ is even,} \\ f(x+y) + f(x-y) \\ - 2p_m f(y) + p_m f(2y), & m \text{ is odd} \end{cases} \end{aligned} \quad (26)$$

for all $x, y \in \mathcal{X}$ for which $p_2 = 4$, $p_3 = 8$ and

$$p_m = \begin{cases} 2p_{m-1} - p_{m-2} + 4 & m \text{ is even,} \\ 2p_{m-1} - p_{m-2} & m \text{ is odd.} \end{cases} \quad (27)$$

Solving the above recurrence equations, we get

$$p_m = \begin{cases} m^2 & m \text{ is even,} \\ m^2 - 1 & m \text{ is odd} \end{cases} \quad (28)$$

for all positive integers $m \geq 2$.

Conversely, assume that $f : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies the functional equations (4) and (22) for each $k \geq m$. Firstly, we

assume that m is even. For $k = m(m-1)$ and for each $x, y \in \mathcal{X}$, we have

$$\begin{aligned} f(x+m(m-1)y) + f(x-m(m-1)y) \\ = 2f(x) - 2m^2f((m-1)y) + m^2f(2(m-1)y) \end{aligned} \quad (29)$$

for all $x, y \in \mathcal{X}$. On the other hand,

$$\begin{aligned} f(x+(m^2-m)y) + f(x-(m^2-m)y) \\ = 2f(x) - 2(m^2-m)^2f(y) + (m^2-m)^2f(2y) \end{aligned} \quad (30)$$

for all $x, y \in \mathcal{X}$. It follows from (29) and (30) that

$$\begin{aligned} -2f((m-1)y) + f(2(m-1)y) \\ = -2(m-1)^2f(y) + (m-1)^2f(2y) \end{aligned} \quad (31)$$

for all $x, y \in \mathcal{X}$. Since $(m+1)(m-1)$ is an odd number, we have

$$\begin{aligned} f(x+(m+1)(m-1)y) + f(x-(m+1)(m-1)y) \\ = f(x+(m-1)y) + f(x-(m-1)y) \\ - 2((m+1)^2-1)f((m-1)y) \end{aligned} \quad (32)$$

for all $x, y \in \mathcal{X}$. Also,

$$\begin{aligned} f(x+(m^2-1)y) + f(x-(m^2-1)y) \\ = f(x+y) + f(x-y) - 2((m^2-1)^2-1)f(y) \\ + ((m^2-1)^2-1)f(2y) \end{aligned} \quad (33)$$

for all $x, y \in \mathcal{X}$. Plugging (32) into (33) and using (31), we get

$$\begin{aligned} f(x+(m-1)y) + f(x-(m-1)y) \\ = f(x+y) + f(x-y) - 2((m-1)^2-1)f(y) \\ + ((m-1)^2-1)f(2y). \end{aligned} \quad (34)$$

For the odd case m , we have

$$\begin{aligned} f(x+(m+1)(m-1)y) + f(x-(m+1)(m-1)y) \\ = 2f(x) - 2(m+1)^2f((m-1)y) \\ + (m+1)^2f(2(m-1)y) \end{aligned} \quad (35)$$

for all $x, y \in \mathcal{X}$. Also

$$\begin{aligned} f(x+(m^2-1)y) + f(x-(m^2-1)y) \\ = 2f(x) - 2(m^2-1)^2f(y) + (m^2-1)^2f(2y) \end{aligned} \quad (36)$$

for all $x, y \in \mathcal{X}$. The comparison of (35) and (36) shows that

$$\begin{aligned} & -2f((m-1)y) + f(2(m-1)y) \\ & = -2(m-1)^2 f(y) + (m-1)^2 f(2y) \end{aligned} \quad (37)$$

for all $x, y \in \mathcal{X}$. For $k = m(m-1)$ and for each $x, y \in \mathcal{X}$, we have

$$\begin{aligned} & f(x + m(m-1)y) + f(x - m(m-1)y) \\ & = f(x + (m-1)y) + f(x - (m-1)y) \\ & \quad - 2(m^2 - 1)f((m-1)y) + (m^2 - 1)f(2(m-1)y) \end{aligned} \quad (38)$$

for all $x, y \in \mathcal{X}$. On the other hand,

$$\begin{aligned} & f(x + (m^2 - m)y) + f(x - (m^2 - m)y) \\ & = 2f(x) - 2(m^2 - m)^2 f(y) + (m^2 - m)^2 f(2y) \end{aligned} \quad (39)$$

for all $x, y \in \mathcal{X}$. Now, by comparing (38) with (39) and applying (37), we obtain

$$\begin{aligned} & f(x + (m-1)y) + f(x - (m-1)y) \\ & = 2f(x) - 2(m-1)^2 f(y) + (m-1)^2 f(2y) \end{aligned} \quad (40)$$

for all $x, y \in \mathcal{X}$. This completes the proof. $\square$

Theorem 3. Let $\mathcal{X}$ and $\mathcal{Y}$ be real vector spaces. A mapping $f: \mathcal{X} \rightarrow \mathcal{Y}$ satisfies either (4) or (5) if and only if there exist a symmetric biadditive mapping $B: \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{Y}$ and an additive mapping $A: \mathcal{X} \rightarrow \mathcal{Y}$ such that $f(x) = B(x, x) + A(x)$ for all $x \in \mathcal{X}$.

Proof. Assume that there exist a symmetric biadditive mapping $B: \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{Y}$ and an additive mapping $A: \mathcal{X} \rightarrow \mathcal{Y}$ such that $f(x) = B(x, x) + A(x)$ for all $x \in \mathcal{X}$. A simple computation shows that the mappings A and B' : $\mathcal{X} \rightarrow \mathcal{Y}$ given by $B'(x) = B(x, x)$ satisfy the functional equations (4) and (5). Therefore the mapping f satisfies (4) and (5).

Conversely, we decompose f into the even part and odd part by setting

$$f_e(x) = \frac{f(x) + f(-x)}{2}, \quad f_o(x) = \frac{f(x) - f(-x)}{2} \quad (41)$$

for all $x \in \mathcal{X}$. Obviously, $f(x) = f_e(x) + f_o(x)$ for all $x \in \mathcal{X}$. One can easily check that the mappings f_e and f_o satisfy (4) and (5). It follows from Lemma 1 that the mappings f_e and f_o are quadratic and additive, respectively. Since f_e is quadratic, by [19], there exists a symmetric biadditive mapping $B: \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{Y}$ such that $f_e(x) = B(x, x)$ for all $x \in \mathcal{X}$. Thus $f(x) = B(x, x) + A(x)$ for all $x \in \mathcal{X}$, where $A(x) = f_o(x)$ for all $x \in \mathcal{X}$. $\square$

3. Hyers-Ulam Stability of (4) and (5)

We recall some basic facts concerning non-Archimedean spaces and some preliminary results.

By a non-Archimedean field, we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Let $\mathcal{X}$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\|: \mathcal{X} \rightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

- (i) $\|x\| = 0$ if and only if $x = 0$;
- (ii) $\|rx\| = |r|\|x\|$, ($x \in \mathcal{X}$, $r \in \mathbb{K}$);
- (iii) the strong triangle inequality (ultrametric), namely,
$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in \mathcal{X}). \quad (42)$$

Then $(\mathcal{X}, \|\cdot\|)$ is called a non-Archimedean normed space. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\|; m \leq j \leq n-1\} \quad (n \geq m), \quad (43)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space $\mathcal{X}$. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.

In [20], Hensel discovered the p -adic numbers as a number theoretical analogue of power series in complex analysis. The most interesting example of non-Archimedean normed spaces is p -adic numbers. A key property of p -adic numbers is that they do not satisfy the Archimedean axiom; for all $x, y > 0$, there exists an integer n such that $x < ny$.

Let p be a prime number. For any nonzero rational number $x = p^r(m/n)$ in which m and n are coprime to the prime number p . Consider the p -adic absolute value $|x|_p = p^{-r}$ on $\mathbb{Q}$. It is easy to check that $|\cdot|_p$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ which is denoted by $\mathbb{Q}_p$ is said to be the p -adic number field. One should remember that if $p > 2$, then $|2^n| = 1$ for all integers n . The stability of some functional equations in non-Archimedean spaces was investigated, for instance, in [21–24] (see also [13, 25]).

Let m be an even positive integer. We use the abbreviation for the given mapping $f: \mathcal{X} \rightarrow \mathcal{Y}$ as follows:

$$\begin{aligned} \mathcal{D}_m f(x, y) &:= f(x + my) + f(x - my) \\ &\quad - 2f(x) + 2m^2 f(y) - m^2 f(2y), \\ \mathcal{D}_3 f(x, y) &:= f(x + 3y) + f(x - 3y) - f(x + y) \\ &\quad - f(x - y) + 16f(y) - 8f(2y). \end{aligned} \quad (44)$$

From now on, we assume that $\mathcal{X}$ is a real vector space and $\mathcal{Y}$ is a complete non-Archimedean space unless otherwise stated explicitly. In the upcoming theorem, we prove the stability of functional equations (4) and (5).

Theorem 4. Let $\phi : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ be a function such that

$$\lim_{k \rightarrow \infty} \frac{1}{|2|^k} \phi(2^k x, 2^k y) = 0 \quad (45)$$

for all $x, y \in \mathcal{X}$. Suppose that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an odd mapping satisfying the inequality

$$\|\mathcal{D}_m f(x, y)\| \leq \phi(x, y) \quad (46)$$

for all $x, y \in \mathcal{X}$, where m is an even positive integer or $m = 3$. Then there exists a unique additive mapping $A : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leq \begin{cases} \frac{1}{|2m^2|} \tilde{\phi}(x), & m \text{ is even,} \\ \frac{1}{|8|} \tilde{\phi}(x), & m = 3 \end{cases} \quad (47)$$

for all $x \in \mathcal{X}$, where $\tilde{\phi}(x) = \sup\{(\phi(0, 2^j x)/|2|^j) : j \in \mathbb{N} \cup \{0\}\}$.

Proof. We prove the result when m is even; another case is similar. Putting $x = 0$ in (46), we have

$$\|2f(y) - f(2y)\| \leq \frac{1}{|m|^2} \phi(0, y) \quad (48)$$

for all $y \in \mathcal{X}$. Replacing y by $2^n x$ in (48) and then dividing both sides by $|2|^{n+1}$, we get

$$\left\| \frac{1}{2^n} f(2^n x) - \frac{1}{2^{n+1}} f(2^{n+1} x) \right\| \leq \frac{1}{|m|^2 |2|^{n+1}} \phi(0, 2^n x) \quad (49)$$

for all $x \in \mathcal{X}$ and all nonnegative integers n . Thus the sequence $\{f(2^n x)/2^n\}$ is Cauchy by (45) and (49). Completeness of the non-Archimedean space $\mathcal{Y}$ allows us to assume that there exists a mapping A , so that

$$\lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n} = A(x). \quad (50)$$

For each $x \in \mathcal{X}$ and nonnegative integers n , we have

$$\begin{aligned} & \left\| \frac{f(2^n x)}{2^n} - f(x) \right\| \\ &= \left\| \sum_{j=0}^{n-1} \frac{f(2^{j+1} x)}{2^{j+1}} - \frac{f(2^j x)}{2^j} \right\| \\ &\leq \max \left\{ \left\| \frac{f(2^{j+1} x)}{2^{j+1}} - \frac{f(2^j x)}{2^j} \right\| : 0 \leq j < n \right\} \\ &\leq \frac{1}{|2m^2|} \max \left\{ \frac{\phi(0, 2^j x)}{|2|^j} : 0 \leq j < n \right\}. \end{aligned} \quad (51)$$

Taking that n tends to approach infinity in (51) and applying (50), we can see that inequality (47) holds when m is even. It follows from (45), (46), and (50) that for all $x, y \in \mathcal{X}$,

$$\begin{aligned} \|\mathcal{D}_m A(x, y)\| &= \lim_{n \rightarrow \infty} \frac{1}{|2|^n} \|\mathcal{D}_m f(2^n x, 2^n y)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{|2|^n} \phi(2^n x, 2^n y) = 0. \end{aligned} \quad (52)$$

Hence, the mapping A satisfies (4). Part (i) of Lemma 1 shows that the mapping A is additive. Now, let $A' : \mathcal{X} \rightarrow \mathcal{Y}$ be another additive mapping satisfying (47). Then we have

$$\begin{aligned} & \|A(x) - A'(x)\| \\ &= \lim_{k \rightarrow \infty} \frac{1}{|2|^k} \|A(2^k x) - A'(2^k x)\| \\ &\leq \lim_{k \rightarrow \infty} \frac{1}{|2|^k} \max \{ \|A(2^k x) - f(2^k x)\|, \\ &\quad \|f(2^k x) - A'(2^k x)\| \} \\ &\leq \frac{1}{|2m^2|} \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \max \left\{ \frac{\phi(0, 2^j x)}{|2|^j} : k \leq j < n+k \right\} \\ &= \frac{1}{|2m^2|} \lim_{k \rightarrow \infty} \sup \left\{ \frac{\phi(0, 2^j x)}{|2|^j} : k \leq j < \infty \right\} = 0 \end{aligned} \quad (53)$$

for all $x \in \mathcal{X}$. This shows the uniqueness of A . $\square$

We have the following result which is analogous to Theorem 4 for the functional equations (4) and (5). We include the proof for (4). The proof of (5) is similar.

Theorem 5. Let $\phi : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ be a function such that

$$\lim_{k \rightarrow \infty} |2|^k \phi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) = 0 \quad (54)$$

for all $x, y \in \mathcal{X}$. Suppose that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an odd mapping satisfying the inequality

$$\|\mathcal{D}_m f(x, y)\|_{\mathcal{Y}} \leq \phi(x, y) \quad (55)$$

for all $x, y \in \mathcal{X}$, where m is an even positive integer or $m = 3$. Then there exists a unique additive mapping $A : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leq \begin{cases} \frac{1}{|m|^2} \tilde{\phi}(x), & m \text{ is even,} \\ \frac{1}{|8|} \tilde{\phi}(x), & m = 3 \end{cases} \quad (56)$$

for all $x \in \mathcal{X}$, where $\tilde{\phi}(x) = \sup\{|2|^j \phi(0, (x/2^{j+1})) : j \in \mathbb{N} \cup \{0\}\}$.

Proof. We only obtain the result for the even integers. Similar to the proof of Theorem 4, we have

$$\|2f(y) - f(2y)\| \leq \frac{1}{|m|^2} \phi(0, y) \quad (57)$$

for all $y \in \mathcal{X}$. If we replace y by $x/2^{n+1}$ in inequality (57) and then multiply both sides of the result to $|2|^n$, we get

$$\left\| 2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) \right\| \leq \frac{|2|^n}{|m|^2} \phi\left(0, \frac{x}{2^{n+1}}\right) \quad (58)$$

for all $x \in \mathcal{X}$ and all nonnegative integers n . Thus, we conclude from (54) and (58) that the sequence $\{2^n f(x/2^n)\}$ is Cauchy. Since the non-Archimedean normed space $\mathcal{Y}$ is complete, this sequence converges in $\mathcal{Y}$ to the mapping A . Indeed,

$$A(x) = \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right) \quad (x \in \mathcal{X}). \quad (59)$$

Using induction and (57), one can show that

$$\begin{aligned} & \left\| 2^n f\left(\frac{x}{2^n}\right) - f(x) \right\| \\ & \leq \frac{1}{|m|^2} \max \left\{ |2|^j \phi\left(0, \frac{x}{2^{j+1}}\right) : 0 \leq j < n \right\} \end{aligned} \quad (60)$$

for all $x \in \mathcal{X}$ and nonnegative integers n . Since the right-hand side of inequality (60) tends to be 0 as n to approach infinity, by applying (59), we deduce inequality (56). Now, similar to the proof of Theorem 4, we can complete the rest of the proof. $\square$

Corollary 6. Let α, r , and s be positive real numbers such that $r, s \neq 1$ and $|2| < 1$. Suppose that $\mathcal{X}$ is a non-Archimedean normed space and $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an odd mapping fulfilling

$$\|\mathcal{D}_m f(x, y)\| \leq \alpha (\|x\|^r + \|y\|^s) \quad (61)$$

for all $x, y \in \mathcal{X}$, where m is an even positive integer or $m = 3$. Then there exists a unique additive mapping $A : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leq \begin{cases} \frac{\alpha \|x\|^s}{|2m^2|} & m \text{ is even, } r, s > 1, \\ \frac{\alpha \|x\|^s}{|8|} & m = 3, \quad r, s > 1, \\ \frac{\alpha \|x\|^s}{|m|^2 |2|^s} & m \text{ is even, } r, s < 1, \\ \frac{\alpha \|x\|^s}{|8| |2|^s} & m = 3, \quad r, s < 1 \end{cases} \quad (62)$$

for all $x \in \mathcal{X}$.

Proof. The result follows from Theorems 4 and 5 by letting $\phi(x, y) = \alpha(\|x\|^r + \|y\|^s)$. $\square$

In the next result, we prove the hyperstability of the functional equations (4) and (5) under some conditions. Recall that a functional equation is called *hyperstable* if every approximate solution is an exact one (see, e.g., [13, 26–29]).

Corollary 7. Let α, r , and s be positive real numbers such that $r + s \neq 1$ and $|2| < 1$. Suppose that $\mathcal{X}$ is a non-Archimedean normed space and $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an odd mapping fulfilling

$$\|\mathcal{D}_m f(x, y)\| \leq \alpha \|x\|^r \|y\|^s \quad (63)$$

for all $x, y \in \mathcal{X}$, where m is an even positive integer or $m = 3$. Then f is an additive mapping.

Proof. Taking $\phi(x, y) = \alpha \|x\|^r \|y\|^s$ in Theorems 4 and 5, we can obtain the desired result. $\square$

Theorem 8. Let $\phi : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ be a function such that

$$\lim_{k \rightarrow \infty} \frac{1}{|4|^k} \phi(2^k x, 2^k y) = 0 \quad (64)$$

for all $x, y \in \mathcal{X}$. Suppose that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an even mapping satisfying the inequality

$$\|\mathcal{D}_m f(x, y)\| \leq \phi(x, y) \quad (65)$$

for all $x, y \in \mathcal{X}$, where m is an even positive integer. Then there exists a unique quadratic mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|} \tilde{\phi}(x) \quad (66)$$

for all $x \in \mathcal{X}$, where $\tilde{\phi}(x) = \sup\{(\phi(0, (2^j x/m)/|4|^j), (\phi(2^j x, (2^j x/m)/|4|^j)) : j \in \mathbb{N} \cup \{0\}\}$.

Proof. It follows from (64) that $\phi(0, 0) = 0$. Thus (65) implies that $f(0) = 0$. Putting $x = 0$ in (65) and interchanging y into x , we have

$$\|2f(mx) + 2m^2 f(x) - m^2 f(2x)\| \leq \phi(0, x) \quad (67)$$

for all $x \in \mathcal{X}$. Substituting x, y by mx, x in (65), respectively, we get

$$\|f(2mx) - 2f(mx) + 2m^2 f(x) - m^2 f(2x)\| \leq \phi(mx, x) \quad (68)$$

for all $x \in \mathcal{X}$. It follows from (67) and (68) that

$$\|f(2mx) - 4f(mx)\| \leq \max\{\phi(0, x), \phi(mx, x)\} \quad (69)$$

for all $x \in \mathcal{X}$. Thus we have

$$\|f(2x) - 4f(x)\| \leq \max\left\{\phi\left(0, \frac{x}{m}\right), \phi\left(x, \frac{x}{m}\right)\right\} \quad (70)$$

for all $x \in \mathcal{X}$. Replacing x by $2^n x$ in (70) and then dividing both sides by $|4|^{n+1}$, we get

$$\begin{aligned} & \left\| \frac{1}{4^{n+1}} f(2^{n+1} x) - \frac{1}{4^n} f(2^n x) \right\| \\ & \leq \frac{1}{|4|} \max \left\{ \frac{\phi(0, (2^n x/m))}{|4|^n}, \frac{\phi(2^n x, (2^n x/m))}{|4|^n} \right\} \end{aligned} \quad (71)$$

for all $x \in \mathcal{X}$ and all nonnegative integers n . Thus the sequence $\{f(2^n x)/4^n\}$ is Cauchy by (64) and (71). Since $\mathcal{Y}$ is complete, the sequence $\{f(2^n x)/4^n\}$ converges in $\mathcal{Y}$ for all $x \in \mathcal{X}$. So one can define the mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ by

$$Q(x) := \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} \quad (x \in \mathcal{X}). \quad (72)$$

For each $x \in \mathcal{X}$ and nonnegative integers n , we have

$$\begin{aligned} & \left\| \frac{f(2^n x)}{4^n} - f(x) \right\| \\ &= \left\| \sum_{j=0}^{n-1} \frac{f(2^{j+1}x)}{4^{j+1}} - \frac{f(2^j x)}{4^j} \right\| \\ &\leq \max \left\{ \left\| \frac{f(2^{j+1}x)}{4^{j+1}} - \frac{f(2^j x)}{4^j} \right\| : 0 \leq j < n \right\} \\ &\leq \frac{1}{|4|} \max \left\{ \frac{\phi(0, (2^j x/m))}{|4|^j}, \right. \\ &\quad \left. \frac{\phi(2^j x, (2^j x/m))}{|4|^j} : 0 \leq j < n \right\}. \end{aligned} \quad (73)$$

Taking n to approach infinity in (73) and using (64) and (72), we find (66). Employing (64), (65), and (72), we obtain

$$\begin{aligned} \|\mathcal{D}_m Q(x, y)\| &= \lim_{n \rightarrow \infty} \frac{1}{|4|^n} \|\mathcal{D}_m f(2^n x, 2^n y)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{|4|^n} \phi(2^n x, 2^n y) = 0. \end{aligned} \quad (74)$$

Hence, the mapping Q satisfies (4). It follows from part (iii) of Lemma 1 that the mapping Q is quadratic. If $Q' : \mathcal{X} \rightarrow \mathcal{Y}$ is another quadratic mapping satisfying (66), then

$$\begin{aligned} & \|Q(x) - Q'(x)\| \\ &= \lim_{k \rightarrow \infty} \frac{1}{|4|^k} \|Q(2^k x) - Q'(2^k x)\| \\ &\leq \lim_{k \rightarrow \infty} \frac{1}{|4|^k} \max \left\{ \|Q(2^k x) - f(2^k x)\|, \right. \\ &\quad \left. \|f(2^k x) - Q'(2^k x)\| \right\} \\ &\leq \frac{1}{|4|} \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \max \left\{ \frac{\phi(0, (2^j x/m))}{|4|^j}, \right. \\ &\quad \left. \frac{\phi(2^j x, (2^j x/m))}{|4|^j} : k \leq j < n+k \right\} \\ &= \frac{1}{|4|} \lim_{k \rightarrow \infty} \sup \left\{ \frac{\phi(0, (2^j x/m))}{|4|^j}, \right. \\ &\quad \left. \frac{\phi(2^j x, (2^j x/m))}{|4|^j} : k \leq j < \infty \right\} = 0 \end{aligned} \quad (75)$$

for all $x \in \mathcal{X}$. Therefore $Q = Q'$. This completes the proof of the uniqueness of Q . $\square$

Theorem 9. Let $\phi : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ be a function such that

$$\lim_{k \rightarrow \infty} |4|^k \phi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) = 0 \quad (76)$$

for all $x, y \in \mathcal{X}$. Suppose that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an even mapping satisfying $f(0) = 0$ and the inequality

$$\|\mathcal{D}_m f(x, y)\| \leq \phi(x, y) \quad (77)$$

for all $x, y \in \mathcal{X}$, where m is an even positive integer. Then there exists a unique quadratic mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|} \tilde{\phi}(x) \quad (78)$$

for all $x \in \mathcal{X}$, where $\tilde{\phi}(x) = \sup\{|4|^n \phi(0, (x/2^n m)), |4|^n \phi((x/2^n), (x/2^n m)) : n \in \mathbb{N}\}$.

Proof. Similar to the proof of Theorem 8, we have

$$\|f(2x) - 4f(x)\| \leq \max \left\{ \phi\left(0, \frac{x}{m}\right), \phi\left(x, \frac{x}{m}\right) \right\} \quad (79)$$

for all $x \in \mathcal{X}$. Then we get

$$\left\| f(x) - 4f\left(\frac{x}{2}\right) \right\| \leq \max \left\{ \phi\left(0, \frac{x}{2m}\right), \phi\left(\frac{x}{2}, \frac{x}{2m}\right) \right\} \quad (80)$$

for all $x \in \mathcal{X}$. Replacing x by $x/2^n$ in (80) and multiplying both sides to $|4|^n$, we get

$$\begin{aligned} & \left\| 4^n f\left(\frac{x}{2^n}\right) - 4^{n+1} f\left(\frac{x}{2^{n+1}}\right) \right\| \\ &\leq \frac{1}{|4|} \max \left\{ |4|^{n+1} \phi\left(0, \frac{x}{2^{n+1}m}\right), |4|^{n+1} \phi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}m}\right) \right\} \end{aligned} \quad (81)$$

for all $x \in \mathcal{X}$ and all nonnegative integers n . Thus the sequence $\{4^n f(x/2^n)\}$ is Cauchy by (76). The completeness of $\mathcal{Y}$ implies that the mentioned sequence is convergent. So we consider the mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ by

$$Q(x) := \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right) \quad (x \in \mathcal{X}). \quad (82)$$

For each $x \in \mathcal{X}$ and nonnegative integers n , we have

$$\begin{aligned} & \left\| f(x) - 4^n f\left(\frac{x}{2^n}\right) \right\| \\ &\leq \frac{1}{|4|} \max \left\{ |4|^j \phi\left(0, \frac{x}{2^j m}\right), |4|^j \phi\left(\frac{x}{2^j}, \frac{x}{2^j m}\right) : 1 \leq j \leq n \right\}. \end{aligned} \quad (83)$$

Letting n approach infinity in (83) and using (76) and (82), we can see that (78) holds. The rest of the proof is similar to the proof of Theorem 8. $\square$

Corollary 10. Let α, r , and s be nonnegative real numbers and $|2| < 1$. Suppose that $\mathcal{X}$ is a non-Archimedean normed space and $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an even mapping fulfilling

$$\|\mathcal{D}_m f(x, y)\| \leq \begin{cases} \alpha \|x\|^r & r \neq 2, \\ \alpha (\|x\|^r + \|y\|^s) & r, s \neq 2 \end{cases} \quad (84)$$

for all $x, y \in \mathcal{X}$, where m is an even positive integer. Then there exists a unique quadratic mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \frac{\alpha \|x\|^r}{|4|} & r > 2, s = 0, \\ \frac{\alpha}{|4|} \left(\|x\|^r + \left\| \frac{x}{m} \right\|^s \right) & r, s > 2, \\ \frac{\alpha \|x\|^r}{|2|^r} & r < 2, s = 0, \\ \alpha \left(\frac{\|x\|^r}{|2|^r} + \left\| \frac{x}{2m} \right\|^s \right) & r, s < 2 \end{cases} \quad (85)$$

for all $x \in \mathcal{X}$.

Theorem 11. Let $\phi : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ be a function such that

$$\lim_{k \rightarrow \infty} \frac{1}{|4|^k} \phi(2^k x, 2^k y) = 0 \quad (86)$$

for all $x, y \in \mathcal{X}$. Suppose that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an even mapping satisfying the inequality

$$\|\mathcal{D}_3 f(x, y)\| \leq \phi(x, y) \quad (87)$$

for all $x, y \in \mathcal{X}$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|} \tilde{\phi}(x) \quad (88)$$

for all $x \in \mathcal{X}$, where $\tilde{\phi}(x) = \sup\{|2|(\phi((2^j x/4), (2^j x/4))/|4|^j), (\phi((5/4)2^j x, (2^j x/4))/|4|^j), (\phi((3/4)2^j x, (2^j x/4))/|4|^j) : j \in \mathbb{N} \cup \{0\}\}$.

Proof. Similar to the proof of Theorem 8, we can show that $f(0) = 0$. Replacing x by $x + y$ and $x - y$ in (87), respectively, we get

$$\begin{aligned} & \|f(x + 4y) + f(x - 2y) - f(x + 2y) \\ & - f(x) + 16f(y) - 8f(2y)\| \leq \phi(x + y, y), \end{aligned} \quad (89)$$

$$\begin{aligned} & \|f(x + 2y) + f(x - 4y) - f(x - 2y) \\ & - f(x) + 16f(y) - 8f(2y)\| \leq \phi(x - y, y) \end{aligned} \quad (90)$$

for all $x, y \in \mathcal{X}$. Inequalities (89) and (90) imply that

$$\begin{aligned} & \|f(x + 4y) + f(x - 4y) - 2f(x) + 32f(y) - 16f(2y)\| \\ & \leq \max\{\phi(x + y, y), \phi(x - y, y)\} \end{aligned} \quad (91)$$

for all $x, y \in \mathcal{X}$. Interchanging x, y into $4x, x$ in (91), respectively, we obtain

$$\begin{aligned} & \|f(8x) - 2f(4x) + 32f(x) - 16f(2x)\| \\ & \leq \max\{\phi(5x, x), \phi(3x, x)\} \end{aligned} \quad (92)$$

for all $x \in \mathcal{X}$. On the other hand, by putting $x = y$ in (87), we can deduce that

$$\|f(4x) - 8f(2x) + 16f(x)\| \leq \phi(x, x) \quad (93)$$

for all $x \in \mathcal{X}$. It follows from (92) and (93) that

$$\|f(8x) - 4f(4x)\| \leq \max\{|2|\phi(x, x), \phi(5x, x), \phi(3x, x)\} \quad (94)$$

for all $x \in \mathcal{X}$. Thus

$$\begin{aligned} & \left\| \frac{1}{4} f(2x) - f(x) \right\| \\ & \leq \frac{1}{|4|} \max\left\{|2|\phi\left(\frac{x}{4}, \frac{x}{4}\right), \phi\left(\frac{5}{4}x, \frac{x}{4}\right), \phi\left(\frac{3}{4}x, \frac{x}{4}\right)\right\} \end{aligned} \quad (95)$$

for all $x \in \mathcal{X}$. Substituting x by $2^n x$ in (95) and then dividing both sides by $|4|^n$, we obtain

$$\begin{aligned} & \left\| \frac{1}{4^{n+1}} f(2^{n+1}x) - \frac{1}{4^n} f(2^n x) \right\| \\ & \leq \frac{1}{|4|} \max\left\{|2| \frac{\phi((2^n x/4), (2^n x/4))}{|4|^n}, \right. \\ & \quad \left. \frac{\phi((5/4)2^n x, (2^n x/4))}{|4|^n}, \frac{\phi((3/4)2^n x, (2^n x/4))}{|4|^n}\right\} \end{aligned} \quad (96)$$

for all $x \in \mathcal{X}$ and all nonnegative integers n . Thus the sequence $\{f(2^n x)/4^n\}$ is Cauchy by (86) and (96). The completeness of $\mathcal{Y}$ implies that the sequence $\{f(2^n x)/4^n\}$ is convergent. Define the mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ via

$$Q(x) := \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} \quad (x \in \mathcal{X}). \quad (97)$$

By a simple computation, one can show that

$$\begin{aligned} & \left\| \frac{f(2^n x)}{4^n} - f(x) \right\| \\ & \leq \frac{1}{|4|} \max\left\{|2| \frac{\phi((2^j x/4), (2^j x/4))}{|4|^j}, \right. \\ & \quad \frac{\phi((5/4)2^j x, (2^j x/4))}{|4|^j}, \\ & \quad \left. \frac{\phi((3/4)2^j x, (2^j x/4))}{|4|^j} : 0 \leq j < n\right\} \end{aligned} \quad (98)$$

for all $x \in \mathcal{X}$ and for all $n \geq 0$. Taking n to approach infinity in (98) and applying (86) and (97), we find (88). By (86), (87), and (97), we have

$$\begin{aligned} & \|\mathcal{D}_3 Q(x, y)\| = \lim_{n \rightarrow \infty} \frac{1}{|4|^n} \|\mathcal{D}_3 f(2^n x, 2^n y)\| \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{|4|^n} \phi(2^n x, 2^n y) = 0. \end{aligned} \quad (99)$$

Hence, the mapping Q satisfies (5). It follows from part (iv) of Lemma 1 that the mapping Q is quadratic. Similar to the proof of Theorem 8, one can show that Q is unique. $\square$

Theorem 12. Let $\phi : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ be a function such that

$$\lim_{k \rightarrow \infty} |4|^k \phi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) = 0 \quad (100)$$

for all $x, y \in \mathcal{X}$. Suppose that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an even mapping satisfying the inequality

$$\|\mathcal{D}_3 f(x, y)\| \leq \phi(x, y) \quad (101)$$

for all $x, y \in \mathcal{X}$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|} \tilde{\phi}(x) \quad (x \in \mathcal{X}), \quad (102)$$

where $\tilde{\phi}(x) = \sup\{|2||4|^j \phi((x/2^{j+3}), (x/2^{j+3})), |4|^j \phi(5(x/2^{j+3}), (x/2^{j+3})), |4|^j \phi(3(x/2^{j+3}), (x/2^{j+3})) : j \in \mathbb{N} \cup \{0\}\}$.

Proof. Similar to the proof of Theorem 8, one can obtain

$$\|f(8x) - 4f(4x)\| \leq \max\{|2|\phi(x, x), \phi(5x, x), \phi(3x, x)\} \quad (103)$$

for all $x \in \mathcal{X}$. Thus

$$\begin{aligned} & \left\| f(x) - 4f\left(\frac{x}{2}\right) \right\| \\ & \leq \max\left\{|2|\phi\left(\frac{x}{8}, \frac{x}{8}\right), \phi\left(\frac{5x}{8}, \frac{x}{8}\right), \phi\left(\frac{3x}{8}, \frac{x}{8}\right)\right\} \end{aligned} \quad (104)$$

for all $x \in \mathcal{X}$. Substituting x by $x/2^n$ in (104) and then multiplying both sides to $|4|^n$, we have

$$\begin{aligned} & \left\| 4^n f\left(\frac{x}{2^n}\right) - 4^{n+1} f\left(\frac{x}{2^{n+1}}\right) \right\| \\ & \leq \max\left\{|2||4|^n \phi\left(\frac{x}{2^{n+3}}, \frac{x}{2^{n+3}}\right), \right. \\ & \quad \left. |4|^n \phi\left(5\frac{x}{2^{n+3}}, \frac{x}{2^{n+3}}\right), |4|^n \phi\left(3\frac{x}{2^{n+3}}, \frac{x}{2^{n+3}}\right)\right\} \end{aligned} \quad (105)$$

for all $x \in \mathcal{X}$ and all nonnegative integers n . The last inequality and (100) imply that the sequence $\{f(2^n x)/4^n\}$ is Cauchy. So, this sequence converges to the mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$. In other words,

$$Q(x) := \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} \quad (x \in \mathcal{X}). \quad (106)$$

We can also show that

$$\begin{aligned} & \left\| f(x) - 4^n f\left(\frac{x}{2^n}\right) \right\| \\ & \leq \frac{1}{|4|} \max\left\{|2||4|^j \phi\left(\frac{x}{2^{j+3}}, \frac{x}{2^{j+3}}\right), |4|^j \phi\left(5\frac{x}{2^{j+3}}, \frac{x}{2^{j+3}}\right), \right. \\ & \quad \left. |4|^j \phi\left(3\frac{x}{2^{j+3}}, \frac{x}{2^{j+3}}\right) : 0 \leq j < n\right\} \end{aligned} \quad (107)$$

for all $x \in \mathcal{X}$ and for all $n \geq 0$. Taking n to approach infinity in (107) and applying (100) and (101), we see that inequality (102) holds. Now, similar to the proof of Theorem 8, one can show that Q is a unique quadratic mapping. $\square$

Corollary 13. Let α and r be positive real numbers such that $r \neq 2$ and $|2| < 1$. Suppose that $\mathcal{X}$ is a non-Archimedean normed space and $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an even mapping fulfilling

$$\|\mathcal{D}_3 f(x, y)\| \leq \alpha \|y\|^r \quad (108)$$

for all $x, y \in \mathcal{X}$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \frac{|2| \|x\|^r}{|4|^{r+1}} \alpha & r > 2, \\ \frac{\|x\|^r}{|2| |8|^r} \alpha & r < 2 \end{cases} \quad (109)$$

for all $x \in \mathcal{X}$.

Proof. Letting $\phi(x, y) = \alpha \|y\|^r$ in Theorems 11 and 12, one can obtain the required result. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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