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Research Article

New Existence Results and Generalizations for Coincidence Points and Fixed Points without Global Completeness

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Some new existence theorems concerning approximate coincidence point property and approximate fixed point property for nonlinear maps in metric spaces without global completeness are established in this paper. By exploiting these results, we prove some new coincidence point and fixed point theorems which generalize and improve Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, Kikkawa-Suzuki's fixed point theorem, and some well known results in the literature. Moreover, some applications of our results to the existence of coupled coincidence point and coupled fixed point are also presented.

1. Introduction

Let us begin with some basic definitions and notations that will be needed in this paper. The symbols $\mathbb{N}$ and $\mathbb{R}$ are used to denote the sets of positive integers and real numbers, respectively. Let (X,d) be a metric space. Denote by $\mathcal{N}(X)$ the family of all nonempty subsets of X, $C(X)$ the class of all nonempty closed subsets of X, and $CB(X)$ the family of all nonempty closed and bounded subsets of X. For each $x \in X$ and $A \subseteq X$, let $d(x,A) = \inf_{y\in A} d(x,y)$. A function $\mathcal{H} : CB(X) \times CB(X) \to [0,\infty)$ defined by

$$\mathcal{H}(A,B) = \max\left\{\sup_{x\in B} d(x,A), \sup_{x\in A} d(x,B)\right\} \tag{1}$$

is said to be the Hausdorff metric on $CB(X)$ induced by the metric d on X.

Let $g : X \to X$ be a self-map and $T : X \to \mathcal{N}(X)$ be a multivalued map. A point v in X is said to be a *coincidence point* (see, for instance, [1–4]) of g and T if $gv \in Tv$. The set of coincidence points of g and T is denoted by $COP(g,T)$. If $v \in Tv$, then v is called a fixed point of T. The set of fixed points of T is denoted by $\mathcal{F}(T)$. The maps g and T are said to have an *approximate coincidence point property* [1, 4] on X provided $\inf_{x\in X} d(gx,Tx) = 0$. The map T is said to have the *approximate fixed point property* [1–5] on X provided $\inf_{x\in X} d(x,Tx) = 0$.

It is obvious that $COP(g,T) \neq \emptyset$ (resp., $\mathcal{F}(T) \neq \emptyset$) implies that T has the approximate coincidence point property (resp., T has the approximate fixed point property). Hussain et al. [1, Theorem 2.6] showed that a generalized multivalued almost contraction T in a metric space (X,d) have $\mathcal{F}(T) \neq \emptyset$ provided either (X,d) is compact and the function $f(x) = d(x,Tx)$ is l.s.c. or T is closed and compact. In [1, Lemma 2.2], the authors had also shown that every generalized multivalued almost contraction in a metric space (X,d) has the approximate fixed point property.

The rapid growth of fixed point theory and its applications over the past decades has led to a number of scholarly essays that examine its nature and its importance in nonlinear analysis, applied mathematical analysis, economics, game theory, and so forth; see [1–47] and references therein. Many authors devoted their attention to investigate its generalizations in various different directions of the celebrated Banach contraction principle. In 2008, Suzuki [6] presented a new type of generalization of the celebrated Banach contraction principle and does characterize the metric completeness.

Theorem 1 (Suzuki [6]). *Define a nonincreasing function θ from $[0,1)$ onto $(1/2,1]$ by*

$$\theta(r)=\begin{cases}1, & \text{if } 0\le r\le \frac{1}{2}\left(\sqrt{5}-1\right),\\ \frac{1-r}{r^2}, & \text{if } \frac{1}{2}\left(\sqrt{5}-1\right)\le r\le \frac{1}{\sqrt{2}},\\ \frac{1}{1+r}, & \text{if } \frac{1}{\sqrt{2}}\le r<1.\end{cases} \tag{2}$$

Then for a metric space (X,d), the following are equivalent:

(1) *X is complete.*

(2) *Every mapping T on X satisfying the following has a fixed point:*

there exists $r\in[0,1)$ such that $\theta(r)d(x,Tx)\le d(x,y)$ implies $d(Tx,Ty)\le rd(x,y)$ for all x, $y\in X$.

(3) *There exists $r\in[0,1)$ such that every mapping T on X satisfying the following has a fixed point:*

$(1/10000)d(x,Tx)\le d(x,y)$ implies $d(Tx,Ty)\le rd(x,y)$ for all $x,y\in X$.

Remark 2 (see [6]). For every $r\in[0,1)$, $\theta(r)$ is the best constant.

Later, Kikkawa and Suzuki [8] proved an interesting generalization of both Theorem 1 and the Nadler fixed point theorem [9] which is an extension of the Banach contraction principle to multivalued maps.

Theorem 3 (Kikkawa and Suzuki [8]). *Define a strictly decreasing function η from $[0,1)$ onto $(1/2,1]$ by*

$$\eta(r)=\frac{1}{1+r}. \tag{3}$$

Let (X,d) be a complete metric space and let T be a map from X into $CB(X)$. Assume that there exists $r\in[0,1)$ such that

$$\eta(r)\,d(x,Tx)\le d(x,y)\ \text{ implies } \mathscr{H}(Tx,Ty)\le rd(x,y) \tag{4}$$

for all $x,y\in X$. Then $\mathscr{F}(T)\neq\emptyset$.

Let f be a real-valued function defined on $\mathbb{R}$. For $c\in\mathbb{R}$, we recall that

$$\limsup_{x\to c^+} f(x)=\inf_{\varepsilon>0}\sup_{c<x<\varepsilon+c} f(x). \tag{5}$$

Definition 4 (see [3, 4, 10–19]). A function $\varphi:[0,\infty)\to[0,1)$ is said to be an *$\mathscr{MT}$-function* (or *$\mathscr{R}$-function*) if $\limsup_{s\to t^+}\varphi(s)<1$ for all $t\in[0,\infty)$.

It is obvious that if $\varphi:[0,\infty)\to[0,1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathscr{MT}$-function. So the set of $\mathscr{MT}$-functions is a rich class.

In 1989, Mizoguchi and Takahashi [19] proved a famous generalization of Nadler's fixed point theorem which gives a partial answer of Problem 9 in Reich [20]. It is worth to mention that the primitive proof of Mizoguchi-Takahashi's fixed point theorem is quite difficult. Recently, Suzuki [21] gave a very simple proof of Mizoguchi-Takahashi's fixed point theorem.

Theorem 5 (Mizoguchi and Takahashi [19]). *Let (X,d) be a complete metric space, $\alpha:[0,\infty)\to[0,1)$ be a $\mathscr{MT}$-function and $T:X\to CB(X)$ be a multivalued map. Assume that*

$$\mathscr{H}(Tx,Ty)\le\alpha(d(x,y))\,d(x,y), \tag{6}$$

for all $x,y\in X$. Then $\mathscr{F}(T)\neq\emptyset$.

In 2007, M. Berinde and V. Berinde [22] proved the following interesting fixed point theorem which generalized and improved Mizoguchi-Takahashi's fixed point theorem.

Theorem 6 (M. Berinde and V. Berinde [22]). *Let (X,d) be a complete metric space, $\alpha:[0,\infty)\to[0,1)$ be a $\mathscr{MT}$-function, $T:X\to CB(X)$ be a multivalued map and $L\ge 0$. Assume that*

$$\mathscr{H}(Tx,Ty)\le\alpha(d(x,y))\,d(x,y)+Ld(y,Tx), \tag{7}$$

for all $x,y\in X$. Then $\mathscr{F}(T)\neq\emptyset$.

Very recently, Du et al. [4] studied the existence of the approximate coincidence point property and the approximate fixed point property for some new nonlinear maps and applied them to metric fixed theory. Some new generalizations of Kikkawa-Suzuki's fixed point theorem, Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, and some well-known results in the literature were established in [4]; for more detail, one can refer to [4].

The paper is organized as follows. In Section 3, we first present some new existence theorems concerning approximate coincidence point property, approximate fixed point property, coincidence point and fixed point for various types of nonlinear maps in metric spaces without global completeness. Section 4 is dedicated to the study of some new coincidence point, and fixed point theorems given by exploiting our results. We establish some generalizations of Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem and others. Some applications of our results to a generalizations of Kikkawa-Suzuki's fixed point theorem and the existence of coupled coincidence point and coupled fixed point are also given in Section 5. Consequently, in this paper, some of our results are original in the literature and we obtain many results in the literature as special cases; see for example, [4–10, 13, 14, 17–23, 30] and references therein.

2. Preliminaries

Recall that a function $p:X\times X\to[0,\infty)$ is called a *w-distance* [5, 7, 10, 14, 15, 24–29, 38–40], if the following are satisfied:

($w1$) $p(x,z)\le p(x,y)+p(y,z)$ for any $x,y,z\in X$;

($w2$) for any $x\in X$, $p(x,\cdot):X\to[0,\infty)$ is l.s.c.;

($w3$) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

A function $p : X \times X \to [0, \infty)$ is said to be a *τ-function* [5, 10, 14, 15, 27–29], first introduced and studied by Lin and Du, if the following conditions hold:

($\tau1$) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;

($\tau2$) if $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ such that $p(x, y_n) \leq M$ for some $M = M(x) > 0$, then $p(x, y) \leq M$;

($\tau3$) for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} p(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$;

($\tau4$) for $x, y, z \in X$, $p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

Note that not either of the implications $p(x, y) = 0 \Leftrightarrow x = y$ necessarily holds and p is nonsymmetric in general. It is well known that the metric d is a w-distance and any w-distance is a τ-function, but the converse is not true; see [27, 29] for more detail.

Definition 7 (see [4]). Let (X, d) be a metric space, p be a τ-function, $g : X \to X$ be a single-valued self-map and $T : X \to \mathcal{N}(X)$ be a multivalued map.

(1) The maps g and T are said to have the *p-approximate coincidence point property* on X provided

$$\inf_{x\in X} p(gx, Tx) = 0. \tag{8}$$

(2) The map T is said to have the *p-approximate fixed point property* on X provided

$$\inf_{x\in X} p(x, Tx) = 0. \tag{9}$$

The following results are crucial in this paper.

Lemma 8 (see [29, Lemma 2.1]). *Let (X, d) be a metric space and $p : X \times X \to [0, \infty)$ be a function. Assume that p satisfies the condition ($\tau3$). If a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence in X.*

For each $x \in X$ and $A \subseteq X$, we denote $p(x, A) = \inf_{y\in A} p(x, y)$.

Lemma 9 (see [10]). *Let A be a closed subset of a metric space (X, d) and $p : X \times X \to [0, \infty)$ be any function. Suppose that p satisfies ($\tau3$) and there exists $u \in X$ such that $p(u, u) = 0$. Then $p(u, A) = 0$ if and only if $u \in A$.*

The concepts of τ^0-functions and τ^0-metrics were introduced in [10] as follows.

Definition 10 (see [10]). Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is called a *τ^0-function* if it is a τ-function on X with $p(x, x) = 0$ for all $x \in X$.

Remark 11. If p is a τ^0-function, then, from ($\tau4$), $p(x, y) = 0$ if and only if $x = y$.

Example 12 (see [10]). Let $X = \mathbb{R}$ with the metric $d(x, y) = |x-y|$ and $0 < a < b$. Define the function $p : X\times X \to [0, \infty)$ by

$$p(x, y) = \max\{a(y - x), b(x - y)\}. \tag{10}$$

Then p is nonsymmetric and hence p is not a metric. It is easy to see that p is a τ^0-function.

Definition 13 (see [10]). Let (X, d) be a metric space and p be a τ^0-function. For any $A, B \in CB(X)$, define a function $\mathcal{D}_p : CB(X) \times CB(X) \to [0, \infty)$ by

$$\mathcal{D}_p(A, B) = \max\left\{\delta_p(A, B), \delta_p(B, A)\right\}, \tag{11}$$

where $\delta_p(A, B) = \sup_{x\in A} p(x, B)$, then $\mathcal{D}_p$ is said to be the *τ^0-metric* on $CB(X)$ induced by p.

Clearly, any Hausdorff metric is a τ^0-metric, but the reverse is not true.

Lemma 14 (see [10]). *Let (X, d) be a metric space and $\mathcal{D}_p$ be a τ^0-metric on $CB(X)$ induced by a τ^0-function p. Then every τ^0-metric $\mathcal{D}_p$ is a metric on $CB(X)$.*

The following characterizations of $\mathcal{MT}$-functions is quite useful for proving our main results.

Lemma 15 (see [18]). *Let $\varphi : [0, \infty) \to [0, 1)$ be a function. Then the following statements are equivalent.*

(a) *φ is an $\mathcal{MT}$-function.*

(b) *For each $t \in [0, \infty)$, there exist $r_t^{(1)} \in [0, 1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \leq r_t^{(1)}$ for all $s \in (t, t + \varepsilon_t^{(1)})$.*

(c) *For each $t \in [0, \infty)$, there exist $r_t^{(2)} \in [0, 1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \leq r_t^{(2)}$ for all $s \in [t, t + \varepsilon_t^{(2)}]$.*

(d) *For each $t \in [0, \infty)$, there exist $r_t^{(3)} \in [0, 1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \leq r_t^{(3)}$ for all $s \in (t, t + \varepsilon_t^{(3)}]$.*

(e) *For each $t \in [0, \infty)$, there exist $r_t^{(4)} \in [0, 1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \leq r_t^{(4)}$ for all $s \in [t, t + \varepsilon_t^{(4)})$.*

(f) *For any nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0, \infty)$, one has $0 \leq \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*

(g) *φ is a function of contractive factor [12]; that is, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0, \infty)$, one has $0 \leq \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*

3. New Nonlinear Conditions for p-Approximate Coincidence Point Property

In Section 3, we will establish some new existence theorems concerning approximate coincidence point property, approximate fixed point property, coincidence point and fixed point for various types of nonlinear maps in metric spaces without global completeness.

Theorem 16. *Let (X,d) be a metric space, p be a τ^0-function, $T : X \to \mathcal{N}(X)$ be a multivalued map, and $f : X \to X$ be a self-map. Suppose that*

(S1) *there exist a nondecreasing function $\tau : [0,\infty) \to [0,\infty)$ and an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ such that for each $x \in X$, if $y \in X$ with $fy \neq fx$ and $fy \in Tx$, then it holds*

$$p(fy,Ty) \le \varphi(\tau(p(fx,fy)))\, p(fx,fy). \tag{12}$$

(S2) $T(X) = \bigcup_{x\in X} T(x) \subseteq f(X)$.

Then the following statements hold.

(a) *There exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that*

$$\begin{aligned}\inf_{n\in\mathbb{N}} p(fx_n, fx_{n+1}) &= \lim_{n\to\infty} p(fx_n, fx_{n+1})\\ &= \lim_{n\to\infty} d(fx_n, fx_{n+1})\\ &= \inf_{n\in\mathbb{N}} d(fx_n, fx_{n+1}) = 0.\end{aligned} \tag{13}$$

(b) $\inf_{x\in X} p(fx,Tx) = \inf_{x\in X} d(fx,Tx) = 0$; *that is, f and T have the p-approximate coincidence point property and approximate coincidence point property on X.*

(c) *If one further assumes the following conditions hold:*

(L1) *$f(X)$ is a complete subspace of X,*

(L2) *for each sequence $\{x_n\}$ in X with $fx_{n+1} \in Tx_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} fx_n = fw$, one has Tw as a closed subset of X and $\lim_{n\to\infty} p(fx_n, Tw) = 0$,*

then $COP(f,T) \neq \emptyset$.

Proof. Let $x_1 \in X$. By (S2), there exists $x_2 \in X$ such that $fx_2 \in Tx_1$. If $fx_1 = fx_2$, then $fx_1 \in Tx_1$ and so

$$\inf_{x\in X} p(fx,Tx) \le p(fx_1,Tx_1) \le p(fx_1,fx_1) = 0, \tag{14}$$

which implies $\inf_{x\in X} p(fx,Tx) = 0$. Clearly, $\inf_{x\in X} d(fx,Tx) = 0$. Let $w_n = x_1$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}\lim_{n\to\infty} p(fw_n, fw_{n+1}) &= \inf_{n\in\mathbb{N}} p(fw_n, fw_{n+1}) = p(fx_1,fx_1) = 0,\\ \lim_{n\to\infty} d(fw_n, fw_{n+1}) &= \inf_{n\in\mathbb{N}} d(fw_n, fw_{n+1}) = d(fx_1,fx_1) = 0.\end{aligned} \tag{15}$$

So, the conclusions (a) and (b) hold in this case. Otherwise, if $fx_2 \neq fx_1$, since p is a τ^0-function, $p(fx_1,fx_2) > 0$. Let $\mu : [0,\infty) \to [0,1)$ be defined by $\mu(t) = (1+\varphi(t))/2$. Clearly, $0 \le \varphi(t) < \mu(t) < 1$ for all $t \in [0,\infty)$. By [3, Lemma 2.1], we know that μ is also an $\mathcal{MT}$-function. From (S1), we get

$$\begin{aligned}p(fx_2,Tx_2) &\le \varphi(\tau(p(fx_1,fx_2)))\, p(fx_1,fx_2)\\ &< \mu(\tau(p(fx_1,fx_2)))\, p(fx_1,fx_2).\end{aligned} \tag{16}$$

Since $\mu(\tau(p(fx_1,fx_2)))p(fx_1,fx_2) > 0$, there exists $\xi \in Tx_2$ such that

$$p(fx_2,\xi) < \mu(\tau(p(fx_1,fx_2)))\, p(fx_1,fx_2). \tag{17}$$

Using (S2) again, there exists $x_3 \in X$ such that $fx_3 = \xi \in Tx_2$. Hence, from (17), we have

$$p(fx_2,fx_3) < \mu(\tau(p(fx_1,fx_2)))\, p(fx_1,fx_2). \tag{18}$$

If $fx_2 = fx_3 \in Tx_2$, then, following a similar argument as above, we can prove the conclusions (a) and (b). Otherwise, if $fx_3 \neq fx_2$, then there exists $x_4 \in X$ such that $fx_4 \in Tx_3$ and

$$p(fx_3,fx_4) < \mu(\tau(p(fx_2,fx_3)))\, p(fx_2,fx_3). \tag{19}$$

By induction, we can obtain a sequences $\{x_n\}$ in X satisfying

$$fx_{n+1} \in Tx_n, \tag{20}$$

$$p(fx_{n+1},fx_{n+2}) < \mu(\tau(p(fx_n,fx_{n+1})))\, p(fx_n,fx_{n+1}), \quad \text{for each } n \in \mathbb{N}. \tag{21}$$

Since $\mu(t) < 1$ for all $t \in [0,\infty)$, we deduces from the inequality (21) that the sequence $\{p(fx_n,fx_{n+1})\}_{n\in\mathbb{N}}$ is strictly decreasing in $[0,\infty)$. Hence

$$\lim_{n\to\infty} p(fx_n,fx_{n+1}) = \inf_{n\in\mathbb{N}} p(fx_n,fx_{n+1}) \ge 0 \text{ exists.} \tag{22}$$

Since τ is nondecreasing, $\{\tau(p(fx_n,fx_{n+1}))\}_{n\in\mathbb{N}}$ is a nonincreasing sequence in $[0,\infty)$. Since μ is an $\mathcal{MT}$-function, by (f) of Lemma 15, we have

$$0 \le \sup_{n\in\mathbb{N}} \mu(\tau(p(fx_n,fx_{n+1}))) < 1. \tag{23}$$

Let $\gamma := \sup_{n\in\mathbb{N}} \mu(\tau(p(fx_n,fx_{n+1})))$. So $\gamma \in [0,1)$. Put $\lambda := (1+\gamma)/2$. Then $0 \le \gamma < \lambda < 1$. By (21), we get

$$\begin{aligned}p(fx_{n+1},fx_{n+2}) &< \mu(\tau(p(fx_n,fx_{n+1})))\, p(fx_n,fx_{n+1})\\ &\le \gamma p(fx_n,fx_{n+1})\\ &< \lambda p(fx_n,fx_{n+1})\\ &< \cdots\\ &< \lambda^n p(fx_1,fx_2) \quad \text{for each } n \in \mathbb{N}.\end{aligned} \tag{24}$$

Since $\lambda \in (0,1)$, $\lim_{n\to\infty} \lambda^n = 0$ and hence it follows from (24) that

$$\lim_{n\to\infty} p(fx_n,fx_{n+1}) = 0. \tag{25}$$

According to (22) and (25), we obtain

$$\inf_{n\in\mathbb{N}} p(fx_n,fx_{n+1}) = \lim_{n\to\infty} p(fx_n,fx_{n+1}) = 0. \tag{26}$$

Next, we verify that $\{fx_n\}$ is a Cauchy sequence in $f(X)$. Let $v_n := fx_n$ for all $n \in \mathbb{N}$. We claim that $\lim_{n\to\infty} \sup\{p(v_n, v_m) : m > n\} = 0$. Put

$$\alpha_n = \left(\frac{\lambda^{n-1}}{1-\lambda}\right) p(v_1, v_2), \quad n \in \mathbb{N}. \tag{27}$$

Then $\alpha_n > 0$ for all $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $m > n$, by (24), we have

$$p(v_n, v_m) \le \sum_{j=n}^{m-1} p(v_j, v_{j+1}) < \alpha_n. \tag{28}$$

Since $\lambda \in (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$ and hence

$$\lim_{n\to\infty} \sup\{p(v_n, v_m) : m > n\} = 0. \tag{29}$$

By Lemma 8, $\{v_n\}$ is a Cauchy sequence in $f(X)$. Hence

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = \lim_{n\to\infty} d(v_n, v_{n+1}) = 0. \tag{30}$$

Since $\inf_{n\in\mathbb{N}} d(fx_n, fx_{n+1}) \le d(fx_m, fx_{m+1})$ for all $m \in \mathbb{N}$ and $\lim_{m\to\infty} d(fx_m, fx_{m+1}) = 0$, one also obtain

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = \inf_{n\in\mathbb{N}} d(fx_n, fx_{n+1}) = 0. \tag{31}$$

Hence (a) is proved. To see (b), since $fx_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$, we have

$$\begin{aligned} &\inf_{x\in X} p(fx, Tx) \le p(fx_n, Tx_n) \le p(fx_n, fx_{n+1}), \\ &\inf_{x\in X} d(fx, Tx) \le d(fx_n, Tx_n) \le d(fx_n, fx_{n+1}), \end{aligned} \tag{32}$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} p(fx_n, fx_{n+1}) = \lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0$, combining (32), we get

$$\inf_{x\in X} p(fx, Tx) = \inf_{x\in X} d(fx, Tx) = 0. \tag{33}$$

Moreover, if we further assume that conditions (L1) and (L2) hold, we want to show $COP(f, T) \ne \emptyset$. Since $\{v_n\}$ is a Cauchy sequence in $f(X)$, by (L1), there exists $\hat{v} \in X$ such that $v_n \equiv fx_n \to f\hat{v}$ as $n \to \infty$. So, by (L2), we have $T\hat{v}$ is a closed subset of X and $\lim_{n\to\infty} p(v_n, T\hat{v}) = 0$. Since $\lim_{n\to\infty} \sup\{p(v_n, v_m) : m > n\} = 0$, there exists $\{a_n\} \subset \{v_n\}$ with $\lim_{n\to\infty} \sup\{p(a_n, a_m) : m > n\} = 0$ and $\{b_n\} \subset T\hat{v}$ such that $\lim_{n\to\infty} p(a_n, b_n) = 0$. By ($\tau$3), $\lim_{n\to\infty} d(a_n, b_n) = 0$. Since $a_n \to f\hat{v}$ as $n \to \infty$ and $d(b_n, f\hat{v}) \le d(b_n, a_n) + d(a_n, f\hat{v})$, it implies $b_n \to f\hat{v}$ as $n \to \infty$. By the closedness of $T\hat{v}$, we have $f\hat{v} \in T\hat{v}$ or $\hat{v} \in COP(f, T)$. The proof is completed. □

Theorem 17. *In Theorem 16, if $f \equiv id$ is the identity map on X, then the following statements hold.*

(a) *There exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that*

$$\begin{aligned} \inf_{n\in\mathbb{N}} p(x_n, x_{n+1}) &= \lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+1}) \\ &= \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0. \end{aligned} \tag{34}$$

(b) $\inf_{x\in X} p(x, Tx) = \inf_{x\in X} d(x, Tx) = 0$; *that is, T has the p-approximate fixed point property and approximate fixed point property on X.*

(c) *If one further assumes the following conditions hold:*

(L1) *$f(X)$ is a complete subspace of X,*

(L3) *for each sequence $\{x_n\}$ in X with $x_{n+1} \in Tx_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = w$, one has Tw as a closed subset of X and $\lim_{n\to\infty} p(x_n, Tw) = 0$,*

then $\mathcal{F}(T) \ne \emptyset$.

As an application of Theorems 16, we can establish the following new existence of approximate coincidence point property easily.

Theorem 18. *Let (X, d) be a metric space, p be a τ^0-function, $\mathcal{D}_p$ be a τ^0-metric on $CB(X)$ induced by p, $T : X \to CB(X)$ be a multivalued map and $f : X \to X$ be a self-map. Suppose that (S2) as in Theorem 16 is satisfied and T further satisfies one of the following conditions:*

(H1) *there exist a nondecreasing function $\tau : [0,\infty) \to [0,\infty)$, an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ and a function $L : X \times X \to [0,\infty)$ such that*

$$\begin{aligned} \mathcal{D}_p(Tx, Ty) \le{}& \varphi(\tau(p(fx, fy)))\, p(fx, fy) \\ &+ L(fx, fy)\, p(fy, Tx) \quad \forall x, y \in X, \end{aligned} \tag{35}$$

(H2) *there exist a nondecreasing function $\tau : [0,\infty) \to [0,\infty)$, an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ and a function $L : X \times X \to [0,\infty)$ such that*

$$\begin{aligned} p(fy, Ty) \le{}& \varphi(\tau(p(fx, fy)))\, p(fx, fy) \\ &+ L(fx, fy)\, p(fy, Tx) \quad \forall x, y \in X. \end{aligned} \tag{36}$$

Then the following statements hold.

(a) *There exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that*

$$\begin{aligned} \inf_{n\in\mathbb{N}} p(fx_n, fx_{n+1}) &= \lim_{n\to\infty} p(fx_n, fx_{n+1}) \\ &= \lim_{n\to\infty} d(fx_n, fx_{n+1}) \\ &= \inf_{n\in\mathbb{N}} d(fx_n, fx_{n+1}) = 0. \end{aligned} \tag{37}$$

(b) $\inf_{x\in X} p(fx, Tx) = \inf_{x\in X} d(fx, Tx) = 0$.

(c) *If one further assumes the following conditions hold:*

(L1) *$f(X)$ is a complete subspace of X,*

(L2) *for each sequence $\{x_n\}$ in X with $fx_{n+1} \in Tx_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} fx_n = fw$, one has Tw as a closed subset of X and $\lim_{n\to\infty} p(fx_n, Tw) = 0$,*

then $COP(f, T) \ne \emptyset$.

Proof. Suppose that (H1) holds. We first notice that for each $x \in X$, by (S2), the set $\{y \in X : fy \in Tx\} \neq \emptyset$. Let $x \in X$ be given and let $y \in X$ with $fy \in Tx$ be arbitrary. Since $p(fy, Tx) = 0$, we have

$$p(fy, Ty) \leq \mathscr{D}_p(Tx, Ty) \leq \varphi(\tau(p(fx, fy)))\, p(fx, fy). \tag{38}$$

Hence (H1) implies (S1). Therefore the conclusion follows from Theorem 16. Similarly, we can prove that (H2) implies (S1) and the desired result follows also from Theorem 16. □

Theorem 19. *In Theorem 18, if $f \equiv id$ is the identity map on X, then the conclusion following statements hold.*

(a) *There exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that*

$$\begin{aligned}\inf_{n\in\mathbb{N}} p(x_n, x_{n+1}) &= \lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+1}) \\ &= \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0.\end{aligned} \tag{39}$$

(b) $\inf_{x\in X} p(x, Tx) = \inf_{x\in X} d(x, Tx) = 0$.

(c) *If one further assumes the following conditions hold:*

(L1) *$f(X)$ is a complete subspace of X,*

(L3) *for each sequence $\{x_n\}$ in X with $x_{n+1} \in Tx_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = w$, one has Tw as a closed subset of X and $\lim_{n\to\infty} p(x_n, Tw) = 0$,*

then $\mathscr{F}(T) \neq \emptyset$.

4. New Generalizations of Berinde-Berinde's Fixed Point Theorem and Mizoguchi-Takahashi's Fixed Point Theorem

In this section, we will establish some new coincidence point theorems which generalize and improve Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem and some main results in [10, 14, 17–19, 22].

Theorem 20. *Let (X, d) be a metric space, p be a τ^0-function, $\mathscr{D}_p$ be a τ^0-metric on $CB(X)$ induced by p, $T : X \to CB(X)$ be a multivalued map and $f : X \to X$ be a self-map. Suppose that*

(i) $T(X) = \bigcup_{x\in X} T(x) \subseteq f(X)$,

(ii) *$f(X)$ is a complete subspace of X,*

(iii) *there exists a nondecreasing function $\tau : [0, \infty) \to [0, \infty)$, an $\mathscr{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ and a function $h : X \to [0, \infty)$ such that*

$$\begin{aligned}\mathscr{D}_p(Tx, Ty) &\leq \varphi(\tau(p(fx, fy)))\, p(fx, fy) \\ &\quad + h(y)\, d(fy, Tx) \quad \forall x, y \in X,\end{aligned} \tag{40}$$

then $COP(f, T) \neq \emptyset$.

Moreover, if we further assume that $fw = ffw$ for any $w \in COP(f, T)$, then T and f have a common fixed point in X.

Proof. Let $x \in X$ be given. Let $y \in X$ with $fy \in Tx$ be arbitrary. Since $d(fy, Tx) = 0$, we have

$$p(fy, Ty) \leq \mathscr{D}_p(Tx, Ty) \leq \varphi(\tau(p(fx, fy)))\, p(fx, fy). \tag{41}$$

Hence (iii) implies (S1). So, following the same argument as the proof of Theorem 16, we can obtain two sequences $\{x_n\}$ and $\{v_n\}$ in X satisfying the following:

(i) $fx_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$;

(ii) $v_n := fx_n$ for all $n \in \mathbb{N}$;

(iii) $p(v_1, v_2) > 0$;

(iv) There exists $\lambda \in (0, 1)$ such that $p(v_{n+1}, v_{n+2}) \leq \lambda^n p(v_1, v_2)$ for each $n \in \mathbb{N}$;

(v) $p(v_n, v_m) < \alpha_n$ for $m, n \in \mathbb{N}$ with $m > n$, where $\alpha_n = (\lambda^{n-1}/(1-\lambda))p(v_1, v_2) > 0$ for $n \in \mathbb{N}$;

(vi) $\lim_{n\to\infty} \sup\{p(v_n, v_m) : m > n\} = 0$;

(vii) $\{v_n\}$ is a Cauchy sequence in $f(X)$.

By (ii), there exists $\overline{v} \in X$ such that $fx_n \equiv v_n \to f\overline{v}$ as $n \to \infty$ or $\lim_{n\to\infty} d(v_n, f\overline{v}) = 0$. Since $p(v_n, v_m) < \alpha_n$ for $m, n \in \mathbb{N}$ with $m > n$, from (τ2), we have

$$p(fx_n, f\overline{v}) = p(v_n, f\overline{v}) \leq \alpha_n \quad \forall n \in \mathbb{N}. \tag{42}$$

So, for each $n \in \mathbb{N}$, it follows from (40) and (42) that

$$\begin{aligned}p(v_{n+1}, T\overline{v}) &= p(fx_{n+1}, T\overline{v}) \\ &\leq \sup_{y\in Tx_n} p(y, T\overline{v}) \\ &\leq \mathscr{D}_p(Tx_n, T\overline{v}) \\ &\leq \varphi(\tau(p(fx_n, f\overline{v})))\, p(fx_n, f\overline{v}) \\ &\quad + h(\overline{v})\, d(f\overline{v}, Tx_n) \\ &< p(fx_n, f\overline{v}) + h(\overline{v})\, d(f\overline{v}, fx_{n+1}) \\ &\leq \alpha_n + h(\overline{v})\, d(f\overline{v}, fx_{n+1}).\end{aligned} \tag{43}$$

The last inequality implies that there exists $y_{n+1} \in T\overline{v}$ such that

$$p(v_{n+1}, y_{n+1}) < \alpha_n + h(\overline{v})\, d(f\overline{v}, fx_{n+1}) \quad \text{for each } n \in \mathbb{N}. \tag{44}$$

Since $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} d(f\overline{v}, fx_{n+1}) = 0$, we get $\lim_{n\to\infty} p(v_{n+1}, y_{n+1}) = 0$. By ($\tau$3), we have $\lim_{n\to\infty} d(v_{n+1}, y_{n+1}) = 0$. Since

$$0 \leq d(y_{n+1}, f\overline{v}) \leq d(y_{n+1}, v_{n+1}) + d(v_{n+1}, f\overline{v}) \quad \forall n \in \mathbb{N}, \tag{45}$$

we obtain $\lim_{n\to\infty} d(y_{n+1}, f\overline{v}) = 0$ or $y_{n+1} \to f\overline{v}$ as $n \to \infty$. Since $y_{n+1} \in T\overline{v}$ for all $n \in \mathbb{N}$ and $T\overline{v}$ is closed, we have $f\overline{v} \in T\overline{v}$ which means that $\overline{v} \in COP(f,T)$. So $COP(f,T) \neq \emptyset$.

Moreover, if we further assume that $fw = ffw$ for all $w \in COP(f,T)$, then we have $f\overline{v} = ff\overline{v}$. For each $n \in \mathbb{N}$, by (40) and (42) again, we have

$$\begin{aligned} p(v_{n+1}, Tf\overline{v}) &= p(fx_{n+1}, Tf\overline{v}) \\ &\le \mathscr{D}_p(Tx_n, Tf\overline{v}) \\ &\le \varphi(\tau(p(fx_n, ff\overline{v})))\, p(fx_n, ff\overline{v}) \\ &\quad + h(f\overline{v})\, d(ff\overline{v}, Tx_n) \\ &< p(fx_n, f\overline{v}) + h(f\overline{v})\, d(f\overline{v}, fx_{n+1}) \\ &\le \alpha_n + h(f\overline{v})\, d(f\overline{v}, fx_{n+1}). \end{aligned} \tag{46}$$

Therefore, there exists $z_{n+1} \in Tf\overline{v}$ such that

$$p(v_{n+1}, z_{n+1}) < \alpha_n + h(f\overline{v})\, d(f\overline{v}, fx_{n+1}) \quad \forall n \in \mathbb{N}. \tag{47}$$

By (47) and $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty} d(f\overline{v}, fx_{n+1}) = 0$, we have $\lim_{n\to\infty} p(v_{n+1}, z_{n+1}) = 0$. By ($\tau 3$), we get $\lim_{n\to\infty} d(v_{n+1}, z_{n+1}) = 0$. Since

$$d(f\overline{v}, z_{n+1}) \le d(f\overline{v}, v_{n+1}) + d(v_{n+1}, z_{n+1}), \tag{48}$$

we have $\lim_{n\to\infty} d(f\overline{v}, z_{n+1}) = 0$. Since $Tf\overline{v}$ is closed and $z_n \in Tf\overline{v}$ for all $n \in \mathbb{N}$, we get $f\overline{v} \in Tf\overline{v}$. Therefore, $f\overline{v} = ff\overline{v} \in Tf\overline{v}$, which means that $f\overline{v}$ is a common fixed point of f and T in X. The proof is completed. □

Corollary 21. *Let (X,d) be a metric space, $T : X \to CB(X)$ be a multivalued map and $f : X \to X$ be a self-map. Suppose that*

(i) $T(X) = \bigcup_{x\in X} T(x) \subseteq f(X)$,

(ii) *$f(X)$ is a complete subspace of X,*

(iii) *there exists a nondecreasing function $\tau : [0,\infty) \to [0,\infty)$, an $\mathscr{MT}$-function $\varphi : [0,\infty) \to [0,1)$ and a function $h : X \to [0,\infty)$ such that*

$$\mathscr{H}(Tx, Ty) \le \varphi(\tau(d(fx, fy)))\, d(fx, fy) + h(y)\, d(fy, Tx) \quad \forall x, y \in X, \tag{49}$$

then $COP(f,T) \neq \emptyset$.

The following results are immediate consequences of Theorem 20. They are generalizations of Berinde-Berinde's fixed point and Mizoguchi-Takahashi's fixed point theorem.

Theorem 22. *Let (X,d) be a complete metric space, p be a τ^0-function, $\mathscr{D}_p$ be a τ^0-metric on $CB(X)$ induced by p, $T : X \to CB(X)$ be a multivalued map, $\tau : [0,\infty) \to [0,\infty)$ be a nondecreasing function, $\varphi : [0,\infty) \to [0,1)$ be an $\mathscr{MT}$-function, and $h : X \to [0,\infty)$ be a function. Suppose that*

$$\mathscr{D}_p(Tx, Ty) \le \varphi(\tau(p(x,y)))\, p(x,y) + h(y)\, d(y, Tx) \quad \forall x, y \in X, \tag{50}$$

then $\mathscr{F}(T) \neq \emptyset$.

Corollary 23. *Let (X,d) be a complete metric space, $T : X \to CB(X)$ be a multivalued map, $\tau : [0,\infty) \to [0,\infty)$ be a nondecreasing function, $\varphi : [0,\infty) \to [0,1)$ be an $\mathscr{MT}$-function, and $h : X \to [0,\infty)$ be a function. Suppose that*

$$\begin{aligned} \mathscr{H}(Tx, Ty) &\le \varphi(\tau(d(fx, fy)))\, d(x,y) \\ &\quad + h(y)\, d(y, Tx) \quad \forall x, y \in X, \end{aligned} \tag{51}$$

then $\mathscr{F}(T) \neq \emptyset$.

Remark 24. (a) It is worth to mention that Theorem 22 is different from [10, Theorem 2.3]. Theorem 22 is comparable to [10, Theorem 2.3] in the following aspects.

(1) In [10, Theorem 2.3], the map T was assumed to satisfy

$$\mathscr{D}_p(Tx, Ty) \le \varphi(p(x,y))\, p(x,y) + Lp(y, Tx) \quad \forall x, y \in X \tag{52}$$

where φ is an $\mathscr{MT}$-function and L is a given nonnegative real number. But in Theorem 22, we assume that

$$\mathscr{D}_p(Tx, Ty) \le \varphi(\tau(p(x,y)))\, p(x,y) + h(y)\, d(y, Tx) \quad \forall x, y \in X, \tag{53}$$

where τ is a nondecreasing function, φ is an $\mathscr{MT}$-function and $h : X \to [0,\infty)$ is any function.

(2) Notice that in [10, Theorem 2.3], the author assumed that T further satisfies one of conditions (D1), (D2), (D3), (D4), and (D5), where

(D1) T is closed;

(D2) the map $f : X \to [0,\infty)$ defined by $f(x) = p(x, Tx)$ is l.s.c.;

(D3) the map $g : X \to [0,\infty)$ defined by $g(x) = d(x, Tx)$ is l.s.c.;

(D4) for any sequence $\{x_n\}$ in X with $x_{n+1} \in Tx_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = v$, we have $\lim_{n\to\infty} p(x_n, Tv) = 0$;

(D5) $\inf\{p(x,z) + p(x,Tx) : x \in X\} > 0$ for every $z \notin \mathscr{F}(T)$.

But Theorem 22 does not require the conditions (D1)–(D5).

(b) If we take $\tau(t) = t$, $t \in [0,\infty)$ and $h(t) = 0$ for all $x \in X$ in Theorem 22, then we obtain [17, Theorem 3.1].

(c) Theorems 20 and 22, and Corollary 21 all generalize Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, Nadler's fixed point theorem, Banach contraction principle and some main results in [10, 14, 18, 19, 22] and references therein.

Here, we give a simple example illustrating Theorem 20.

Example 25. Let $X = (-9876, -10) \cup [0, \infty)$ with the metric $d(x, y) = |x - y|$ for $x, y \in X$. Then (X, d) is a metric space. Let $p : X \times X \to [0, \infty)$ be defined by

$$p(x, y) = \max\{2(x - y), 3(y - x)\}, \tag{54}$$

for all $x, y \in X$. By Example 12, we know that p is a τ^0-function. Let $\varphi : [0, \infty) \to [0, 1)$ and $\tau : [0, \infty) \to [0, \infty)$ be defined by $\varphi(t) = 1/2$ and $\tau(t) = 5$, respectively. Then φ is an $\mathscr{MT}$-function and τ is a nondecreasing function. Let $f : X \to X$ be defined by

$$f(x) := \begin{cases} 0, & \text{if } x \in (-9876, -10), \\ 2x, & \text{if } x \in [0, \infty). \end{cases} \tag{55}$$

Then $f(X) = [0, \infty)$ is a proper complete subspace of X. Define $T : X \to CB(X)$ by

$$T(x) := \begin{cases} \{0\}, & \text{if } x \in (-9876, -10), \\ [0, x], & \text{if } x \in [0, \infty). \end{cases} \tag{56}$$

Clearly, $T(X) = \bigcup_{x \in X} T(x) \subseteq f(X)$. Let $h : X \to [0, \infty)$ be defined by $h(x) = e^x$. We claim that

$$\begin{aligned} &\mathscr{D}_p(Tx, Ty) \\ &\quad \le \varphi(\tau(p(fx, fy)))\, p(fx, fy) + h(y)\, d(fy, Tx), \end{aligned} \tag{$*$}$$

for all $x, y \in X$. We consider the following six possible cases.

Case 1. Clearly, inequality $(*)$ holds for $x = y \in X$.

Case 2. If $x, y \in (-9876, -10)$, then $\mathscr{D}_p(Tx, Ty) = 0 = \varphi(\tau(p(fx, fy)))p(fx, fy) + h(y)d(fy, Tx)$.

Case 3. If $x \in (-9876, -10)$ and $y \in [0, \infty)$, then $Tx = \{0\}$ and $Ty = [0, y]$. So

$$\begin{aligned} p(fx, fy) &= \max\{2(fx - fy), 3(fy - fx)\} \\ &= \max\{-4y, 6y\} = 6y, \\ \mathscr{D}_p(Tx, Ty) &= \max\left\{\sup_{z \in Tx} p(z, Ty), \sup_{z \in Ty} p(z, Tx)\right\} \\ &= 2y = \frac{1}{3} p(fx, fy) \\ &< \varphi(\tau(p(fx, fy)))\, p(fx, fy) \\ &\quad + h(y)\, d(fy, Tx). \end{aligned} \tag{57}$$

Case 4. If $x \in [0, \infty)$ and $y \in (-9876, -10)$, then $Tx = [0, x]$, $Ty = \{0\}$ and $p(fy, Tx) = 0$. Hence, we have

$$\begin{aligned} p(fx, fy) &= \max\{2(fx - fy), 3(fy - fx)\} \\ &= \max\{4x, -6x\} = 4x, \\ \mathscr{D}_p(Tx, Ty) &= \max\left\{\sup_{z \in Tx} p(z, Ty), \sup_{z \in Ty} p(z, Tx)\right\} \\ &= 2x = \frac{1}{2} p(fx, fy) \\ &= \varphi(\tau(p(fx, fy)))\, p(fx, fy) \\ &\quad + h(y)\, d(fy, Tx). \end{aligned} \tag{58}$$

Case 5. If $0 \le x < y$, then $Tx = [0, x]$ and $Ty = [0, y]$. So, we have

$$\begin{aligned} p(fx, fy) &= \max\{2(fx - fy), 3(fy - fx)\} = 6(y - x), \\ \mathscr{D}_p(Tx, Ty) &= \max\left\{\sup_{z \in Tx} p(z, Ty), \sup_{z \in Ty} p(z, Tx)\right\} \\ &= 2(y - x) \\ &< \frac{1}{2} p(fx, fy) \\ &\le \varphi(\tau(p(fx, fy)))\, p(fx, fy) \\ &\quad + h(y)\, d(fy, Tx). \end{aligned} \tag{59}$$

Case 6. If $0 \le y < x$, then $Tx = [0, x]$, $Ty = [0, y]$. Thus we obtain

$$\begin{aligned} p(fx, fy) &= \max\{2(fx - fy), 3(fy - fx)\} = 4(x - y), \\ \mathscr{D}_p(Tx, Ty) &= \max\left\{\sup_{z \in Tx} p(z, Ty), \sup_{z \in Ty} p(z, Tx)\right\} \\ &= 2(x - y) \\ &= \frac{1}{2} p(fx, fy) \\ &\le \varphi(\tau(p(fx, fy)))\, p(fx, fy) \\ &\quad + h(y)\, d(fy, Tx). \end{aligned} \tag{60}$$

By Cases 1–6, we verify that inequality $(*)$ holds for all $x, y \in X$. So all the hypotheses of Theorem 20 are fulfilled. It is therefore possible to apply Theorem 20 to get $COP(f, T) \neq \emptyset$. In fact, $COP(f, T) = (-9876, -10) \cup \{0\}$.

Moreover, since $fw = ffw = 0$ for any $w \in COP(f, T)$, by Theorem 20 again, we know that f and T have a common fixed point in X (precisely speaking, 0 is the unique common fixed point of f and T).

5. Some Applications to New Coupled Coincidence Point Theorems and a Generalization of Kikkawa-Suzuki's Fixed Point Theorem

Let (X, d) be a metric space. We endow the product space $X \times X$ with the metric ρ defined by

$$\rho((x,y),(u,v)) = d(x,u) + d(y,v) \quad \text{for any } (x,y),(u,v) \in X \times X. \tag{61}$$

Let $g : X \to X$ be a self-map. Recall that an element $(x, y) \in X \times X$ is called a *coupled coincidence point* of the maps F and g if

$$F(x,y) = gx, \qquad F(y,x) = gy \quad \text{when } F : X \times X \longrightarrow X \text{ is a single-valued map} \tag{62}$$

or

$$gx \in F(x,y), \qquad gy \in F(y,x) \quad \text{when } F : X \times X \longrightarrow X \text{ is a multivalued map.} \tag{63}$$

In particular, if we take $g \equiv id$ (the identity map) in (62) and (63), then (x, y) is called a *coupled fixed point* of F. The existence of coupled coincidence point and coupled fixed point has been investigated by several authors recently in [4, 11, 12, 23, 30–37] and references therein.

As an interesting application of Corollary 21 (or Theorem 20), we establish the following new coupled coincidence point theorems. It is worth to mention that the technique in the proof of Theorem 26 is quite different in the well known literature.

Theorem 26. *Let (X, d) be a metric space, $F : X \times X \to CB(X)$ be a multivalued map and $f : X \to X$ be a self-map satisfying $\bigcup_{x\in X} F(x, x) \subseteq f(X)$ and $f(X)$ is a complete subspace of X. Assume that there exists an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ such that for any $(x, y), (u, v) \in X \times X$,*

$$\mathcal{H}(F(x,y), F(u,v)) \le \frac{1}{2}\varphi(\rho((fx,fy),(fu,fv)))\,\rho((fx,fy),(fu,fv)), \tag{64}$$

where $\rho((x, y), (u, v)) = d(x, u) + d(y, v)$ for $(x, y), (u, v) \in X \times X$. Then there exists $v \in X$ such that $fv \in F(v, v)$; that is, $(v, v) \in X \times X$ is a coupled coincidence point of F and f.

Proof. Define $T : X \to CB(X)$ by

$$T(x) = F(x,x). \tag{65}$$

Then $T(X) = \bigcup_{x\in X} T(x) \subseteq f(X)$. Let $\tau(t) = 2t$, $\forall t \in [0,\infty)$ and $h(x) = 0$, $\forall x \in X$. So the inequality (64) implies

$$\begin{aligned}\mathcal{H}(T(x), T(u)) &\le \varphi(2d(fx,fu))\,d(fx,fu)\\ &= \varphi(\tau(d(fx,fu)))\,d(fx,fu)\\ &\quad + h(u)\,d(fu,Tx) \quad \forall x,u \in X.\end{aligned} \tag{66}$$

Therefore, all the assumptions of Corollary 21 are fulfilled. Hence we can apply Corollary 21 to show that there exists $v \in X$ such that $fv \in Tv = F(v, v)$. □

The following conclusion is immediate from Theorem 26 with $f \equiv id$ (the identity map).

Corollary 27 (see [4]). *Let (X, d) be a complete metric space and $F : X \times X \to CB(X)$ be a multivalued map. Assume that there exists an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ such that for any $(x, y), (u, v) \in X \times X$,*

$$\mathcal{H}(F(x,y), F(u,v)) \le \frac{1}{2}\varphi(\rho((x,y),(u,v)))\,\rho((x,y),(u,v)), \tag{67}$$

where $\rho((x, y), (u, v)) = d(x, u) + d(y, v)$ for $(x, y), (u, v) \in X \times X$. Then there exists $v \in X$ such that $v \in F(v, v)$; that is, $(v, v) \in X \times X$ is a coupled fixed point of F.

Applying Theorem 26, we obtain the following coupled coincidence point theorem.

Theorem 28. *Let (X, d) be a metric space, $F : X \times X \to X$ be a map and $f : X \to X$ be a self-map satisfying $\bigcup_{x\in X} F(x, x) \subseteq f(X)$ and $f(X)$ is a complete subspace of X. Assume that there exists an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ such that for any $(x, y), (u, v) \in X \times X$,*

$$d(F(x,y), F(u,v)) \le \frac{1}{2}\varphi(\rho((fx,fy),(fu,fv)))\,\rho((fx,fy),(fu,fv)). \tag{68}$$

Then there exists $v \in X$ such that (v, v) is the unique coupled coincidence point of F and f.

Proof. Applying Theorem 26, there exists $\hat{v} \in X$ such that $f\hat{v} = F(\hat{v}, \hat{v})$. To complete the proof, it suffices to show the uniqueness of the coupled coincidence point of F and f. On the contrary, suppose that there exists $(\hat{x}, \hat{y}) \in X \times X$, such that $f\hat{x} = F(\hat{x}, \hat{y})$ and $f\hat{y} = F(\hat{y}, \hat{x})$. By (68), we have

$$\begin{aligned} d(f\hat{v}, f\hat{x}) &= d(F(\hat{v},\hat{v}), F(\hat{x},\hat{y})) < \frac{1}{2}[d(f\hat{v}, f\hat{x}) + d(f\hat{v}, f\hat{y})],\\ d(f\hat{v}, f\hat{y}) &= d(F(\hat{v},\hat{v}), F(\hat{y},\hat{x})) < \frac{1}{2}[d(f\hat{v}, f\hat{y}) + d(f\hat{v}, f\hat{x})]. \end{aligned} \tag{69}$$

So, combining (69), we get

$$d(f\hat{v}, f\hat{x}) + d(f\hat{v}, f\hat{y}) < d(f\hat{v}, f\hat{x}) + d(f\hat{v}, f\hat{y}), \tag{70}$$

which leads a contradiction. The proof is completed. □

Corollary 29 (see [4]). *Let (X,d) be a complete metric space and $F : X \times X \to X$ be a map. Assume that there exists an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ such that for any (x,y), $(u,v) \in X \times X$,*

$$d(F(x,y),F(u,v)) \le \frac{1}{2}\varphi(\rho((x,y),(u,v)))\rho((x,y),(u,v)). \tag{71}$$

Then there exists $v \in X$ such that (v,v) is the unique coupled fixed point of F.

Applying Theorem 18, we can prove the following coincidence point theorem which is a generalization of Kikkawa-Suzuki's fixed point theorem.

Theorem 30. *Define a strictly decreasing function η from $[0,1)$ onto $(1/2,1]$ by*

$$\eta(r) = \frac{1}{1+r}. \tag{72}$$

Let (X,d) be a metric space, $T : X \to CB(X)$ be a multivalued map, and $f : X \to X$ be an injective self-map such that $T(X) = \bigcup_{x\in X} T(x) \subseteq f(X)$. Assume that

(KS) *there exists $\gamma \in [0,1)$ such that*

$$\eta(\gamma)d(fx,Tx) \le d(fx,fy) \quad \text{implies } \mathcal{H}(Tx,Ty) \le \gamma d(fx,fy), \tag{73}$$

for all $x, y \in X$.

Then f and T have the approximate coincidence property on X.

Moreover, if we further assume $f(X)$ is a complete subspace of X, then $COP(f,T) \ne \emptyset$.

Proof. Let $x \in X$ be arbitrary. If $fx \in Tx$, then $x \in COP(f,T)$ and hence f and T have the approximate coincidence property. So the theorem is finished in this case. Suppose $fx \notin Tx$. Let $\tau : [0,\infty) \to [0,\infty)$ be defined by $\tau(t) = t$. Then τ is a nondecreasing function. By (KS), we can define an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ by $\varphi(t) = \gamma$, $\forall t \in [0,\infty)$. Let $y \in X$ with $fy \ne fx$ and $fy \in Tx$. Since

$$\eta(\gamma)d(fx,Tx) \le d(fx,Tx) \le d(fx,fy). \tag{74}$$

by (KS) again, we have

$$\begin{aligned} d(fy,Ty) &\le \mathcal{H}(Tx,Ty) \le \gamma d(fx,fy) \\ &= \varphi(\tau(d(fx,fy)))d(fx,fy), \end{aligned} \tag{75}$$

which shows that (H1) as in Theorem 18 holds. Therefore T have the approximate coincidence property by applying Theorem 18.

Now, suppose that $f(X)$ is a complete subspace of X. We shall prove $COP(f,T) \ne \emptyset$. It suffices to prove that the condition (L2) as in Theorem 18 holds. Let $\{x_n\}$ in X with $fx_{n+1} \in Tx_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} fx_n = fw$. Clearly, Tw is a closed subset of X. We will proceed with the following claims to prove $\lim_{n\to\infty} d(fx_n,Tw) = 0$.

Claim 1. $d(fw,Tx) \le \gamma d(fw,fx)$ for all $x \in X$ with $fx \ne fw$.

For $x \in X$ with $fx \ne fw$, since $\lim_{n\to\infty} fx_n = fw$, there exists $n_0 \in \mathbb{N}$, such that

$$d(fw,fx_n) \le \frac{1}{3}d(fx,fw) \quad \forall n \in \mathbb{N} \text{ with } n \ge n_0. \tag{76}$$

For $n \in \mathbb{N}$ with $n \ge n_0$, it follows from (76) that

$$\begin{aligned} \eta(\gamma)d(fx_n,Tx_n) &\le d(fx_n,Tx_n) \\ &\le d(fx_n,fx_{n+1}) \\ &\le d(fx_n,fw) + d(fw,fx_{n+1}) \\ &\le \frac{2}{3}d(fx,fw) \\ &= d(fx,fw) - \frac{1}{3}d(fx,fw) \\ &\le d(fx,fw) - d(fx_n,fw) \\ &\le d(fx_n,fx). \end{aligned} \tag{77}$$

By our hypothesis (KS), we have

$$d(fx_{n+1},Tx) \le \mathcal{H}(Tx_n,Tx) \le \gamma d(fx_n,fx) \quad \forall n \in \mathbb{N} \text{ with } n \ge n_0. \tag{78}$$

Since $\lim_{n\to\infty} fx_n = fw$, the last inequality implies

$$d(fw,Tx) \le \gamma d(fw,fx) \quad \forall x \in X \text{ with } fx \ne fw. \tag{79}$$

Claim 2. $\mathcal{H}(Tx,Tw) \le \gamma d(fx,fw)$ for all $x \in X$.

Let $x \in X$. It is quite obvious that the desired inequality holds if $x = w$. Suppose $x \ne w$. Since f is one-to-one, we have $fx \ne fw$. For every $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(fw,y_n) < d(fw,Tx) + \frac{1}{n}d(x,w). \tag{80}$$

Using (80) and Claim 1, we obtain

$$\begin{aligned} d(fx,Tx) &\le d(fx,y_n) \\ &\le d(fx,fw) + d(fw,y_n) \\ &< d(fx,fw) + d(fw,Tx) + \frac{1}{n}d(x,w) \\ &\le d(fx,fw) + \gamma d(fw,fx) + \frac{1}{n}d(x,w) \\ &= (1+\gamma)d(fx,fw) + \frac{1}{n}d(x,w) \quad \forall n \in \mathbb{N}. \end{aligned} \tag{81}$$

The last inequality implies

$$\eta(\gamma)d(fx,Tx) = \frac{1}{1+\gamma}d(fx,Tx) \le d(fx,fw). \tag{82}$$

By (KS), we get

$$\mathscr{H}(Tx, Tw) \leq \gamma d(fx, fw) \quad \forall x \in X. \tag{83}$$

Claim 3. $\lim_{n\to\infty} d(fx_n, Tw) = 0$.

Since $fx_{n+1} \in Tx_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} fx_n = fw$, applying Claim 2, we have

$$\begin{aligned}\lim_{n\to\infty} d(fx_{n+1}, Tw) &\leq \lim_{n\to\infty} \mathscr{H}(Tx_n, Tw) \\ &\leq \lim_{n\to\infty} \gamma d(fx_n, fw) = 0,\end{aligned} \tag{84}$$

which implies $\lim_{n\to\infty} d(fx_n, Tw) = 0$.

By our Claims, we prove that (L2) holds. Therefore, by applying Theorem 18, we prove $COP(f, T) \neq \emptyset$. □

Take $f \equiv id$ (the identity map) in Theorem 30, we obtain the following existence theorem which is also a generalized Kikkawa-Suzuki's fixed point theorem.

Corollary 31. *Define a strictly decreasing function η from $[0, 1)$ onto $(1/2, 1]$ by*

$$\eta(r) = \frac{1}{1+r}. \tag{85}$$

Let (X, d) be a metric space and let T be a map from X into $CB(X)$. Assume that there exists $\gamma \in [0, 1)$ such that

$$\eta(\gamma) d(x, Tx) \leq d(x, y) \text{ implies } \mathscr{H}(Tx, Ty) \leq \gamma d(x, y), \tag{86}$$

for all $x, y \in X$. Then T has the approximate fixed property.

Moreover, if we further assume X is complete, then $\mathscr{F}(T) \neq \emptyset$.

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