Research Article

On Strong Convergence for Weighted Sums of a Class of Random Variables

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Let \( \{X_n, n \geq 1\} \) be a sequence of random variables satisfying the Rosenthal-type maximal inequality. Complete convergence is studied for linear statistics that are weighted sums of identically distributed random variables under a suitable moment condition. As an application, the Marcinkiewicz-Zygmund-type strong law of large numbers is obtained. Our result generalizes the corresponding one of Zhou et al. (2011) and improves the corresponding one of Wang et al. (2011, 2012).

1. Introduction

Throughout the paper, let \( I(A) \) be the indicator function of the set \( A \). \( C \) denotes a positive constant which may be different in various places, and \( a_n = O(b_n) \) stands for \( a_n \leq C b_n \). Denote \( \log x = \ln \max(x, e) \).

Let \( \{X, X_n, n \geq 1\} \) be a sequence of identically distributed random variables and \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) an array of constants. The strong convergence results for weighted sums \( \sum_{i=1}^{n} a_{ni} X_i \) have been studied by many authors; see, for example, Choi and Sung [1], Cuzick [2], Wu [3], Bai and Cheng [4], Chen and Gan [5], Cai [6], Sung [7, 8], Shen [9], Wang et al. [10–14], Zhou et al. [15], Wu [16–18], Xu and Tang [19], and so forth. Many useful linear statistics are these weighted sums. Examples include least squares estimators, nonparametric regression function estimators, and jackknife estimates among others. Bai and Cheng [4] proved the strong law of large numbers for weighted sums:

\[
\lim_{n \to \infty} \frac{1}{b_n} \sum_{i=1}^{n} a_{ni} X_i \rightarrow 0, \text{ a.s.,} 
\]

when \( \{X, X_n, n \geq 1\} \) is a sequence of independent and identically distributed random variables with \( EX = 0 \) and \( E \exp(h|X|^\gamma) < \infty \) for some \( h > 0 \) and \( \gamma > 0 \), and \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) is an array of constants satisfying

\[
A_\alpha = \limsup_{n \to \infty} A_\alpha, n < \infty, \quad A_\alpha, n = \frac{1}{n} \sum_{i=1}^{n} |a_{ni}|^\alpha, \quad (2)
\]

for some \( 1 < \alpha < 2 \), where \( b_n = n^{\frac{1}{\alpha}} (\log n)^{1/\gamma \left(\alpha - 1\right)/\alpha \left(1 + \gamma\right)} \).

Cai [6] generalized the result of Bai and Cheng [4] to the case of negatively associated (NA, in short) random variables and obtained the following complete convergence result for weighted sums of identically distributed NA random variables.

Theorem 1. Let \( \{X, X_n, n \geq 1\} \) be a sequence of NA random variables with identical distributions. And let \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) be a triangular array of constants satisfying \( \sum_{i=1}^{n} |a_{ni}|^\alpha = O(n) \) for \( 0 < \alpha \leq 2 \). Let \( b_n = n^{\frac{1}{\alpha}} \log^{1/\gamma} n \) for some \( \gamma > 0 \). Furthermore, assume that \( EX = 0 \) when \( 1 < \alpha \leq 2 \). If \( E \exp(h|X|^\gamma) < \infty \) for some \( h > 0 \), then

\[
\sum_{n=1}^{\infty} \frac{1}{n} P \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{n} a_{ni} X_i \right| > \varepsilon b_n \right) < \infty, \quad \forall \varepsilon > 0. \quad (3)
\]

for NA random variables under much weaker conditions. Zhou et al. [15] generalized the result of Sung [8] to the case of $\rho^*$-mixing random variables when $\alpha > \gamma$. The technique used in Sung [8] is the result of Chen et al. [20] for NA random variables, which is not proved for $\rho^*$-mixing random variables. The main purpose of the paper is to further study the strong convergence for a class of random variables satisfying the Rosenthal-type maximal inequality by using a different method from that of Sung [8]. We not only generalize the result of Zhou et al. [15] for $\rho^*$-mixing random variables to the case of a sequence of random variables satisfying the Rosenthal-type maximal inequality, but also consider the case of $\alpha \leq \gamma$. In addition, our main result improves the corresponding one of Wang et al. [11, 14], since the exponential moment condition is weakened to moment condition.

2. Main Results

In this section, we will study the strong convergence for a class of random variables satisfying the Rosenthal-type maximal inequality by using a different method from that of Sung [8]. As an application, the Marcinkiewicz-Zygmund-type strong law of large numbers is obtained.

Theorem 2. Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed random variables. Let $\{a_n, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying $\sum_{i=1}^{\infty} |a_n|^\alpha = O(n)$ for some $0 < \alpha \leq 2$. $E X_n = 0$ when $1 < \alpha \leq 2$. Let $b_n = n^{1/\alpha} \log^{1/\gamma} n$ for some $\gamma > 0$. Assume that for any $q \geq 2$, there exists a positive constant $C_q$ depending only on $q$ such that

$$E \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} (Y_{ni} - EY_{ni}) \right|^q \right) \leq C_q \left( \sum_{i=1}^{n} E |Y_{ni}|^q + \left( \sum_{i=1}^{n} E Y_{ni}^2 \right)^{q/2} \right),$$

(4)

where $Y_{ni} = -b_n I(a_n X_i < -b_n) + a_n X_i I(|a_n X_i| \leq b_n) + b_n I(a_n X_i > b_n)$ or $Y_{ni} = a_n X_i I(|a_n X_i| \leq b_n)$. Furthermore, suppose that

$$\sum_{n=1}^{\infty} \left[ \sum_{i=1}^{n} P \left( |a_n X_i| > b_n \right) \right]^{q/2} < \infty$$

(5)

for $Y_{ni} = -b_n I(a_n X_i < -b_n) + a_n X_i I(|a_n X_i| \leq b_n) + b_n I(a_n X_i > b_n)$. If

$$E|X_i|^\alpha < \infty, \quad \text{for } \alpha > \gamma,$$

$$E|X_i|^\alpha \log |X_i| < \infty, \quad \text{for } \alpha = \gamma,$$

$$E|X_i|^{\alpha \gamma} < \infty, \quad \text{for } \alpha < \gamma,$$

then (3) holds.

Proof. We only need to prove that (3) holds for $Y_{ni} = -b_n I(a_n X_i < -b_n) + a_n X_i I(|a_n X_i| \leq b_n) + b_n I(a_n X_i > b_n)$. The proof for $Y_{ni} = a_n X_i I(|a_n X_i| \leq b_n)$ is analogous.

Without loss of generality, we may assume that $\sum_{i=1}^{n} |a_n|^\alpha \leq n$. It is easy to check that for any $\epsilon > 0$,

$$\left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_n X_i \right| > \epsilon b_n \right) \leq \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_n X_i \right| > b_n \right) \cup \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} Y_{ni} \right| > \epsilon b_n \right),$$

which implies that

$$P \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_n X_i \right| > \epsilon b_n \right) \leq P \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_n X_i \right| > b_n \right) + P \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} Y_{ni} \right| > \epsilon b_n \right) \leq \sum_{i=1}^{n} P \left( |a_n X_i| > b_n \right) + P \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} (Y_{ni} - EY_{ni}) \right| > \epsilon b_n - \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} EY_{ni} \right| \right).$$

(8)

Firstly, we will show that

$$b_n^{-1} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} EY_{ni} \right| \to 0, \quad \text{as } n \to \infty.$$

(9)

When $1 < \alpha \leq 2$, we have by $E X_n = 0$, Markov’s inequality and (6) that

$$b_n^{-1} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} EY_{ni} \right| \leq \sum_{i=1}^{n} P \left( |a_n X_i| > b_n \right) + b_n^{-1} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_n EX_i I(|a_n X_i| > b_n) \right| \leq \sum_{i=1}^{n} P \left( |a_n X_i| > b_n \right) + b_n^{-1} \sum_{i=1}^{n} E |a_n X_i| I(|a_n X_i| > b_n) \leq b_n^{-\alpha} \sum_{i=1}^{n} |a_n|^\alpha E |X_i|^\alpha + b_n^{-\alpha} \sum_{i=1}^{n} |a_n|^\alpha E |X_i|^\alpha \leq 2E |X_1|^\alpha \left( \log n \right)^{-\alpha/\gamma} \to 0, \quad \text{as } n \to \infty.$$
When $0 < \alpha \leq 1$, we have by Markov’s inequality and (6) again that
\[
b^{-1} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} E Y_{ni} \right| \\
\leq \sum_{i=1}^{n} P \left( |a_{ni}X_{i}| > b_{n} \right) + b^{-1} \sum_{i=1}^{n} E \left| a_{ni}X_{i} \right| I \left( |a_{ni}X_{i}| \leq b_{n} \right) \\
= \sum_{i=1}^{n} P \left( |a_{ni}X_{i}| > b_{n} \right) + b^{-1} \sum_{i=1}^{n} E \left| a_{ni}X_{i} \right| I \left( |a_{ni}X_{i}| \leq b_{n} \right) \\
\leq b^{-1} \sum_{i=1}^{n} |a_{ni}|^{\alpha} E \left| X_{i} \right|^{\alpha} + b^{-1} b_{n} \sum_{i=1}^{n} E \left| a_{ni}X_{i} \right|^{\alpha} I \left( |a_{ni}X_{i}| \leq b_{n} \right) \\
\leq b_{n}^{-\alpha} n E \left| X_{1} \right|^{\alpha} + b_{n}^{-\alpha} n E \left| X_{1} \right|^{\alpha} \\
= 2 E \left| X_{1} \right|^{\alpha} \left( \log n \right)^{-\alpha/\gamma} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{11}
\]

By (10) and (11), we can get (9) immediately. Hence, for $n$ large enough,
\[
P \left( \max_{1 \leq j \leq n} \sum_{i=1}^{j} a_{ni}X_{i} > \varepsilon b_{n} \right) \\
\leq \sum_{i=1}^{n} P \left( |a_{ni}X_{i}| > b_{n} \right) + P \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} \left( Y_{ni} - E Y_{ni} \right) \right| > \frac{\varepsilon}{2} b_{n} \right). \tag{12}
\]

To prove (3), we only need to show that
\[
I \doteq \sum_{n=1}^{\infty} n^{-1} \left( \sum_{i=1}^{n} P \left( |a_{ni}X_{i}| > b_{n} \right) < \infty, \tag{13}
\]
\[
J \doteq \sum_{n=1}^{\infty} n^{1-\alpha} P \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} \left( Y_{ni} - E Y_{ni} \right) \right| > \frac{\varepsilon}{2} b_{n} \right) < \infty. \tag{14}
\]

Firstly, we will prove (13). By $\sum_{i=1}^{n} |a_{ni}|^{\alpha} \leq n$ and (6), we can get that
\[
I \leq \sum_{n=1}^{\infty} n^{-1} b_{n}^{-\alpha} \sum_{i=1}^{n} E \left| a_{ni}X_{i} \right|^{\alpha} I \left( |a_{ni}X_{i}| > b_{n} \right) \\
\leq \sum_{n=1}^{\infty} n^{-2} \left( \log n \right)^{-\alpha/\gamma} \\
\times \sum_{i=1}^{n} E \left| a_{ni}X_{i} \right|^{\alpha} I \left( \sum_{i=1}^{n} |a_{ni}X_{i}|^{\alpha} > n \left( \log n \right)^{\alpha/\gamma} \right)
\]
\[
\leq \sum_{n=1}^{\infty} n^{-2} \left( \log n \right)^{-\alpha/\gamma} \\
\times \sum_{i=1}^{n} E \left| a_{ni}X_{i} \right|^{\alpha} I \left( \left| X_{1} \right|^{\alpha} > \left( \log n \right)^{\alpha/\gamma} \right)
\]
\[
\leq \sum_{n=1}^{\infty} n^{-1} \left( \log n \right)^{-\alpha/\gamma} \\
\times \sum_{i=1}^{n} E \left| a_{ni}X_{i} \right|^{\alpha} I \left( \left| X_{1} \right|^{\alpha} > \left( \log n \right)^{\alpha/\gamma} \right)
\]
\[
= \sum_{n=1}^{\infty} n^{-1} \left( \log n \right)^{-\alpha/\gamma} \\
\times \sum_{i=1}^{n} E \left| X_{1} \right|^{\alpha} I \left( \log m < \left| X_{1} \right|^{\alpha} \leq \log (m+1) \right)
\]
\[
= \sum_{m=1}^{\infty} E \left| X_{1} \right|^{\alpha} I \left( \log m < \left| X_{1} \right|^{\alpha} \leq \log (m+1) \right)
\]

which implies (13).

In the following, we will prove (14). Let $q > \max \{ 2, \alpha, \gamma, 2\gamma/\alpha \}$. By Markov’s inequality and condition (4), we have
\[
I \leq C \sum_{n=1}^{\infty} n^{-1} b_{n}^{-q} E \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} \left( Y_{ni} - E Y_{ni} \right) \right|^{q} \right)
\]
\[
\leq C \sum_{n=1}^{\infty} n^{-1} b_{n}^{-q} \frac{n}{\sum_{i=1}^{n} E \left| Y_{ni} \right|^{q}} \\
+ C \sum_{n=1}^{\infty} n^{-1} b_{n}^{-q} \left( \sum_{i=1}^{n} E Y_{ni}^{2} \right)^{q/2}
\]
\[
= I_{1} + I_{2}.
\]

To prove (14), it suffices to show that $I_{1} < \infty$ and $I_{2} < \infty$. 

For $j \geq 1$ and $n \geq 2$, denote

$$I_{nj} = \left\{ 1 \leq i \leq n : \frac{n}{j + 1} < |a_{ni}|^{\alpha} \leq \frac{n}{j} \right\}. \quad (17)$$

In view of $\sum_{i=1}^{n} |a_{ij}|^{\alpha} \leq n$, it is easy to see that $\{I_{nj}, j \geq 1\}$ are disjoint and $\bigcup_{j=1}^{\infty} I_{nj} = \{ 1 \leq i \leq n : a_{ni} \neq 0 \}$. Hence, we have for all $m \geq 1$

$$n \geq \sum_{j=1}^{m} \sum_{i=1}^{n} |a_{ni}|^{\alpha} \geq n \sum_{j=1}^{m} (j + 1)^{-1} \#I_{nj} \quad (18)$$

$$\geq n \sum_{j=1}^{m} (j + 1)^{-q/\alpha} (j + 1)^{q/\alpha - 1} \#I_{nj} \quad (19)$$

$$\geq n \sum_{j=m}^{\infty} (j + 1)^{-q/\alpha} (m + 1)^{q/\alpha - 1} \#I_{nj},$$

which implies that for all $m \geq 1$,

$$\sum_{j=m}^{\infty} (j + 1)^{-q/\alpha} \#I_{nj} \leq C m^{-q/\alpha}, \quad n \geq 2. \quad (20)$$

By $C_{r}$’s inequality, (13) and (17), we can get that

$$J_{1} \leq C \sum_{n=2}^{\infty} n^{-1} \sum_{i=1}^{n} P \left( |a_{ni} X_{i} | > b_{n} \right)$$

$$+ C \sum_{n=1}^{\infty} n^{-1} b_{n}^{-q/\alpha} \sum_{i=1}^{n} E |a_{ni} X_{i} |^{q} \left( |a_{ni} X_{i} | \leq b_{n} \right)$$

$$\leq C \sum_{n=2}^{\infty} n^{-1} b_{n}^{-q/\alpha} \sum_{i=1}^{n} E |a_{ni} X_{i} |^{q} \left( |a_{ni} X_{i} | \leq b_{n} \right)$$

$$\leq C \sum_{n=2}^{\infty} n^{-1- q/\alpha} (\log n)^{-q/\gamma}$$

$$\times \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} E |a_{ni} X_{i} |^{q} \left( |X_{i} | \leq (j + 1)^{1/\alpha} (\log n)^{1/\gamma} \right)$$

$$\leq C \sum_{n=2}^{\infty} n^{-1- q/\alpha} (\log n)^{-q/\gamma}$$

$$\times \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} E |X_{i} |^{q} \left( |X_{i} | \leq (j + 1)^{1/\alpha} (\log n)^{1/\gamma} \right) I_{nj}$$
\[ \leq C \sum_{m=2}^{\infty} (\log m)^{1-\eta/\alpha} E|X_1|^\eta I \]
\[ \times \left( (\log (m-1))^{1/\gamma} < |X_1| \leq (\log m)^{1/\gamma} \right) \]
\[ \leq C \sum_{m=2}^{\infty} E|X_1|^\eta I \left( (\log (m-1))^{1/\gamma} < |X_1| \leq (\log m)^{1/\gamma} \right) \]
\[ \leq CE|X_1|^\eta < \infty. \quad (22) \]

By (21) and (22), we can get that \( J_1 < \infty \). Next, we will prove that \( J_1 < \infty \).

It follows by (6) and (19) again that
\[
J_{12} = C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-\eta/\gamma} \]
\[ \times \sum_{k=1}^{\infty} E|X_1|^\eta I \left( k^{1/\alpha} (\log n)^{1/\gamma} < |X_1| \leq (k+1)^{1/\alpha} (\log n)^{1/\gamma} \right) \]
\[ \times \sum_{j=k}^{\infty} j^{-\eta/\alpha} I_{nj} \]
\[ \leq C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-\eta/\gamma} \]
\[ \times \sum_{k=1}^{\infty} k^{1-\eta/\alpha} E|X_1|^\eta I \]
\[ \times \left( k^{1/\alpha} (\log n)^{1/\gamma} < |X_1| \leq (k+1)^{1/\alpha} (\log n)^{1/\gamma} \right) \]
\[ \leq C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-\eta/\gamma} \]
\[ \times \sum_{k=1}^{\infty} E|X_1|^\eta I \left( k^{1/\alpha} (\log n)^{1/\gamma} < |X_1| \leq (k+1)^{1/\alpha} (\log n)^{1/\gamma} \right) \]
\[ = C \sum_{m=2}^{\infty} E|X_1|^\eta I \left( (\log m)^{1/\gamma} < |X_1| \leq (\log (m+1))^{1/\gamma} \right) \]
\[ \times \sum_{n=2}^{m} n^{-1} (\log n)^{-\eta/\gamma} \]
\[ \leq \left\{ \begin{array}{ll}
C \sum_{m=1}^{\infty} E|X_1|^\eta I \left( \log m < |X_1| \leq \log (m+1) \right), \\
\quad \text{for } \alpha > \gamma,
\end{array} \right.
\]
\[ \leq \left\{ \begin{array}{ll}
C \sum_{m=1}^{\infty} E|X_1|^\eta I \left( \log m < |X_1| \leq \log (m+1) \right) \times \log \log m, \\
\quad \text{for } \alpha = \gamma,
\end{array} \right.
\]
\[ \leq \left\{ \begin{array}{ll}
C \sum_{m=1}^{\infty} E|X_1|^\eta I \left( \log m < |X_1| \leq \log (m+1) \right) \times (\log m)^{1-\eta/\alpha}, \\
\quad \text{for } \alpha < \gamma.
\end{array} \right.
\]
\[ \leq \left\{ \begin{array}{ll}
CE|X_1|^\eta, \\
\quad \text{for } \alpha > \gamma,
\end{array} \right.
\]
\[ \leq \left\{ \begin{array}{ll}
CE|X_1|^\eta \log |X_1|, \\
\quad \text{for } \alpha = \gamma,
\end{array} \right.
\]
\[ \leq CE|X_1|^\eta, \quad \text{for } \alpha < \gamma \]
\[ < \infty. \quad (23) \]

By \( J_{11} < \infty \) and \( J_{12} < \infty \), we can get that \( J_1 < \infty \).

To prove (14), it suffices to show that \( J_2 < \infty \). By \( C_r \)'s inequality, conditions (5) and (6), we can get that
\[
J_2 \leq C \sum_{n=1}^{\infty} n^{-1} \left[ \sum_{i=1}^{n} P \left( |a_i X_i| > b_n \right) \right] \left( \frac{\eta}{2} \right)
+ C \sum_{n=1}^{\infty} n^{-1} \left[ \sum_{i=1}^{n} E|a_i X_i|^\eta I \left( |a_i X_i| \leq b_n \right) \right] \left( \frac{\eta}{2} \right)
\leq C \sum_{n=1}^{\infty} n^{-1} \left[ \sum_{i=1}^{n} b_n^{-\eta} E|a_i X_i|^\eta I \left( |a_i X_i| \leq b_n \right) \right] \left( \frac{\eta}{2} \right)
\leq C \left( E|X_1|^\eta \right) \left( \frac{\eta}{2} \right) \sum_{n=1}^{\infty} n^{-1} (\log n)^{-\eta/2} < \infty.
\]

Therefore, (14) follows from (16) and \( J_1 < \infty \), \( J_2 < \infty \) immediately. This completes the proof of the theorem. \( \square \)

The following result provides the Marcinkiewicz-Zygmund-type strong law of large numbers for weighted sums \( \sum_{i=1}^{n} a_i X_i \) of a class of random variables satisfying the Rosenthal-type maximal inequality.

**Theorem 3.** Let \( \{X_n, n \geq 1\} \) be a sequence of identically distributed random variables. Let \( \{a_n, n \geq 1\} \) be a sequence of constants satisfying \( \sum_{i=1}^{n} |a_i|^\alpha = O(n) \) for some \( 0 < \alpha \leq 2 \).

\[ EX_n = 0 \text{ when } 1 < \alpha \leq 2. \text{ Let } b_{n} = n^{1/\alpha} \log^{1/\gamma} n \text{ for some } \gamma > 0. \text{ Assume that for any } q \geq 2, \text{ there exists a positive constant } C_q \text{ depending only on } q \text{ such that } (4) \text{ holds, where} \]

\[ \leq CE|X_1|^\eta \log |X_1|, \quad \text{for } \alpha = \gamma,
\]

\[ CE|X_1|^\eta, \quad \text{for } \alpha < \gamma \]
\[ Y_n = -b_n I(a_i X_i < -b_n) + a_i X_i I(|a_i X_i| \leq b_n) + b_n I(a_i X_i > b_n) \]

or \[ Y_n = a_i X_i I(|a_i X_i| \leq b_n). \]

Further, \( Y_n \) satisfies the conditions of Theorem 2. So it suffices to show (27). Denote \( Y_n = \sum_{i=1}^{n} a_i X_i \) for each \( n \geq 1 \). It follows by (26) that

\[
\begin{align*}
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} a_i X_i & \to 0 \quad \text{a.s., as } n \to \infty. \\
\max_{i \leq j \leq n} |S_j| & \to 0 \quad \text{a.s.}
\end{align*}
\]

For all positive integers \( n \), there exists a positive integer \( i_0 \) such that \( 2^{i_0} \leq n < 2^{i_0+1} \). We have by (29) that

\[
\frac{|S_n|}{b_n} \leq \max_{2^{i_0} \leq i \leq 2^{i_0+1}} |S_i| \leq \frac{2^{2i_0}}{2^{(i_0+1)n} \log(2^{i_0+1})^{1/\gamma}} \to 0 \quad \text{a.s., as } i_0 \to \infty,
\]

which implies (27). This completes the proof of the theorem.

If the Rosenthal type inequality for the maximal partial sum is replaced by the partial sum, then we can get the following complete convergence result for a class of random variables. The proof is similar to that of Theorem 2. So the details are omitted.

**Theorem 4.** Let \( \{X_n, n \geq 1\} \) be a sequence of identically distributed random variables. Let \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) be an array of constants satisfying \( \sum_{i=1}^{n} |a_{ni}|^\alpha = O(n) \) for some \( 0 < \alpha \leq 2 \). \( E X_n = 0 \) when \( 1 < \alpha \leq 2 \). Let \( b_n = n^{\gamma/\alpha} \log^{1/\gamma} n \) for some \( \gamma > 0 \). Assume that for any \( q \geq 2 \), there exists a positive constant \( C_q \) depending only on \( q \) such that

\[
E \left( \sum_{i=1}^{n} (Y_n - E Y_n)^q \right) \leq C_q \left[ \sum_{i=1}^{n} E |Y_n|^q + \left( \sum_{i=1}^{n} E Y_n^q \right)^{q/2} \right],
\]

where \( Y_n = -b_n I(a_{ni} X_i < -b_n) + a_{ni} X_i I(|a_{ni} X_i| \leq b_n) + b_n I(a_{ni} X_i > b_n) \) or \( Y_n = a_{ni} X_i I(|a_{ni} X_i| \leq b_n) \). Furthermore, suppose that (5) holds for \( Y_n \) as in (6).

\[
\sum_{i=1}^{n} P \left( \max_{1 \leq j \leq n} |S_j| > \varepsilon b_n \right) < \infty, \quad \forall \varepsilon > 0.
\]

Remark 5. There are many sequences of dependent random variables satisfying (4) for all \( q \geq 2 \). Examples include sequences of NA random variables (see Shao [21]), \( \rho^* \)–mixing random variables (see Utev and Peligrad [22]), \( \varphi \)–mixing random variables with the mixing coefficients satisfying certain conditions (see Wang et al. [23]), \( \rho \)–mixing random variables with the mixing coefficients satisfying certain conditions (see Wang and Lu [24]), and asymptotically almost negatively associated random variables (see Yuan and An [25]). There are many sequences of dependent random variables satisfying (31) for all \( q \geq 2 \). Examples not only include the sequences of above, but also include sequences of NOD random variables (see Asadian et al. [26]) and extended negatively dependent random variables (see Shen [27]).

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