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Research Article

Algorithmic Approach to the Split Problems

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This paper deals with design algorithms for the split variational inequality and equilibrium problems. Strong convergence theorems are demonstrated.

1. Introduction

Let $\mathbb{H}$ be a real Hilbert space. Let $\mathbb{C}$ and $\mathbb{Q}$ be two nonempty closed convex subsets of $\mathbb{H}$. Consider the following problem.

Problem 1. Find a point $u^{\S} \in \mathbb{C}$ such that

$$\Psi\left(u^{\S}\right) \in \mathbb{Q}. \tag{1}$$

This problem is called split feasibility problem when Ψ is a bounded linear operator. In this case, Problem 1 can be applied to many practical problems such as signal processing and image reconstruction. Specifically, we can find the prototype of Problem 1 in intensity-modulated radiation therapy; see, for example, [1–3]. Based on this relation, many mathematicians were devoted to study the split feasibility problem and develop its iterative algorithms. Related works can be found in [4–8] and the references therein.

Let $\mathbb{A}, \Psi : \mathbb{C} \to \mathbb{H}$ be two mappings. Consider the variational inequality of finding $u^{\dagger} \in \mathbb{C}, \Psi(u^{\dagger}) \in \mathbb{C}$ such that

$$\left\langle \mathbb{A}u^{\dagger}, \Psi(u) - \Psi\left(u^{\dagger}\right)\right\rangle \geq 0, \tag{2}$$

for all $\Psi(u) \in \mathbb{C}$. We use $\mathrm{VI}(\mathbb{A}, \Psi)$ to denote the set of solutions of (2). Variational inequality problems have important applications in many fields such as elasticity, optimization, economics, transportation, and structural analysis, and various numerical methods have been studied by many researchers; see, for instance, [9–17].

Let $\varrho : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ be an equilibrium bifunction; that is, $\varrho(u, u) = 0$ for each $u \in \mathbb{C}$. Consider the equilibrium problem which is to find $u^* \in \mathbb{C}$ such that

$$\varrho\left(u^*, v\right) \geq 0, \quad \forall v \in \mathbb{C}. \tag{3}$$

Denote the set of solutions of (3) by $\mathrm{EP}(\varrho, \mathbb{C})$. The equilibrium problems include fixed point problems, optimization problems, and variational inequality problems as special cases. Some algorithms have been proposed to solve the equilibrium problems; see, for example, [18–22]. Thus it is an interesting topic associated with algorithmic approach to the variational inequality and equilibrium problems. In this paper, our main purpose is to study the following split problem involved in the variational inequality and equilibrium problems. Find a point $x^{\natural}$ such that

$$\begin{gathered} x^{\natural} \in \mathrm{VI}(\mathbb{A}, \Psi), \\ \Psi\left(x^{\natural}\right) \in \mathrm{EP}(\varrho, \mathbb{C}). \end{gathered} \tag{4}$$

We are devoted to study (4) with operator Ψ being a nonlinear mapping. For this purpose, we develop an iterative algorithm for solving the split problem (4). We can compute $x^{\natural}$ iteratively by using our algorithm. Convergence analysis is given under some mild assumptions.

2. Basic Concepts

Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. An operator $\mathbb{B} : \mathbb{C} \to \mathbb{H}$ is said to be

(i) monotone $\twoheadrightarrow \langle u - v, \mathbb{B}u - \mathbb{B}v\rangle \geq 0$ for all $u, v \in \mathbb{C}$;

(ii) strongly monotone $\twoheadrightarrow$ $\langle u-v, \mathbb{B}u-\mathbb{B}v\rangle \geq \zeta\|u-v\|^2$ for some constant $\zeta > 0$ and for all $u, v \in \mathbb{C}$;

(iii) inverse-strongly monotone $\twoheadrightarrow$ $\langle u-v, \mathbb{B}u-\mathbb{B}v\rangle \geq \varsigma\|\mathbb{B}u-\mathbb{B}v\|^2$ for some $\varsigma > 0$ and for all $u, v \in \mathbb{C}$; in this case, $\mathbb{B}$ is called ς-inverse strongly monotone;

(iv) ς-inverse strongly θ-monotone $\twoheadrightarrow$ $\langle \theta(u)-\theta(v), \mathbb{B}u-\mathbb{B}v\rangle \geq \varsigma\|\mathbb{B}u-\mathbb{B}v\|^2$ for all $u, v \in \mathbb{C}$ and for some $\varsigma > 0$, where $\theta : \mathbb{C} \to \mathbb{C}$ is a mapping.

A mapping $\vartheta : \mathbb{C} \to \mathbb{H}$ is said to be

(i) nonexpansive $\twoheadrightarrow$ $\|\vartheta u-\vartheta v\| \leq \|u-v\|$ for all $u, v \in \mathbb{C}$;

(ii) firmly nonexpansive $\twoheadrightarrow$ $\|\vartheta u-\vartheta v\|^2 \leq \langle u-v, \vartheta u-\vartheta v\rangle$ for all $u, v \in \mathbb{C}$;

(iii) L-Lipschitz continuous $\twoheadrightarrow$ $\|\vartheta u-\vartheta v\| \leq L\|u-v\|$ for some constant $L > 0$ and for all $u, v \in \mathbb{C}$. In such a case, ϑ is said to be L-Lipschitz continuous.

In the sequel, we use $\mathrm{Fix}(\vartheta)$ to denote the set of fixed points of ϑ.

Let $\mathbb{A} : \mathbb{H} \to 2^{\mathbb{H}}$ be a multivalued mapping. The effective domain of $\mathbb{A}$ is denoted by $\mathrm{dom}(\mathbb{A})$. $\mathbb{A}$ is said to be

(i) monotone $\twoheadrightarrow$ $\langle x-y, u-v\rangle \geq 0$ for all $x, y \in \mathrm{dom}(\mathbb{A})$, $u \in \mathbb{A}x$, and $v \in \mathbb{A}y$;

(ii) maximal monotone $\twoheadrightarrow$ $\mathbb{A}$ is monotone and its graph is not strictly contained in the graph of any other monotone operator on $\mathbb{H}$.

A function $f : \mathbb{H} \to \mathbb{R}$ is said to be convex if for any $u, v \in \mathbb{H}$ and for any $\tau \in [0,1]$, $f(\tau u+(1-\tau)v) \leq \tau f(u)+(1-\tau)f(v)$.

Let $\mathrm{proj}_{\mathbb{C}} : \mathbb{C} \to \mathbb{H}$ be the metric projection from $\mathbb{H}$ onto $\mathbb{C}$. It is known that $\mathrm{proj}_{\mathbb{C}}$ satisfies the following inequality:

$$\langle x-\mathrm{proj}_{\mathbb{C}}x, y-\mathrm{proj}_{\mathbb{C}}x\rangle \leq 0. \tag{5}$$

for all $x \in \mathbb{H}$ and $y \in \mathbb{C}$. From this characteristic inequality, we can deduce that $\mathrm{proj}_{\mathbb{C}}$ is firmly nonexpansive.

3. Useful Lemmas

In this section, we present several lemmas which will be used in the next section.

Lemma 2 (see [19]). *Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let $\varrho : \mathbb{C}\times\mathbb{C} \to \mathbb{R}$ be a bifunction. Assume that ϱ satisfies the following conditions:*

($\mathfrak{F}1$) *$\varrho(u,u) = 0$ for all $u \in \mathbb{C}$;*

($\mathfrak{F}2$) *ϱ is monotone, that is, $\varrho(u,v)+\varrho(v,u) \leq 0$ for all $u, v \in \mathbb{C}$;*

($\mathfrak{F}3$) *for each $u, v, w \in \mathbb{C}$, $\lim_{t\downarrow 0}\varrho(tw+(1-t)u, v) \leq \varrho(u,v)$;*

($\mathfrak{F}4$) *for each $u \in \mathbb{C}$, $v \mapsto \varrho(u,v)$ is convex and lower semicontinuous.*

Let $\varpi > 0$ and $u \in \mathbb{C}$. Then there exists $w \in \mathbb{C}$ such that

$$\varrho(w,v)+\frac{1}{\varpi}\langle v-w, w-u\rangle \geq 0, \quad \forall v \in \mathbb{C}. \tag{6}$$

Set $F_{\varpi}(u) = \{w \in \mathbb{C} : \varrho(w,v)+(1/\varpi)\langle v-w, w-u\rangle \geq 0$ for all $v \in \mathbb{C}\}$. Then one have the following:

(i) *F_{ϖ} is single valued and F_{ϖ} is firmly nonexpansive,*

(ii) *$\mathrm{EP}(\varrho,\mathbb{C})$ is closed and convex and $\mathrm{EP}(\varrho,\mathbb{C}) = \mathrm{Fix}(F_{\varpi})$.*

Lemma 3 (see [23]). *Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. For $x \in \mathbb{H}$, let the mapping F_{ϖ} be the same as in Lemma 2. Then for $\mu, \nu > 0$ and $x \in \mathbb{H}$, one has*

$$\left\|F_{\mu}(x)-F_{\nu}(x)\right\|^2 \leq \frac{\mu-\nu}{\mu}\left\langle F_{\mu}(x)-F_{\nu}(x), F_{\mu}(x)-x\right\rangle. \tag{7}$$

Lemma 4 (see [24]). *Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in a Banach space $\mathbb{E}$, and let $\{\kappa_n\}$ be a sequence in $[0,1]$ satisfying $0 < \liminf_{n\to\infty}\kappa_n \leq \limsup_{n\to\infty}\kappa_n < 1$. Suppose $u_{n+1} = (1-\kappa_n)v_n+\kappa_n u_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|v_{n+1}-v_n\|-\|u_{n+1}-u_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|u_n-v_n\| = 0$.*

Lemma 5 (see [25]). *Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let $\mathbb{S} : \mathbb{C} \to \mathbb{C}$ be a nonexpansive mapping with $\mathrm{Fix}(\mathbb{S}) \neq \emptyset$. Then $\mathbb{S}$ is demiclosed on $\mathbb{C}$.*

Lemma 6 (see [26]). *Let $\{a_n\} \subset [0,\infty)$ be a sequence. Assume that $a_{n+1} \leq (1-\gamma_n)a_n+\delta_n\gamma_n$, where $\{\gamma_n\}$ is a sequence in $(0,1)$, and $\{\delta_n\}$ is a sequence satisfying $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leq 0$ (or $\sum_{n=1}^{\infty}|\delta_n\gamma_n| < \infty$). Then $\lim_{n\to\infty}a_n = 0$.*

4. Main Results

In this section, we firstly present our problem and algorithm constructed. Consequently, we give the convergence analysis of the presented algorithm.

Problem 7. Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Assume that

(1) $\Psi : \mathbb{C} \to \mathbb{C}$ is a weakly continuous and ζ-strongly monotone mapping such that $R(\Psi) = \mathbb{C}$;

(2) $\mathbb{A} : \mathbb{C} \to \mathbb{H}$ is an ς-inverse strongly Ψ-monotone mapping;

(3) $\varrho : \mathbb{C}\times\mathbb{C} \to \mathbb{R}$ is a bifunction satisfying conditions ($\mathfrak{F}1$)–($\mathfrak{F}4$) in Lemma 2.

Our objective is to

$$\text{find } x^{\natural} \in \mathrm{VI}(\mathbb{A},\Psi) \text{ such that } \Psi\left(x^{\natural}\right) \in \mathrm{EP}(\varrho,\mathbb{C}). \tag{8}$$

We use Υ to denote the set of solutions of (8). In the following, we assume that Υ is nonempty. For solving Problem 7, we introduce the following algorithm.

Algorithm 8.
Step 0 (initialization). Let

$$u_0 \in \mathbb{C}. \tag{9}$$

Step 1. For given $\{u_n\}$, let the sequence $\{v_n\}$ be generated iteratively by

$$v_n = \mathrm{proj}_{\mathbb{C}}\left(\Psi\left(u_n\right) - \mu_n \mathbb{A} u_n\right), \quad n \geq 0, \tag{10}$$

where $\mathrm{proj}_{\mathbb{C}}$ is the metric projection and $\{\mu_n\}$ is a real number sequence.

Step 2. For given $\{v_n\}$, find $\{z_n\}$ such that

$$\varrho\left(z_n, y\right) + \frac{1}{\varpi_n}\left\langle y - z_n, z_n - \left(1 - \alpha_n\right) v_n\right\rangle \geq 0, \quad \forall y \in \mathbb{C}, \tag{11}$$

where $\{\varpi_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$ are two real number sequences.

Step 3. For the previous sequences $\{u_n\}$ and $\{z_n\}$, let the $(n+1)$th sequence $\{u_{n+1}\}$ be generated by

$$\Psi\left(u_{n+1}\right) = \kappa_n \Psi\left(u_n\right) + \left(1 - \kappa_n\right) z_n, \quad n \geq 0, \tag{12}$$

where $\{\kappa_n\} \subset [0, 1]$ is a real number sequence.

Theorem 9. *Assume that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty}\kappa_n \leq \limsup_{n\to\infty}\kappa_n < 1$;

(C3) $\varpi_n \in (\eta_1, \eta_2) \subset (0, \infty)$, $\mu_n \in (\xi_1, \xi_2) \subset (0, 2\varsigma)$, *and* $\zeta \in (0, 2\varsigma)$;

(C4) $\lim_{n\to\infty}(\mu_{n+1} - \mu_n) = 0$ *and* $\lim_{n\to\infty}(\varpi_{n+1} - \varpi_n) = 0$.

Then the sequence $\{u_n\}$ *generated by Algorithm 8 converges strongly to* $x^* \in \Upsilon$.

Proof. Let $\check{x} \in \Upsilon$. Hence $\check{x} \in \mathrm{VI}(\mathbb{A}, \Psi)$ and $\Psi(\check{x}) \in \mathrm{EP}(\varrho, \mathbb{C})$, noting that $\check{x} \in \mathrm{VI}(\mathbb{A}, \Psi)$ implies $\Psi(\check{x}) = \mathrm{proj}_{\mathbb{C}}(\Psi(\check{x}) - \nu\mathbb{A}\check{x})$ for all $\nu > 0$. Hence $\Psi(\check{x}) = \mathrm{proj}_{\mathbb{C}}(\Psi(\check{x}) - \mu_n\mathbb{A}\check{x})$ for all $n \geq 0$. Thus, from (10), we have

$$\begin{aligned}
&\left\|v_n - \Psi(\check{x})\right\|^2 \\
&\quad = \left\|\mathrm{proj}_{\mathbb{C}}\left(\Psi\left(u_n\right) - \mu_n \mathbb{A} u_n\right) - \mathrm{proj}_{\mathbb{C}}\left(\Psi(\check{x}) - \mu_n \mathbb{A}\check{x}\right)\right\|^2 \\
&\quad \leq \left\|\left(\Psi\left(u_n\right) - \mu_n \mathbb{A} u_n\right) - \left(\Psi(\check{x}) - \mu_n \mathbb{A}\check{x}\right)\right\|^2 \\
&\quad = \left\|\Psi\left(u_n\right) - \Psi(\check{x})\right\|^2 - 2\mu_n\left\langle \mathbb{A} u_n - \mathbb{A}\check{x}, \Psi\left(u_n\right) - \Psi(\check{x})\right\rangle \\
&\qquad + \mu_n^2\left\|\mathbb{A} u_n - \mathbb{A}\check{x}\right\|^2 \\
&\quad \leq \left\|\Psi\left(u_n\right) - \Psi(\check{x})\right\|^2 \\
&\qquad - 2\mu_n\varsigma\left\|\mathbb{A} u_n - \mathbb{A}\check{x}\right\|^2 + \mu_n^2\left\|\mathbb{A} u_n - \mathbb{A}\check{x}\right\|^2 \\
&\quad \leq \left\|\Psi\left(u_n\right) - \Psi(\check{x})\right\|^2 + \mu_n\left(\mu_n - 2\varsigma\right)\left\|\mathbb{A} u_n - \mathbb{A}\check{x}\right\|^2.
\end{aligned} \tag{13}$$

Condition (C3) and (13) imply that

$$\left\|v_n - \Psi(\check{x})\right\| \leq \left\|\Psi\left(u_n\right) - \Psi(\check{x})\right\|. \tag{14}$$

From Lemma 2 and (11), we get $z_n = \digamma_{\varpi_n}(1 - \alpha_n)v_n$ for all $n \geq 0$. Since $\Psi(\check{x}) \in \mathrm{EP}(\varrho, \mathbb{C})$, from Lemma 2 we deduce that $\Psi(\check{x}) = \digamma_{\varpi_n}\Psi(\check{x})$ for all $n \geq 0$. So,

$$\begin{aligned}
&\left\|z_n - \Psi(\check{x})\right\| \\
&\quad = \left\|\digamma_{\varpi_n}\left(1 - \alpha_n\right) v_n - \digamma_{\varpi_n}\Psi(\check{x})\right\| \\
&\quad \leq \left\|\left(1 - \alpha_n\right) v_n - \Psi(\check{x})\right\| \\
&\quad \leq \left(1 - \alpha_n\right)\left\|v_n - \Psi(\check{x})\right\| + \alpha_n\left\|\Psi(\check{x})\right\| \\
&\quad \text{by (14)} \\
&\quad \leq \left(1 - \alpha_n\right)\left\|\Psi\left(u_n\right) - \Psi(\check{x})\right\| + \alpha_n\left\|\Psi(\check{x})\right\|.
\end{aligned} \tag{15}$$

It follows that

$$\begin{aligned}
&\left\|\Psi\left(u_{n+1}\right) - \Psi(\check{x})\right\| \\
&\quad \leq \kappa_n\left\|\Psi\left(u_n\right) - \Psi(\check{x})\right\| + \left(1 - \kappa_n\right)\left\|z_n - \Psi(\check{x})\right\| \\
&\quad \leq \kappa_n\left\|\Psi\left(u_n\right) - \Psi(\check{x})\right\| \\
&\qquad + \left(1 - \kappa_n\right)\left(1 - \alpha_n\right)\left\|\Psi\left(u_n\right) - \Psi(\check{x})\right\| \\
&\qquad + \left(1 - \kappa_n\right)\alpha_n\left\|\Psi(\check{x})\right\| \\
&\quad = \left[1 - \left(1 - \kappa_n\right)\alpha_n\right]\left\|\Psi\left(u_n\right) - \Psi(\check{x})\right\| \\
&\qquad + \left(1 - \kappa_n\right)\alpha_n\left\|\Psi(\check{x})\right\|.
\end{aligned} \tag{16}$$

By induction

$$\left\|\Psi\left(u_n\right) - \Psi(\check{x})\right\| \leq \max\left\{\left\|\Psi\left(u_0\right) - \Psi(\check{x})\right\|, \left\|\Psi(\check{x})\right\|\right\}. \tag{17}$$

Hence, $\{\Psi(u_n)\}$ is bounded. Since Ψ is ζ-strongly monotone, we can get $\zeta\|u_n - \check{x}\| \leq \|\Psi(u_n) - \Psi(\check{x})\|$. So, $\|u_n - \check{x}\| \leq (1/\zeta)\|\Psi(u_n) - \Psi(\check{x})\| \leq (1/\zeta)\max\{\|\Psi(u_0) - \Psi(\check{x})\|, \|\Psi(\check{x})\|\}$. This implies that $\{u_n\}$ is bounded. Next, we show $\|u_{n+1} - u_n\| \to 0$. From $z_n = \digamma_{\varpi_n}(1 - \alpha_n)v_n$, we have

$$\begin{aligned}
&\left\|z_{n+1} - z_n\right\| \\
&\quad = \left\|\digamma_{\varpi_{n+1}}\left(1 - \alpha_{n+1}\right) v_{n+1} - \digamma_{\varpi_n}\left(1 - \alpha_n\right) v_n\right\| \\
&\quad \leq \left\|\digamma_{\varpi_{n+1}}\left(1 - \alpha_{n+1}\right) v_{n+1} - \digamma_{\varpi_{n+1}}\left(1 - \alpha_n\right) v_n\right\| \\
&\qquad + \left\|\digamma_{\varpi_{n+1}}\left(1 - \alpha_n\right) v_n - \digamma_{\varpi_n}\left(1 - \alpha_n\right) v_n\right\| \\
&\quad \leq \left\|\left(1 - \alpha_{n+1}\right) v_{n+1} - \left(1 - \alpha_n\right) v_n\right\| \\
&\qquad + \left\|\digamma_{\varpi_{n+1}}\left(1 - \alpha_n\right) v_n - \digamma_{\varpi_n}\left(1 - \alpha_n\right) v_n\right\|.
\end{aligned} \tag{18}$$

Using Lemma 3, we obtain

$$\begin{aligned}
&\left\|F_{\varpi_{n+1}}(1-\alpha_n)v_n - F_{\varpi_n}(1-\alpha_n)v_n\right\|^2 \\
&\quad\le \frac{\varpi_{n+1}-\varpi_n}{\varpi_{n+1}} \\
&\quad\times \Big\langle F_{\varpi_{n+1}}(1-\alpha_n)v_n \\
&\qquad - F_{\varpi_n}(1-\alpha_n)v_n, F_{\varpi_{n+1}}(1-\alpha_n)v_n \\
&\qquad - (1-\alpha_n)v_n\Big\rangle \\
&\quad\le \frac{|\varpi_{n+1}-\varpi_n|}{\varpi_{n+1}}\left\|F_{\varpi_{n+1}}(1-\alpha_n)v_n - F_{\varpi_n}(1-\alpha_n)v_n\right\| \\
&\quad\times \left\|F_{\varpi_{n+1}}(1-\alpha_n)v_n - (1-\alpha_n)v_n\right\|.
\end{aligned} \tag{19}$$

Then

$$\begin{aligned}
&\left\|F_{\varpi_{n+1}}(1-\alpha_n)v_n - F_{\varpi_n}(1-\alpha_n)v_n\right\| \\
&\quad\le \frac{|\varpi_{n+1}-\varpi_n|}{\varpi_{n+1}}\left\|F_{\varpi_{n+1}}(1-\alpha_n)v_n - (1-\alpha_n)v_n\right\|.
\end{aligned} \tag{20}$$

By condition $(C3)$, we have $\varpi_n > \eta_1 > 0$. So,

$$\begin{aligned}
&\|z_{n+1}-z_n\| \\
&\quad\le \|(1-\alpha_{n+1})v_{n+1} - (1-\alpha_n)v_n\| \\
&\qquad + \frac{|\varpi_{n+1}-\varpi_n|}{\varpi_{n+1}} \\
&\qquad\times \left\|F_{\varpi_{n+1}}(1-\alpha_n)v_n - (1-\alpha_n)v_n\right\| \\
&\quad\le (1-\alpha_{n+1})\|v_{n+1}-v_n\| \\
&\qquad + |\alpha_{n+1}-\alpha_n|\|v_n\| + \frac{|\varpi_{n+1}-\varpi_n|}{\eta_1} \\
&\qquad\times \left\|F_{\varpi_{n+1}}(1-\alpha_n)v_n - (1-\alpha_n)v_n\right\|.
\end{aligned} \tag{21}$$

From (10), we have

$$\begin{aligned}
&\|v_{n+1}-v_n\| \\
&\quad= \|\mathrm{proj}_{\mathbb{C}}(\Psi(u_{n+1}) - \mu_{n+1}\mathbb{A}u_{n+1}) \\
&\qquad -\mathrm{proj}_{\mathbb{C}}(\Psi(u_n) - \mu_n\mathbb{A}u_n)\| \\
&\quad\le \|\Psi(u_{n+1}) - \mu_{n+1}\mathbb{A}u_{n+1} \\
&\qquad - (\Psi(u_n) - \mu_{n+1}\mathbb{A}u_n)\| + |\mu_{n+1}-\mu_n|\|\mathbb{A}(u_n)\| \\
&\quad\le \|\Psi(u_{n+1}) - \Psi(u_n)\| + |\mu_{n+1}-\mu_n|\|\mathbb{A}(u_n)\|.
\end{aligned} \tag{22}$$

Therefore,

$$\begin{aligned}
&\|z_{n+1}-z_n\| \le (1-\alpha_{n+1})\|\Psi(u_{n+1}) - \Psi(u_n)\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\|v_n\| + |\mu_{n+1}-\mu_n|\|\mathbb{A}(u_n)\| \\
&\quad + \frac{1}{\eta_1}|\varpi_{n+1}-\varpi_n|\left\|F_{\varpi_{n+1}}(1-\alpha_n)v_n - (1-\alpha_n)v_n\right\|.
\end{aligned} \tag{23}$$

It follows that

$$\begin{aligned}
&\|z_{n+1}-z_n\| - \|\Psi(u_{n+1}) - \Psi(u_n)\| \\
&\quad\le |\alpha_{n+1}-\alpha_n|\|v_n\| + |\mu_{n+1}-\mu_n|\|\mathbb{A}(u_n)\| \\
&\qquad + \frac{1}{\eta_1}|\varpi_{n+1}-\varpi_n|\left\|F_{\varpi_{n+1}}(1-\alpha_n)v_n - (1-\alpha_n)v_n\right\|.
\end{aligned} \tag{24}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}(\mu_{n+1}-\mu_n) = 0$, and $\lim_{n\to\infty}(\varpi_{n+1}-\varpi_n) = 0$ and the sequences $\{\Psi(u_n)\}$, $\{z_n\}$, $\{v_n\}$, and $\{\mathbb{A}u_n\}$ are bounded, we deduce that

$$\limsup_{n\to\infty}\left(\|z_{n+1}-z_n\| - \|\Psi(u_{n+1}) - \Psi(u_n)\|\right) \le 0. \tag{25}$$

Applying Lemma 4, we obtain

$$\lim_{n\to\infty}\|z_n - \Psi(u_n)\| = 0. \tag{26}$$

Thus,

$$\lim_{n\to\infty}\|\Psi(u_{n+1}) - \Psi(u_n)\| = \lim_{n\to\infty}(1-\kappa_n)\|z_n - \Psi(u_n)\| = 0. \tag{27}$$

This together with the ζ-strong monotonicity of Ψ implies that

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| = 0. \tag{28}$$

From (13) and (16), we derive

$$\begin{aligned}
&\|\Psi(u_{n+1}) - \Psi(\check{x})\|^2 \\
&\quad\le \kappa_n\|\Psi(u_n) - \Psi(\check{x})\|^2 + (1-\kappa_n)\|z_n - \Psi(\check{x})\|^2 \\
&\quad\le \kappa_n\|\Psi(u_n) - \Psi(\check{x})\|^2 + (1-\kappa_n) \\
&\qquad\times \left[(1-\alpha_n)\|v_n - \Psi(\check{x})\|^2 + \alpha_n\|\Psi(\check{x})\|^2\right]
\end{aligned}$$

$$\begin{aligned}
&\le (1-\kappa_n) \\
&\quad \times \left[(1-\alpha_n)\left\|(\Psi(u_n)-\mu_n\mathbb{A}u_n)-(\Psi(\check{x})-\mu_n\mathbb{A}\check{x})\right\|^2\right. \\
&\qquad \left.+\alpha_n\|\Psi(\check{x})\|^2\right]+\kappa_n\left\|\Psi(u_n)-\Psi(\check{x})\right\|^2 \\
&\le \kappa_n\left\|\Psi(u_n)-\Psi(\check{x})\right\|^2+(1-\kappa_n)(1-\alpha_n) \\
&\quad \times\left(\left\|\Psi(u_n)-\Psi(\check{x})\right\|^2+\mu_n(\mu_n-2\varsigma)\left\|\mathbb{A}u_n-\mathbb{A}\check{x}\right\|^2\right) \\
&\quad +\alpha_n\|\Psi(\check{x})\|^2 \\
&\le \left\|\Psi(u_n)-\Psi(\check{x})\right\|^2 \\
&\quad +(1-\kappa_n)(1-\alpha_n)\mu_n(\mu_n-2\varsigma)\left\|\mathbb{A}u_n-\mathbb{A}\check{x}\right\|^2 \\
&\quad +\alpha_n\|\Psi(\check{x})\|^2.
\end{aligned} \tag{29}$$

Hence,

$$\begin{aligned}
&(1-\kappa_n)(1-\alpha_n)\mu_n(2\varsigma-\mu_n)\left\|\mathbb{A}u_n-\mathbb{A}\check{x}\right\|^2 \\
&\quad \le \left\|\Psi(u_n)-\Psi(\check{x})\right\|^2-\left\|\Psi(u_{n+1})-\Psi(\check{x})\right\|^2 \\
&\qquad +\alpha_n\|\Psi(\check{x})\|^2 \\
&\quad \le \left(\left\|\Psi(u_n)-\Psi(\check{x})\right\|+\left\|\Psi(u_{n+1})-\Psi(\check{x})\right\|\right) \\
&\qquad \times\left\|\Psi(u_{n+1})-\Psi(u_n)\right\|+\alpha_n\|\Psi(\check{x})\|^2.
\end{aligned} \tag{30}$$

Since $\alpha_n \to 0$, $\|\Psi(u_{n+1})-\Psi(u_n)\| \to 0$, and $\liminf_{n\to\infty}(1-\kappa_n)(1-\alpha_n)\mu_n(2\varsigma-\mu_n)>0$, we obtain

$$\lim_{n\to\infty}\left\|\mathbb{A}u_n-\mathbb{A}\check{x}\right\|=0. \tag{31}$$

Set $y_n=\Psi(u_n)-\mu_n\mathbb{A}u_n-(\Psi(\check{x})-\mu_n\mathbb{A}\check{x})$ for all n. By using the firm nonexpansivity of projection, we get

$$\begin{aligned}
&\left\|v_n-\Psi(\check{x})\right\|^2 \\
&\quad =\left\|\mathrm{proj}_{\mathbb{C}}(\Psi(u_n)-\mu_n\mathbb{A}u_n)^2\right. \\
&\qquad \left.-\mathrm{proj}_{\mathbb{C}}(\Psi(\check{x})-\mu_n\mathbb{A}\check{x})\right\|^2 \\
&\quad \le \langle y_n, v_n-\Psi(\check{x})\rangle \\
&\quad =\frac{1}{2}\left\{\|y_n\|^2+\left\|v_n-\Psi(\check{x})\right\|^2-\left\|y_n-v_n+\Psi(\check{x})\right\|^2\right\} \\
&\quad \le \frac{1}{2}\left\{\left\|\Psi(u_n)-\Psi(\check{x})\right\|^2+\left\|v_n-\Psi(\check{x})\right\|^2\right. \\
&\qquad \left.-\left\|\Psi(u_n)-v_n-\mu_n(\mathbb{A}u_n-\mathbb{A}\check{x})\right\|^2\right\} \\
&\quad =\frac{1}{2}\left\{\left\|\Psi(u_n)-\Psi(\check{x})\right\|^2+\left\|v_n-\Psi(\check{x})\right\|^2\right. \\
&\qquad -\left\|\Psi(u_n)-v_n\right\|^2-\mu_n^2\left\|\mathbb{A}u_n-\mathbb{A}\check{x}\right\| \\
&\qquad \left.+2\mu_n\langle\Psi(u_n)-v_n,\mathbb{A}u_n-\mathbb{A}\check{x}\rangle\right\}.
\end{aligned} \tag{32}$$

It follows that

$$\begin{aligned}
\left\|v_n-\Psi(\check{x})\right\|^2 &\le \left\|\Psi(u_n)-\Psi(\check{x})\right\|^2-\left\|\Psi(u_n)-v_n\right\|^2 \\
&\quad +2\mu_n\left\|\Psi(u_n)-v_n\right\|\left\|\mathbb{A}u_n-\mathbb{A}\check{x}\right\|.
\end{aligned} \tag{33}$$

From (29) and (32), we have

$$\begin{aligned}
&\left\|\Psi(u_{n+1})-\Psi(\check{x})\right\|^2 \\
&\quad \le \kappa_n\left\|\Psi(u_n)-\Psi(\check{x})\right\|^2+(1-\kappa_n) \\
&\qquad \times\left[(1-\alpha_n)\left\|v_n-\Psi(\check{x})\right\|^2+\alpha_n\|\Psi(\check{x})\|^2\right] \\
&\quad \le \kappa_n\left\|\Psi(u_n)-\Psi(\check{x})\right\|^2+(1-\alpha_n)(1-\kappa_n) \\
&\qquad \times\left\|\Psi(u_n)-\Psi(\check{x})\right\|^2-(1-\kappa_n)\left\|\Psi(u_n)-v_n\right\|^2 \\
&\qquad +(1-\kappa_n)\alpha_n\|\Psi(\check{x})\|^2+2\mu_n(1-\kappa_n) \\
&\qquad \times\left\|\Psi(u_n)-v_n\right\|\left\|\mathbb{A}u_n-\mathbb{A}\check{x}\right\| \\
&\quad \le \left\|\Psi(u_n)-\Psi(\check{x})\right\|^2-(1-\kappa_n)\left\|\Psi(u_n)-v_n\right\|^2 \\
&\qquad +2\mu_n\left\|\Psi(u_n)-v_n\right\|\left\|\mathbb{A}u_n-\mathbb{A}\check{x}\right\|+\alpha_n\|\Psi(\check{x})\|^2.
\end{aligned} \tag{34}$$

Then, we obtain

$$\begin{aligned}
&(1-\kappa_n)\left\|\Psi(u_n)-v_n\right\|^2 \\
&\quad \le \left(\left\|\Psi(u_n)-\Psi(\check{x})\right\|+\left\|\Psi(u_{n+1})-\Psi(\check{x})\right\|\right) \\
&\qquad \times\left\|\Psi(u_{n+1})-\Psi(x_n)\right\| \\
&\qquad +2\mu_n\left\|\Psi(u_n)-v_n\right\|\left\|\mathbb{A}u_n-\mathbb{A}\check{x}\right\|+\alpha_n\|\Psi(\check{x})\|^2.
\end{aligned} \tag{35}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\|\Psi(u_{n+1})-\Psi(u_n)\|=0$, and $\lim_{n\to\infty}\|\mathbb{A}u_n-\mathbb{A}\check{x}\|=0$, we deduce that

$$\lim_{n\to\infty}\left\|\Psi(u_n)-v_n\right\|=0. \tag{36}$$

Next, we prove $\limsup_{n\to\infty}\langle\Psi(x^*),v_n-\Psi(x^*)\rangle\ge 0$, where x^* satisfies (GVI): $\langle\Psi(x^*),\Psi(x)-\Psi(x^*)\rangle\ge 0$, for all $x\in\Upsilon$ (note that Ψ is ζ-strongly monotone; we can easily deduce that the solution of (GVI) is unique). We take a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle\Psi(x^*),v_n-\Psi(x^*)\rangle \\
&\quad =\lim_{i\to\infty}\left\langle\Psi(x^*),v_{n_i}-\Psi(x^*)\right\rangle \\
&\quad =\lim_{i\to\infty}\left\langle\Psi(x^*),\Psi(u_{n_i})-\Psi(x^*)\right\rangle.
\end{aligned} \tag{37}$$

By the boundedness of $\{u_{n_i}\}$, we can choose a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ such that $u_{n_{i_j}}\to z$ weakly. For the convenience, we may assume that $u_{n_i}\rightharpoonup z$. This implies that $\Psi(u_{n_i})\rightharpoonup\Psi(z)$ due to the weak continuity of Ψ. Now, we show $z\in\Upsilon$. We firstly show $\Psi(z)\in\mathrm{EP}(\varrho,\mathbb{C})$.

Note that $\varpi_n \in (\eta_1, \eta_2)$. Then we choose a subsequence $\{\varpi_{n_i}\}$ of $\{\varpi_n\}$ such that $\lim_{i\to\infty}\varpi_{n_i} = \varpi \in (\eta_1, \eta_2)$. From (26) and (36), we deduce that $\|z_n - v_n\| = \|F_{\varpi_n}(1-\alpha_n)v_n - v_n\| \to 0$. Thus, $\|z_{n_i} - v_{n_i}\| = \|F_{\varpi_{n_i}}(1-\alpha_{n_i})v_{n_i} - v_{n_i}\| \to 0$. From Lemma 2, we know that F_ϖ is nonexpansive. By demiclosed principle (Lemma 5), we get immediately that $\Psi(z) \in \mathrm{Fix}(F_\varpi) = \mathrm{EP}(\varrho, \mathbb{C})$.

Next we prove $z \in \mathrm{VI}(\mathbb{A}, \Psi)$. Set

$$Rv = \begin{cases} \mathbb{A}v + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{38}$$

By [27], we know that R is maximal Ψ-monotone. Let $(v, w) \in G(R)$. Since $w - \mathbb{A}v \in N_C(v)$ and $u_n \in C$, we have $\langle \Psi(v) - \Psi(u_n), w - \mathbb{A}v\rangle \geq 0$. Noting that $v_n = \mathrm{proj}_{\mathbb{C}}(\Psi(u_n) - \mu_n \mathbb{A}u_n)$, we get

$$\langle \Psi(v) - v_n, v_n - (\Psi(u_n) - \mu_n \mathbb{A}u_n)\rangle \geq 0. \tag{39}$$

It follows that

$$\left\langle \Psi(v) - v_n, \frac{v_n - \Psi(u_n)}{\mu_n} + \mathbb{A}u_n \right\rangle \geq 0. \tag{40}$$

Then,

$$\begin{aligned}
&\left\langle \Psi(v) - \Psi(u_{n_i}), w \right\rangle \\
&\quad \geq \left\langle \Psi(v) - \Psi(u_{n_i}), \mathbb{A}v \right\rangle \\
&\quad \geq \left\langle \Psi(v) - \Psi(u_{n_i}), \mathbb{A}v \right\rangle \\
&\qquad - \left\langle \Psi(v) - v_{n_i}, \frac{v_{n_i} - \Psi(u_{n_i})}{\mu_{n_i}} \right\rangle \\
&\qquad - \left\langle \Psi(v) - v_{n_i}, \mathbb{A}u_{n_i} \right\rangle \\
&\quad = \left\langle \Psi(v) - \Psi(u_{n_i}), \mathbb{A}v - \mathbb{A}u_{n_i} \right\rangle \\
&\qquad + \left\langle \Psi(v) - \Psi(u_{n_i}), \mathbb{A}u_{n_i} \right\rangle \\
&\qquad - \left\langle \Psi(v) - v_{n_i}, \frac{v_{n_i} - \Psi(u_{n_i})}{\mu_{n_i}} \right\rangle \\
&\qquad - \left\langle \Psi(v) - v_{n_i}, \mathbb{A}u_{n_i} \right\rangle \\
&\quad \geq - \left\langle \Psi(v) - v_{n_i}, \frac{v_{n_i} - \Psi(u_{n_i})}{\mu_{n_i}} \right\rangle \\
&\qquad - \left\langle \Psi(u_{n_i}) - v_{n_i}, \mathbb{A}u_{n_i} \right\rangle.
\end{aligned} \tag{41}$$

Since $\|\Psi(u_{n_i}) - v_{n_i}\| \to 0$ and $\Psi(u_{n_i}) \rightharpoonup \Psi(z)$, we deduce that $\langle \Psi(v) - \Psi(z), w\rangle \geq 0$ by taking $i \to \infty$ in (41). Thus, $z \in R^{-1}0$ by the maximal Ψ-monotonicity of R. Hence, $z \in \mathrm{VI}(\mathbb{A}, \Psi)$. Therefore, $z \in \Upsilon$. From (37), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \langle \Psi(x^*), v_n - \Psi(x^*)\rangle \\
&\quad = \lim_{i\to\infty} \langle \Psi(x^*), \Psi(u_{n_i}) - \Psi(x^*)\rangle \\
&\quad = \langle \Psi(x^*), \Psi(z) - \Psi(x^*)\rangle \geq 0.
\end{aligned} \tag{42}$$

From (12), we have

$$\begin{aligned}
&\|\Psi(u_{n+1}) - \Psi(x^*)\|^2 \\
&\quad \leq \kappa_n \|\Psi(u_n) - \Psi(x^*)\|^2 \\
&\qquad + (1-\kappa_n)\|(1-\alpha_n)v_n - \Psi(x^*)\|^2 \\
&\quad \leq \kappa_n \|\Psi(u_n) - \Psi(x^*)\|^2 + (1-\kappa_n) \\
&\qquad \times \Big[(1-\alpha_n)\|v_n - \Psi(x^*)\|^2 \\
&\qquad\quad - 2\alpha_n(1-\alpha_n)\langle \Psi(x^*), v_n - \Psi(x^*)\rangle \\
&\qquad\quad + \alpha_n^2 \|\Psi(x^*)\|^2\Big] \\
&\quad \leq \kappa_n \|\Psi(u_n) - \Psi(x^*)\|^2 + (1-\kappa_n) \\
&\qquad \times \Big[(1-\alpha_n)\|\Psi(u_n) - \Psi(x^*)\|^2 - 2\alpha_n(1-\alpha_n) \\
&\qquad\quad \times \langle \Psi(x^*), v_n - \Psi(x^*)\rangle + \alpha_n^2 \|\Psi(x^*)\|^2\Big] \\
&\quad = [1 - (1-\kappa_n)\alpha_n]\|\Psi(u_n) - \Psi(x^*)\|^2 + (1-\kappa_n)\alpha_n \\
&\qquad \times \{2(1-\alpha_n)\langle -\Psi(x^*), v_n - \Psi(x^*)\rangle \\
&\qquad\quad + \alpha_n \|\Psi(x^*)\|^2\}.
\end{aligned} \tag{43}$$

Using Lemma 6, we conclude that $\Psi(u_n) \to \Psi(x^*)$, and hence $u_n \to x^*$. This completes the proof. □

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