Research Article

Analysis of Stability of Traveling Wave for Kadomtsev-Petviashvili Equation

Jun Liu,¹ Xi Liu,² Gui Mu,¹ Chunyan Zhu,¹ and Jie Fu¹

¹ College of Mathematics and Information Science, Qujing Normal University, Qujing, Yunnan 655011, China

² College of Information Science and Engineering, Yunnan University, Kunming, Yunnan 650091, China

Correspondence should be addressed to Jun Liu; liujunxei@126.com

Received 31 January 2013; Accepted 4 February 2013

Academic Editor: de Dai

Copyright © 2013 Jun Liu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper presents the boundedness and uniform boundedness of traveling wave solutions for the Kadomtsev-Petviashvili (KP) equation. They are discussed by means of a traveling wave transformation and Lyapunov function.

1. Introduction

We consider the Kadomtsev-Petviashvili (KP) equation:

$$u_{tx} + 6u_x u_{xx} + u_{xxxx} + u_{yy} + cu = 0. \quad (1)$$

It is well known that Kadomtsev-Petviashvili equation arises in a number of remarkable nonlinear problems both in physics and mathematics. By using various methods and techniques, exact traveling wave solutions, solitary wave solutions, doubly periodic solutions, and some numerical solutions have been obtained in [1–6].

In this paper, (1) can be changed into an ordinary differential equation by using traveling wave transformation; the boundedness and uniform boundedness of solution for the resulting ordinary differential equation are discussed using the method of Lyapunov function.

2. The Boundedness

Taking a traveling wave transformation $\xi = \alpha x + \beta y + \gamma t$ in (1), then (1) can be transformed into the following form:

$$u^{(4)} + \left(\frac{\gamma}{\alpha^3} + \frac{\beta^2}{\alpha^4} + \frac{6}{\alpha^2} u \right) u'' + \frac{6}{\alpha^2} u'^2 + \frac{c}{\alpha^4} u = 0. \quad (2)$$

In general, we use the following system, which is equivalent to (2):

$$\begin{aligned} u^{(4)} + au''' + f(t, u, u'') + g(u') + du \\ = p(t, u, u', u'', u'''), \end{aligned} \quad (3)$$

where

$$\begin{aligned} f(t, u, u') &= \left(\frac{\gamma}{\alpha^3} + \frac{\beta^2}{\alpha^4} + \frac{6}{\alpha^2} u \right) u'', & g(u') &= \frac{6}{\alpha^2} u'^2, \\ p(t, u, u', u'', u''') &= -au''', & d &= \frac{c}{\alpha^4}. \end{aligned} \quad (4)$$

We consider the following system, which is equivalent to (3):

$$\begin{aligned} x'_1 &= x_2, & x'_2 &= x_3, & x'_3 &= x_4, \\ x'_4 &= -ax_4 - f(t, x_1, x_2, x_3) - g(x_2) - dx_1 \\ &+ p(t, x_1, x_2, x_3, x_4). \end{aligned} \quad (5)$$

Theorem 1. If the following conditions hold for the system (5):

(i) there are positive constants a, b, d, δ, k , and λ such that

$$k \leq b^2 \lambda, \quad ab \frac{g(x_2)}{x_2} - \left[\frac{g(x_2)}{x_2} \right]^2 - a^2 d \geq \delta, \quad (x_2 \neq 0). \quad (6)$$

$$(ii) \quad f(t, x_1, x_2, 0) = 0, \quad 0 \leq f(t, x_1, x_2, x_3)/x_3 - b \leq 2\delta\lambda/k \quad (x_2 \neq 0).$$

$$(iii) \quad x_3 f'_t(t, x_1, x_2, x_3) + x_2 x_3 f'_{x_1}(t, x_1, x_2, x_3) + x_3^2 f'_{x_2}(t, x_1, x_2, x_3) \leq 0.$$

$$(iv) \quad |p(t, x_1, x_2, x_3, x_4)| \leq q(t)(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2},$$

where $q(t)$ is a nonnegative continuous function and $\int_0^\infty q(t)dt < \infty$.

Then, all the solutions of system (5) are bounded.

Proof. We first construct the Lyapunov function $V = V(t, x_1, x_2, x_3, x_4)$ defined by

$$\begin{aligned} V &= b^2(2x_4 + ax_3 + bx_2)^2 + 2bd(2x_3 + ax_2 + bx_1)^2 \\ &\quad + (b^2 - 4d)(ax_4 + bx_2)^2 + 4ab^2 \\ &\quad \times \int_0^{x_2} \left[\frac{g(x_2)}{x_2} - \frac{ad}{b} \right] x_2 dx_2 \\ &\quad + [2b(b^2 - 4d) + 4a^2d] x_3^2 \\ &\quad + 8b^2 \int_0^{x_3} \left[\frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] x_3 dx_3. \end{aligned} \quad (7)$$

It follows from conditions (i) and (ii) that

$$\begin{aligned} b^2 - 4d &\geq 0, \\ 0 &\leq \int_0^{x_2} \left[\frac{g(x_2)}{x_2} - \frac{ad}{b} \right] x_2 dx_2 \leq \frac{a(b^2 - d)}{2b} x_2^2, \\ 0 &\leq \int_0^{x_3} \left[\frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] x_3 dx_3 \leq \frac{\delta\lambda}{k} x_3^2. \end{aligned} \quad (8)$$

Summing up the above discussions, we get

$$V \geq 2b(b^2 - 4d)x_3^2 + 4a^2dx_3^2. \quad (9)$$

Thus, we deduce that the function $V(t, x_1, x_2, x_3, x_4)$ defined in (7) is a positive definite function which has infinite inferior limit and infinitesimal upper limit. Hence, there exists a positive constant $\varepsilon(>0)$ such that

$$V(t, x_1, x_2, x_3, x_4) \geq \varepsilon(x_1^2 + x_2^2 + x_3^2 + x_4^2). \quad (10)$$

Taking the total derivative of (7) with respect to t along the trajectory of (5), we obtain

$$\begin{aligned} \frac{dV}{dt} &= -2ab^2 \left[x_4 + \frac{1}{a}g(x_2) \right]^2 \\ &\quad - 2b^3 x_2 x_3 \left[\frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] \\ &\quad - 2ab^2 \left[\frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] x_3^2 - \frac{2b^2}{a} \\ &\quad \times \left[ab \frac{g(x_2)}{x_2} - \frac{g^2(x_2)}{x_2^2} - a^2d \right] x_2^2 \\ &\quad + 4b^2 \int_0^{x_3} f'_t(t, x_1, x_2, x_3) dx_3 \\ &\quad + 4b^2 x_2 \int_0^{x_3} f'_{x_1}(t, x_1, x_2, x_3) dx_3 \\ &\quad + 4b^2 x_3 \int_0^{x_3} f'_{x_2}(t, x_1, x_2, x_3) dx_3 \\ &\quad + 2b^2(bx_2 + ax_3 + 2x_4)p(t, x, x_2, x_3, x_4). \end{aligned} \quad (11)$$

By using conditions (i) and (iii), it follows that

$$\begin{aligned} \frac{dV}{dt} &\leq -\frac{2b^2\delta}{a}x_2^2 - 2b^3x_2x_3 \left[\frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] \\ &\quad - 2ab^2 \left[\frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] x_3^2 \\ &\quad + 2b^2(bx_2 + ax_3 + 2x_4)p(t, x, x_2, x_3, x_4). \end{aligned} \quad (12)$$

According to (ii), we have

$$\begin{aligned} &2b^3x_2x_3 \left[\frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] \\ &\quad + 2ab^2 \left[\frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] x_3^2 \\ &= -\frac{b^4}{2a} \left[\frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] x_2^2 \\ &\quad + \frac{2b^2}{a} \left[\frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] \cdot \left(ax_3 + \frac{b}{2}x_2 \right)^2 \\ &\geq -\frac{b^4}{2a} \left[\frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] x_2^2 \\ &= -\frac{b^4}{2a} \cdot \frac{2\delta\lambda}{k} x_2^2 = -\frac{b^4\delta\lambda}{ak} x_2^2. \end{aligned} \quad (13)$$

Hence,

$$\begin{aligned}
 \frac{dV}{dt} &\leq -\frac{2b^2\delta}{a}x_2^2 + \frac{b^4\delta\lambda}{ak}x_2^2 \\
 &\quad + 2b^2(bx_2 + ax_3 + 2x_4)p(t, x, x_2, x_3, x_4) \\
 &= -\frac{b^2\delta}{a}x_2^2 + \frac{b^2\delta}{ak}(b^2\lambda - k)x_2^2 \\
 &\quad + 2b^2(4 + a^2 + b^2)^{1/2}(x_2^2 + x_3^2 + x_4^2)^{1/2} \\
 &\quad \times (x_2^2 + x_3^2 + x_4^2)^{1/2}q(t) \\
 &\leq -\frac{b^2\delta}{a}x_2^2 + 2b^2(4 + a^2 + b^2)^{1/2}(x_2^2 + x_3^2 + x_4^2)q(t) \\
 &\leq -\frac{b^2\delta}{a}x_2^2 + 2b^2(4 + a^2 + b^2)^{1/2} \cdot q(t) \cdot \frac{V}{\varepsilon} \\
 &\leq 2b^2(4 + a^2 + b^2)^{1/2} \cdot \frac{q(t)}{\varepsilon} \cdot V \equiv \varphi(V, t).
 \end{aligned} \tag{14}$$

Thus, all the solutions of system (5) are bounded. $\square$

Theorem 2. Let conditions (i)–(iv) of Theorem 1 be satisfied for the system (5), and let the following condition hold:

$$(4 + a^2 + b^2)^{1/2} \cdot \frac{q(t)}{\varepsilon} \cdot V - \frac{\delta}{a}x_2^2 \leq 0. \tag{15}$$

Then, all the solutions of system (5) are uniformly bounded.

Proof. It is clear that the function $V(t, x_1, x_2, x_3, x_4)$ defined in (7) satisfies the conditions (15), therefore, all the solutions of system (5); are uniformly bounded [7]. $\square$

Acknowledgments

This work was financially supported by the Chinese Natural Science Foundation (11061028) and Yunnan Natural Science Foundation (2010CD086).

References

- [1] Z. Dai, J. Liu, and Z. Liu, “Exact periodic kink-wave and degenerative soliton solutions for potential Kadomtsev-Petviashvili equation,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 9, pp. 2331–2336, 2010.
- [2] D.-s. Li and H.-q. Zhang, “New soliton-like solutions to the potential Kadomtsev-Petviashvili (PKP) equation,” *Applied Mathematics and Computation*, vol. 146, no. 2-3, pp. 381–384, 2003.
- [3] X. Zeng, Z. Dai, D. Li, S. Han, and H. Zhou, “Some exact periodic soliton solution and resonance for the potential Kadomtsev-Petviashvili equation,” *Chaos, Solitons & Fractals*, vol. 42, pp. 657–661, 2009.
- [4] I. E. Inan and D. Kaya, “Some exact solutions to the potential Kadomtsev-Petviashvili equation and to a system of shallow water wave equations,” *Physics Letters A*, vol. 355, no. 4-5, pp. 314–318, 2006.
- [5] D. Kaya and S. M. El-Sayed, “Numerical soliton-like solutions of the potential Kadomtsev-Petviashvili equation by the decomposition method,” *Physics Letters A*, vol. 320, no. 2-3, pp. 192–199, 2003.
- [6] Z. Li, Z. Dai, and J. Liu, “New periodic solitary-wave solutions to the 3 + 1-dimensional Kadomtsev-Petviashvili equation,” *Mathematical & Computational Applications*, vol. 15, no. 5, pp. 877–882, 2010.
- [7] C. Tunç, “On the uniform boundedness of solutions of some non-autonomous differential equations of the fourth order,” *Applied Mathematics and Mechanics*, vol. 20, no. 6, pp. 622–628, 1999.