1. Introduction and Main Results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We will use the standard notations of Nevanlinna’s value distribution theory such as $T(r, f)$, $N(r, f)$, $N_1(r, f)$, and $m(r, f)$, as explained in Hayman [1], Yang [2], and Yang and Yi [3]. We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$, as $r \to \infty$ possibly outside a set of finite linear measures. For $f$ meromorphic in $\mathbb{C}$, denote by $S(f)$ the family of all meromorphic functions $a(z)$ that satisfy $T(r, a) = o(T(r, f))$ for $r \to \infty$ outside a possible exceptional set of finite linear measure. In addition, we denote by $\rho(f)$ and $\rho_1(f)$ the order of $f$ and the hyper-order of $f$ [3, 4]. Moreover, we define difference operators by $\Delta_c f = f(z + c) - f(z)$ where $c$ is a nonzero constant. If $c = 1$, we use the usual difference notation $\Delta f = \Delta_1 f$.

Let $f$ and $g$ be two nonconstant meromorphic functions and $a$ be a finite complex number. We say that $f, g$ share the value $a$ CM (counting multiplicities) if $f, g$ have the same $a$-points with the same multiplicities, and we say that $f, g$ share the value $a$ IM (ignoring multiplicities) if we do not consider the multiplicities. We denote by $N_{12}(r, 1/(f - a))$ the counting function for $a$-points of both $f$ and $g$ about which $f$ has larger multiplicity than $g$, with multiplicity not being counted. Similarly, we have the notation $N_{11}(r, 1/(g - a))$. Next, we denote by $N_{0}(r, 1/F')$ the counting function of those zeros of $F'$ that are not the zeros of $F(F - 1)$ and denote by $N_{11}(r, 1/(f - a))$ the counting function for common simple 1-point of both $f$ and $g$. In addition, we need the following three definitions.

**Definition 1.** Let $k$ be a positive integer. Let $f$ and $g$ be two nonconstant meromorphic functions such that $f'$ and $g'$ share the value 1 IM. Let $z_0$ be a 1-point of $f$ with multiplicity $p$ and a 1-point of $g$ with multiplicity $q$. We denote by $N_{f\geq k}(r, 1/(g - 1))$ the reduced counting function of those 1-points of $f$ and $g$ such that $p > q = k$. $N_{g\leq k}(r, 1/(f - 1))$ is defined analogously.

**Definition 2** (see [5]). Let $k$ be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $m + 1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that $f, g$ share the value $a$ with weight $k$.

The definition implies that if $f, g$ share a value $a$ with weight $k$, then $z_0$ is an $a$-point of $f$ with multiplicity $m(\leq k)$ if and only if it is an $a$-point of $g$ with multiplicity $m(\leq k)$ and $z_0$ is an $a$-point of $f$ with multiplicity $m(> k)$ if and only if it is an $a$-point of $g$ with multiplicity $m(> k)$, where $m$ is not necessarily equal to $n$.

We write that $f, g$ share $(a, k)$ to mean that $f, g$ share the value $a$ with weight $k$. Clearly if $f, g$ share $(a, k)$, then $f, g$ share $(a, p)$ for any integer $p$, $0 \leq p < k$. Also we note that $f, g$ share a value $a$ IM or CM if and only if $f, g$ share $(a, 0)$ or $(a, \infty)$, respectively.

**Definition 3.** Let $f$ be a nonconstant meromorphic function, and let $p$ be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. Then, by
Abstract and Applied Analysis

Let \( f \) and \( g \) be transcendental entire functions of finite order, let \( t \) be a nonzero complex constant, and set \( F(z) = f(z)^n \Delta_c f \); then

\[
nt'(r, f) + S(r, f) \leq nT(r, F) \leq (n + 1)T(r, f) + S(r, f).
\]

Proof. Since

\[
T(r, F) = T'(r, f(z)^n \Delta_c f) \leq nT'(r, f) + T'(r, \Delta_c f) \\
\leq nT'(r, f) + m(r, \Delta_c f) \leq nT'(r, f) + m(r, f) + S(r, f) \\
= (n + 1)T(r, f) + S(r, f),
\]

then

\[
(n + 1)T(r, f) = T'(r, f(z)^{n+1}) = m(r, f(z)^{n+1}) \\
\leq m(r, \frac{f(z)^{n+1}}{F}) + m(r, F) + S(r, f) \\
\leq m(r, \frac{f(z)}{\Delta_c f}) + m(r, F) + S(r, f) \\
\leq T'(r, \frac{f(z)}{\Delta_c f}) + T'(r, F) + S(r, f) \\
\leq T'(r, \frac{\Delta_c f}{f(z)}) + T'(r, F) + S(r, f) \\
= m(r, \frac{\Delta_c f}{f(z)}) + N(r, \frac{\Delta_c f}{f(z)}) \\
+ T(r, F) + S(r, f) \\
\leq T(r, F) + N\left( r, \frac{1}{f(z)} \right) + S(r, f) \\
\leq T(r, F) + T(r, f) + S(r, f).
\]

Lemma 8 (see [9]). Let \( f \) be a meromorphic function of finite order, and let \( c \in \mathbb{C} \) and \( \delta \in (0, 1) \). Then

\[
m\left( r, \frac{f(z+c)}{f(z)} \right) + m\left( r, \frac{f(z)}{f(z+c)} \right) = o\left( \frac{T(r, f)}{r^\delta} \right) = S(r, f).
\]

Lemma 9 (see [10]). Let \( f_1, f_2, \) and \( f_3 \) be nonconstant meromorphic functions such that \( f_1 + f_2 + f_3 = 1 \). If \( f_1, f_2, \) and \( f_3 \) are linearly independent, then

\[
T(r, f_1) \leq \sum_{j=1}^{3} N_2\left( r, \frac{1}{f_j} \right) + \sum_{j=1}^{3} N\left( r, f_j \right) + o(T(r)),
\]

where \( T(r) = \max_{1 \leq j \leq 3} T(r, f_j), r \notin E, \) and \( E \) denote a set of positive real numbers of finite linear measure.

Lemma 10. Let \( f \) be transcendental entire functions of finite order, let \( c \) be a nonzero complex constant, and set \( F(z) = f(z)^n \Delta_c f; \) then

\[
nT(r, f) + S(r, f) \leq T(r, F) \leq (n + 1)T(r, f) + S(r, f).
\]

2. Some Lemmas

Lemma 7 (see [8]). Let \( f \) be a nonconstant meromorphic function of finite order \( \sigma \), and let \( c \) be a nonzero constant. Then, for each \( \varepsilon > 0 \),

\[
T(r, f(z+c)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r).
\]
That is,
\[ nT(r, f) + S(r, f) \leq T(r, F) \leq (n + 1)T(r, f) + S(r, f). \]
(7)

**Lemma 11** (see [11]). Let \( f_1 \) and \( f_2 \) be two nonconstant meromorphic functions. If \( c_1 f_1 + c_2 f_2 = c_3 \), where \( c_1, c_2, \) and \( c_3 \) are nonzero constants, then
\[ T(r, f_1) \leq N(\gamma, f) + N\left( r, \frac{1}{f_1} \right) + S(r, f_1). \]
(8)

**Lemma 12** (see [12]). Let \( f(z) \) be a nonconstant meromorphic function, and let \( k \) be a positive integer. Suppose that \( f^{(k)} \neq 0 \); then
\[ N\left(r, \frac{1}{f^{(k)}} \right) \leq N\left( r, \frac{1}{f} \right) + kN(r, f) + S(r, f). \]
(9)

**Lemma 13** (see [13]). Let \( f, g \) share \((1, 0)\). Then
(i) \( N_{f^{(1)}}(r, \frac{1}{(g-1)}) \leq N_{f^{(1)}}(r, \frac{1}{f}) + N_{f^{(1)}}(r, f) - N_{f^{(1)}}(r, \frac{1}{f'}), \)
(ii) \( N_{g^{(1)}}(r, \frac{1}{(f-1)}) \leq N_{g^{(1)}}(r, g) + N_{g^{(1)}}(r, g) - N_{g^{(1)}}(r, \frac{1}{g'}). \)

**Lemma 14.** Let \( f(z) \) and \( g(z) \) be two nonconstant entire functions. If \( f \) and \( g \) share 1IM, then one of the following cases holds:
(i) \( T(r, g) \leq N_2(r, 1/g) + N_2(r, 1/f) + N(r, 1/f) + 2N(r, 1/g) + S(r, f) + S(r, g), \)
(ii) \( f \equiv (Ag + B)/(Cg + D), \) where \( A, B, C, \) and \( D \) are finite complex numbers satisfying \( AD \neq BC. \)

**Proof.** Let
\[ \Phi(z) = \frac{f''}{f'} - \frac{2f' f''}{f - 1} - \frac{g'g''}{g - 1} + 2 \frac{g'}{g - 1}. \]
(10)
Clearly \( m(r, \Phi) = S(r, f) + S(r, g). \) We consider the cases \( \Phi(z) \neq 0 \) and \( \Phi(z) \equiv 0. \)

If \( \Phi(z) \neq 0, \) then if \( z_0 \) is a common simple l-point of \( f' \) and \( g', \) substituting their Taylor series at \( z_0 \) into (10), we see that \( z_0 \) is a zero of \( \Phi(z). \) Thus, we have
\[ N_{11}(r, \frac{1}{f - 1}) = N_{11}(r, \frac{1}{g - 1}) \leq N\left( r, \frac{1}{f'} \right) \leq T(r, \Phi) + O(1) \leq N(r, \Phi) + S(r, f) + S(r, g). \]
(11)

Our assumptions are that \( \Phi(z) \) has poles; all are simple only at zeros of \( f' \) and \( g' \) and poles of \( f \) and \( g, \) and l-points of \( f \) whose multiplicities are not equal to the multiplicities of the corresponding l-points of \( g. \) Thus, we deduce from (10) that
\[ N(r, \Phi) \leq N_{12}(r, \frac{1}{f'}) + N_{12}(r, \frac{1}{g'}) + N_0\left( r, \frac{1}{f'} \right) + N_0\left( r, \frac{1}{g'} \right) + N_{12}(r, \frac{1}{g - 1}) + N_{12}(r, \frac{1}{f - 1}). \]
(12)

where \( N_0(r, 1/f') \) is the counting function which only counts those points such that \( f' = 0, \) but \( f(f - 1) \neq 0. \) By the second fundamental theorem, we have
\[ T(r, g) \leq N\left( r, \frac{1}{g - 1} \right) \leq N_{g^{(1)}}(r, \frac{1}{g' - 1}) + N_{g^{(1)}}(r, \frac{1}{g - 1}) \]
(13)

Thus, we deduce from (11)–(14) that
\[ T(r, g) \leq N\left( r, \frac{1}{g - 1} \right) + N\left( r, \frac{1}{g' - 1} \right) \leq N_{g^{(1)}}(r, \frac{1}{g' - 1}) + N_{g^{(1)}}(r, \frac{1}{g - 1}) \]
(14)

Since
\[ N\left( r, \frac{1}{g - 1} \right) = N_{11}(r, \frac{1}{g - 1}) + N_{12}(r, \frac{1}{g' - 1}) + N_{12}(r, \frac{1}{g - 1}) \]
(15)

From the definition of \( N_0(r, 1/f'), \) we see that
\[ N_0\left( r, \frac{1}{f'} \right) + N_{12}(r, \frac{1}{g' - 1}) + N_{12}(r, \frac{1}{g - 1}) \]
(16)

The above inequality and Lemma 12 give
\[ N_0\left( r, \frac{1}{f'} \right) + N_{12}(r, \frac{1}{g' - 1}) \leq N\left( r, \frac{1}{f'} \right) \leq N\left( r, \frac{1}{g' - 1} \right) \]
(17)
Substituting (17) in (15), we get

\[ T(r, g) \leq N\left( r, \frac{1}{g} \right) + N_2\left( r, \frac{1}{f} \right) + N_2\left( r, \frac{1}{g} \right) + N\left( r, \frac{1}{f-1} \right) + N\left( r, \frac{1}{g-1} \right) + N_g\left( r, \frac{1}{f-1} \right) + S\left( r, f \right) + S\left( r, g \right) \leq N\left( r, \frac{1}{g} \right) + N_2\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{g} \right) + N\left( r, \frac{1}{f-1} \right) + N\left( r, \frac{1}{g-1} \right) + N_g\left( r, \frac{1}{f-1} \right) + S\left( r, f \right) + S\left( r, g \right), \]

(18)

since

\[ N_L\left( r, \frac{1}{f-1} \right) \leq N\left( r, \frac{1}{f-1} \right) = N\left( r, \frac{1}{f} \right) - N\left( r, \frac{1}{f} \right) \leq N\left( r, \frac{1}{f} \right) \leq N\left( r, \frac{1}{f} \right) + S\left( r, f \right) \]

(19)

\[ \leq N\left( r, \frac{1}{f} \right) + S\left( r, f \right). \]

Similarly,

\[ N_L\left( r, \frac{1}{g-1} \right) \leq N\left( r, \frac{1}{g-1} \right) \leq N\left( r, \frac{1}{g} \right) + S\left( r, g \right). \]

(20)

Combining the above inequalities, Lemma 13, and (18), we obtain

\[ T(r, g) \leq N_2\left( r, \frac{1}{g} \right) + N_2\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{f} \right) + 2N\left( r, \frac{1}{g} \right) - N_0\left( r, \frac{1}{g} \right) + S\left( r, f \right) + S\left( r, g \right) \leq N_2\left( r, \frac{1}{g} \right) + N_2\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{f} \right) + 2N\left( r, \frac{1}{g} \right) + S\left( r, f \right) + S\left( r, g \right). \]

(21)

Thus, we obtain (i).

If \( \Phi(z) \equiv 0 \), then by (10), we have

\[ \frac{f''}{f'} - \frac{2f'}{f-1} \equiv \frac{g''}{g'} - \frac{2g'}{g-1}. \]

(22)

By integrating two sides of the above equality, we obtain

\[ f \equiv \frac{Ag + B}{Cg + D}, \]

(23)

where \( A, B, C, \) and \( D \) are finite complex numbers satisfying \( AD \neq BC \). This proves the lemma.

\[ \square \]

\[ \text{Lemma 15 (see [14]). Let } f(z) \text{ be a nonconstant meromorphic function, } s, k \text{ be two positive integers; then} \]

\[ N_{rk}\left( r, \frac{1}{f(k)} \right) \leq T(r, f) + S\left( r, f \right) \]

(24)

\[ + N_{sk}\left( r, \frac{1}{f(k) f} \right) + S\left( r, f \right). \]

Clearly, \( N(r, 1/f(k)) = N_1(r, 1/f^{(k)}) \).

\[ \text{Lemma 16 (see [15]). Let } a_0(z), a_1(z), \ldots, a_n(z), b(z) \text{ be polynomials such that } a_0(z)a_n(z) \neq 0; \text{ let } c_i \text{ be constants and} \]

\[ \deg \left( \sum_{\deg a_i \neq 0} a_i \right) = d, \]

(25)

where \( d = \max_{0 \leq j \leq n} \{ \deg a_j \} \). If \( f(z) \) is a transcendental meromorphic solution of

\[ \sum_{j=0}^{n} a_j \left( z + c_j \right) = b(z), \]

(26)

then \( \rho(f) \geq 1 \).

3. Proof of Theorems

3.1. Proof of Theorem 4. Let \( G(z) = f(z)^n(f(z + c) - 1)\Delta_z f \).

Since \( f \) is a transcendental entire function of finite order, from Lemma 7, we have

\[ (n+2)T(r, f(z)) \leq T(r, f(z) + S(r, f)) \leq nT(r, f(z) - 1) + S(r, f) \leq mT(r, f(z) + S(r, f)) \leq mT(r, f(z) + S(r, f)) \leq T(r, f) + S(r, f). \]

(27)

By the second main theorem, we deduce that

\[ T(r, G) \leq N(r, G) + N\left( r, \frac{1}{G} \right) + N\left( r, \frac{1}{G - a} \right) + S(r, G) \]

\[ \leq N\left( r, \frac{1}{G - a} \right) + N\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{f(z + c)} \right) + S(r, f) \]

\[ + N\left( r, \frac{1}{G} \right) + S(r, f). \]

(28)

Abstract and Applied Analysis
\[
\leq N\left(r, \frac{1}{G-a}\right) + N\left(r, \frac{1}{f}\right) + T\left(r, f(z+c) - 1\right) \\
+ T\left(r, \Delta_c f\right) + S\left(r, f\right) \\
\leq N\left(r, \frac{1}{G-a}\right) + N\left(r, \frac{1}{f}\right) + T\left(r, f(z+c) - 1\right) \\
+ m\left(r, \frac{\Delta_c f}{f}\right) + S\left(r, f\right) \\
\leq N\left(r, \frac{1}{G-a}\right) + N\left(r, \frac{1}{f}\right) + 3T\left(r, f\right) + S\left(r, f\right). 
\]

(28)

According to (27) and (28), we have

\[
(n-1)T\left(r, f\right) \leq N\left(r, \frac{1}{G-a}\right) + S\left(r, f\right). 
\]

(29)

Noting that \( n \geq 2 \), we get that \( G-a \) has infinitely many zeros. This completes the proof of Theorem 4.

3.2. Proof of Theorem 5. Since \( [f(z)^n\Delta_c f]^{(k)} \) and \( [g(z)^n\Delta_c g]^{(k)} \) share 1 CM, we have

\[
\frac{[f(z)^n\Delta_c f]^{(k)} - 1}{[g(z)^n\Delta_c g]^{(k)} - 1} = e^{h(z)}, 
\]

where \( h(z) \) is a polynomial. Set \( F = f(z)^n\Delta_c f, G = g(z)^n\Delta_c g, \)

\[
F_1 = F^{(k)}, \quad F_2 = -e^{h(z)}G^{(k)}, \quad F_3 = e^{h(z)}, 
\]

then \( F_1 + F_2 + F_3 = 1 \).

\[
T\left(r\right) = \max_{1 \leq j \leq 3} T\left(r, F_j\right), \quad S\left(r\right) = o\left(T\left(r\right)\right). 
\]

(31)

Next, we will prove that \( F_1, F_2, \) and \( F_3 \) are linearly independent and either \( F_2 \) or \( F_3 \) is a constant.

Now, we suppose that neither \( F_2 \) nor \( F_3 \) is a constant and \( F_1, F_2, \) and \( F_3 \) are linearly independent; then by Lemma 9, we have

\[
T\left(r, F_i\right) \leq \sum_{j=1}^{3} N_2\left(r, \frac{1}{F_j}\right) + \sum_{j=1}^{3} N\left(r, F_j\right) + o\left(T\left(r\right)\right). 
\]

(32)

Since \( F_j \) (\( j = 1, 2, 3 \)) are entire functions, by the above inequality, we get

\[
T\left(r, F_i\right) \leq N_2\left(r, \frac{1}{F^{(k)}}\right) + N_2\left(r, \frac{1}{G^{(k)}}\right) + o\left(T\left(r\right)\right). 
\]

(33)

From (33) and the first main theorem, we have

\[
T\left(r, \frac{1}{F^{(k)}}\right) = T\left(r, F^{(k)}\right) + O\left(1\right) = T\left(r, F_i\right) + O\left(1\right) \\
\leq N_2\left(r, \frac{1}{F^{(k)}}\right) + N_2\left(r, \frac{1}{G^{(k)}}\right) + o\left(T\left(r\right)\right) \\
\leq N\left(r, \frac{1}{F^{(k)}}\right) \\
- \left[N_3\left(r, \frac{1}{F^{(k)}}\right) - 2N_3\left(r, \frac{1}{G^{(k)}}\right)\right] \\
+ N\left(r, \frac{1}{G^{(k)}}\right) \\
- \left[N_3\left(r, \frac{1}{G^{(k)}}\right) - 2N_3\left(r, \frac{1}{F^{(k)}}\right)\right] \\
+ o\left(T\left(r\right)\right). 
\]

(34)

Assuming that \( z_0 \) is zero of \( f(z) \) (or \( g(z) \)) with multiplicity \( p \), if \( z_0 \) is zero of \( f(z+c) \) (or \( g(z+c) \)) with multiplicity \( q(\geq 1) \), let \( m = \min\{p, q\} \), then \( z_0 \) is a zero of \( F^{(k)} \) (or \( G^{(k)} \)) with multiplicity \( np + m - k \geq np - k \geq 3 \), and if \( z_0 \) is not zero of \( f(z+c) \) (or \( g(z + c) \)), then \( z_0 \) is a zero of \( F^{(k)} \) (or \( G^{(k)} \)) with multiplicity \( np - k \geq 3 \). Therefore, we get that

\[
N_3\left(r, \frac{1}{F^{(k)}}\right) - 2N_3\left(r, \frac{1}{G^{(k)}}\right) \geq \left(n-k-2\right)N\left(r, \frac{1}{f}\right), 
\]

(35)

\[
N_3\left(r, \frac{1}{G^{(k)}}\right) - 2N_3\left(r, \frac{1}{F^{(k)}}\right) \geq \left(n-k-2\right)N\left(r, \frac{1}{g}\right), 
\]

(36)

since

\[
m\left(r, \frac{1}{f}\right) = m\left(r, \frac{1}{F^{(k)}}\right) = m\left(r, \frac{\Delta_c f}{F}\right) \\
\leq m\left(r, \frac{1}{F}\right) + m\left(r, \frac{\Delta_c f}{F}\right) \cdot f \\
\leq m\left(r, \frac{F^{(k)}}{F}\right) \cdot \frac{1}{F^{(k)}} \quad + m\left(r, \Delta_c f \frac{1}{F}\right) \\
+ m\left(r, f\right) + S\left(r, f\right) \\
\leq m\left(r, \frac{1}{F^{(k)}}\right) + T\left(r, f\right) + S\left(r, f\right) \\
+ T\left(r, \frac{1}{F^{(k)}}\right) - N\left(r, \frac{1}{F^{(k)}}\right) + T\left(r, f\right) + S\left(r, f\right). 
\]

(37)

Therefore, from (34), (35), (36), (37), and Lemma 12,

\[
(n-1)T\left(r, f\right) \leq (k+2)N\left(r, \frac{1}{f}\right) + (k+2)N\left(r, \frac{1}{g}\right) \\
+ T\left(r, g\right) + o\left(T\left(r\right)\right). 
\]

(38)
On the other hand, from (30), we have $A^{(k)} + e^{-h} - e^{-h}F^{(k)} = 1$. Obviously, according to our assumptions, neither $e^{-h}$ nor $e^{-h}F^{(k)}$ is a constant and $F_1, F_2$, and $F_3$ are linearly independent. Similarly, we have

$$(n-1)T(r,g) \leq (k+2)N\left(r, \frac{1}{g}\right) + (k+2)N\left(r, \frac{1}{f}\right) + T(r,f) + o(T(r)).$$

From (38) and (39), we obtain that

$$[n-2k-6](T(r,f)+T(r,g)) \leq o(T(r)),$$

which is a contradiction to $n \geq 2k + 7$.

Therefore, $F_1, F_2$, and $F_3$ are linearly dependent, and there exist constants $C_1, C_2, C_3$ which are not all equal to zero such that

$$C_1 F_1 + C_2 F_2 + C_3 F_3 = 0.$$

Suppose that $C_1 = 0$; we have $C_2 F_2 + C_3 F_3 = 0$. If $C_2 \neq 0$, we get $F_2 = -(C_3/C_2) F_3$; that is, $G^{(k)} = C_3/C_2$; thus $g(z)$ is a polynomial; it is impossible. Similarly, if $C_3 = 0$, we also deduce a contradiction.

Suppose that $C_1 \neq 0$, from (41); we know that $(C_2, C_3) \neq (0, 0)$. If $C_2 \neq 0$, from (41), we have

$$\left(1 - \frac{C_2}{C_1}\right) F_2 + \left(1 - \frac{C_3}{C_1}\right) F_3 = 1$$

and $C_1 \neq C_2, C_1 \neq C_3$. That is,

$$(1 - C_2/C_1) G^{(k)} + 1/\epsilon = 1 - C_3/C_1.$$

From Lemma II, we have

$$T(r, G^{(k)}) \leq N\left(r, \frac{1}{G^{(k)}}\right) + N\left(r, G^{(k)}\right) + N\left(r, e^h\right) + S(r, g)$$

$$= N\left(r, \frac{1}{G^{(k)}}\right) + S(r, g) \leq N\left(r, \frac{1}{G^{(k)}}\right)$$

$$- \left[N_2\left(r, \frac{1}{G^{(k)}}\right) - N_2\left(r, \frac{1}{G^{(k)}}\right)\right] + S(r, g).$$

By the similar argument in (37), we have

$$\left(n-1\right)T\left(r, \frac{1}{g}\right) \leq \left(n-1\right)N\left(r, \frac{1}{G^{(k)}}\right) + T\left(r, \frac{1}{g}\right) + S(r, g).$$

From $n \geq 2k+7 > k+2$, if $z_0$ is zero of $g(z)$ with multiplicity $p$, then $z_0$ is a zero of $G^{(k)}$ with multiplicity $np-k \geq 2$, and we get

$$N_2\left(r, \frac{1}{G^{(k)}}\right) - N_2\left(r, \frac{1}{G^{(k)}}\right) \geq (n-k-1)N\left(r, \frac{1}{g}\right).$$

According to (44), (45), and (46), we have

$$(n-1)T(r,g) \leq (k+1)N\left(r, \frac{1}{g}\right) + S(r,g),$$

which is a contradiction to $n \geq 2k + 7$.

Therefore, $C_2 = 0, C_3 \neq 0$, which gives $(1-C_1/C_3)F_1 + F_2 = 1$. Similarly, we derive a contradiction by calculation.

Hence, we deduce that either $F_2$ or $F_3$ is a constant.

Suppose $F_2 = c \neq 1$; from $F_1 + F_2 + F_3 = 1$, we have $F^{(k)} + e^h = 1 - c$; in the same manner as above, we get a contradiction. Therefore, $c = 1$; that is, $F_2 = 0$. Suppose $F_3 = c \neq 1$; similarly as above, we get $c = 1$; that is, $F_3 = 1$.

Therefore, we conclude that $F_2 = 0$ or $F_3 = 1$.

If $F_2 = 1$, since $F_1 + F_2 + F_3 = 1$, we have $F_1 = -F_3 = -e^{h(z)}$.

That is

$$\left[f^n \Delta \epsilon \cdot g^m \Delta \epsilon \right]^{(k)} = 1.$$
Method as above, we also deduce a contradiction. Therefore, there are not transcendental entire functions $f(z)$ and $g(z)$ satisfying (48).

If $F_3 = 1$, that is, $e^{h(z)} = 1$, from (30), we get

$$[f^n \Delta_c f]^{(k)} \equiv [g^n \Delta_c g]^{(k)}.$$  \hfill (54)

From (54), we have

$$f^n \Delta_c f \equiv g^n \Delta_c g + p(z),$$ \hfill (55)

where $p(z)$ is a polynomial of degree at most $k - 1$. Suppose $p(z) \not\equiv 0$; then we get

$$\frac{f^n \Delta_c f}{p(z)} = \frac{g^n \Delta_c g}{p(z)} + 1.$$ \hfill (56)

Therefore, from the second main theorem, we have

$$(n + 1) T(r, f) \leq T\left(\frac{f^n \Delta_c f}{p(z)}\right) + S(r, f)$$

$$\leq N\left(\frac{f^n \Delta_c f}{p(z)}\right) + N\left(\frac{p(z)}{f^n \Delta_c f}\right) + N\left(\frac{1}{\Delta_c f}\right) + N\left(\frac{1}{g}\right) + S(r, f)$$

$$\leq 2T(r, f) + 2T(r, g) + S(r, f).$$ \hfill (57)

Similarly, we have

$$(n + 1) T(r, g) \leq 2T(r, f) + 2T(r, g) + S(r, f).$$ \hfill (58)

Therefore,

$$(n + 1) [T(r, f) + T(r, g)] \leq 4 [T(r, f) + T(r, g)] + S(r, f) + S(r, g),$$ \hfill (59)

which is a contradiction to $n \geq 2k + 7$. Thus, $p(z) \equiv 0$, which implies that

$$f^n \Delta_c f \equiv g^n \Delta_c g.$$ \hfill (60)

Let $f/g = h$; if $h$ is not a constant, then by (60), we have

$$h^{n+1} = \frac{f}{\Delta_c f} \cdot \frac{\Delta_c g}{g}.$$ \hfill (61)

Thus,

$$(n + 1) T(r, h) \leq T\left(r, \frac{\Delta_c f}{f}\right) + T\left(r, \frac{\Delta_c g}{g}\right) + O(1)$$

$$\leq N\left(r, \frac{\Delta_c f}{f}\right) + N\left(r, \frac{\Delta_c g}{g}\right) + S(r, f) + S(r, g)$$

$$\leq T(r, f) + T(r, g) + S(r, f) + S(r, g).$$ \hfill (62)

Combining $T(r, h) = T(r, f/g) = T(r, f) + T(r, g) + O(1)$, we obtain $n(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g)$, which is impossible.

Therefore, $h$ is a constant; then substituting $f = gh$ into (60), we have $h^{n+1} \equiv 1$. Hence $f(z) = tg(z)$, where $t$ is a constant and $f^{n+1} \equiv 1$.

The proof of Theorem 5 is complete.

3.3. Proof of Theorem 6. Let

$$F(z) = [f(z)^n \Delta_c f]^{(k)}, \quad G(z) = [g(z)^n \Delta_c g]^{(k)},$$

$$F_1(z) = f(z)^n \Delta_c f, \quad G_1(z) = g(z)^n \Delta_c g.$$ \hfill (63)

Then $F(z)$ and $G(z)$ share 1 IM, and $F_1^{(k)} = F, G_1^{(k)} = G$. By Lemma 10, we have

$$nT(r, f) + S(r, f) \leq T(r, F_1) \leq (n + 1) T(r, f) + S(r, f),$$ \hfill (64)

$$nT(r, g) + S(r, g) \leq T(r, G_1) \leq (n + 1) T(r, g) + S(r, g).$$ \hfill (65)

Since $f$ is transcendental entire, by the definition of $F$, we have

$$N_2\left(r, \frac{1}{F}\right) = N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{F}\right)$$

$$= N\left(r, \frac{1}{F}\right) - \left[N\left(r, \frac{1}{F}\right) - 2N\left(r, \frac{1}{F}\right)\right].$$ \hfill (66)

Using the argument in (35), we have

$$N_{13}\left(r, \frac{1}{F}\right) - 2N_{13}\left(r, \frac{1}{F}\right) \geq (n - k - 2) N\left(r, \frac{1}{F}\right).$$ \hfill (67)

It follows from Lemma 12 and (66), (67), we have

$$N_2\left(r, \frac{1}{F}\right) \leq N\left(r, \frac{1}{F}\right) - (n - k - 2) N\left(r, \frac{1}{F}\right)$$

$$\leq N\left(r, \frac{1}{f^n \Delta_c f}\right) - (n - k - 2) N\left(r, \frac{1}{f}\right)$$

$$+ S(r, f) \leq n N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\Delta_c f}\right)$$

$$- (n - k - 2) N\left(r, \frac{1}{f}\right) + S(r, f)$$

$$\leq (k + 3) T(r, f) + S(r, f).$$ \hfill (68)

From Lemma 15, we have

$$N\left(r, \frac{1}{F}\right) \leq N_{k+1}\left(r, \frac{1}{f^n \Delta_c f}\right) + S(r, f)$$

$$\leq (k + 1) N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\Delta_c f}\right) + S(r, f)$$

$$\leq (k + 2) T(r, f) + S(r, f).$$ \hfill (69)
Similarly,
\[
N_2 \left( r, \frac{1}{G} \right) \leq (k + 3) T(r, g) + S(r, g),
\]

\[
N \left( r, \frac{1}{G} \right) \leq (k + 2) T(r, g) + S(r, f).
\]

By Lemma 14, one of the following cases holds:

(i) \( T(r, G) \leq N_2(r, 1/G) + N_2(r, 1/F) + N(r, 1/F) + 2N(r, 1/G) + S(r, F) + S(r, G), \) the same inequality holding for \( T(r, F) \);

(ii) \( F \equiv \frac{AG + B}{CG + D} \).

For case (i), we have
\[
T(r, G) \leq N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) + 2N \left( r, \frac{1}{G} \right) + S(r, F) + S(r, G),
\]

\[
= N \left( r, \frac{1}{G} \right) + N \left( r, \frac{1}{F} \right) + S(r, F) + S(r, G).
\]

Therefore, we get
\[
T(r, F) + T(r, G) \leq 2 \left[ N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) \right] + 3 \left[ N \left( r, \frac{1}{G} \right) + N \left( r, \frac{1}{F} \right) \right] + S(r, F) + S(r, G).
\]

By (64) and Lemma 15, we have
\[
n T(r, f) \leq T(r, F_1) + S(r, f) \leq T(r, F) - N_2 \left( r, \frac{1}{F} \right) + N_{k+2} \left( r, \frac{1}{F} \right) + S(r, f)
\]

\[
\leq T(r, F) - N_2 \left( r, \frac{1}{F} \right) + (k + 2) N \left( r, \frac{1}{F} \right)
\]

\[
+ N \left( r, \frac{1}{\Delta \omega f} \right) + S(r, f)
\]

\[
\leq T(r, F) - N_2 \left( r, \frac{1}{F} \right) + (k + 3) T(r, f) + S(r, f).
\]

Similarly,
\[
n T(r, g) \leq T(r, G) - N_2 \left( r, \frac{1}{G} \right) + (k + 3) T(r, g) + S(r, g).
\]

By (70), (72), (73), and (74), we obtain
\[
(n - 5k - 12) \left[ T(r, f) + T(r, g) \right] \leq S(r, f) + S(r, g),
\]

which is a contradiction since \( n \geq 5k + 13 \).

For case (ii), we have
\[
F \equiv \frac{AG + B}{CG + D},
\]

where \( A, B, C, \) and \( D \) are finite complex numbers satisfying \( AD \neq BC \). Therefore, by the first fundamental theorem, \( T(r, F) = T(r, G) + S(r, F) \).

Next, we consider three cases.

**Case I.** \( AC \neq 0; \) from (76), we get
\[
F - \frac{A}{C} = \frac{B - AD}{CG + D}.
\]

By the second fundamental theorem and (69), we have
\[
T(r, F) \leq N \left( r, \frac{1}{F - A/C} \right) + N \left( r, \frac{1}{F} \right) + S(r, F)
\]

\[
= N \left( r, g \right) + (k + 2) T(r, f) + S(r, F)
\]

\[
\leq (k + 2) T(r, f) + S(r, F).
\]

From (73), we obtain \( (n - 2k - 5)T(r, f) \leq S(r, f), \) contradicting to \( n \geq 5k + 13 \).

**Case 2.** \( A \neq 0, \) and \( C = 0 \). Then, \( F \equiv AG + B/D \).

If \( B \neq 0 \), by the second fundamental theorem and (69), (70), we have
\[
T(r, F) \leq N \left( r, \frac{1}{F - B/D} \right) + N \left( r, \frac{1}{F} \right) + S(r, F)
\]

\[
= N \left( r, G \right) + (k + 2) T(r, f) + S(r, F)
\]

\[
\leq (k + 2) T(r, f) + S(r, F).
\]

Similarly,
\[
T(r, G) \leq (k + 2) T(r, f) + (k + 2) T(r, g) + S(r, G).
\]

From (73), (74), (79), and (80), we get
\[
(n - 3k - 7) \left[ T(r, f) + T(r, g) \right] \leq S(r, f) + S(r, g),
\]

which is a contradiction to \( n \geq 5k + 13 \).

If \( B = 0 \), then \( F \equiv AG/D\). If \( A/D = 1 \), then \( F \equiv G \); that is, \( \left[ f^n \Delta \omega f \right]^{[k]} = \left[ g^n \Delta \omega g \right]^{[k]} \); using the argument in (54) and noting that \( n \geq 5k + 13 \), we obtain \( f(z) = tg(z) \), where \( t \) is a constant and \( t^{n+1} = 1 \). If \( A/D \neq 1 \), by the condition that \( F \) and \( G \) share 1 IM, then \( F \neq 1 \) and \( G \neq 1 \). We obtain then \( F \neq 1 \) and \( F \neq A/D \). By the second fundamental theorem, we have
\[
T(r, F) \leq N \left( r, \frac{1}{F - A/D} \right) + N \left( r, \frac{1}{F - 1} \right) + S(r, F) \leq S(r, F),
\]

which is impossible.
Case 3. $A = 0$, and $C \neq 0$. Then, $F \equiv B/(CG + D)$.

If $D \neq 0$, by the second fundamental theorem and (69), (70), we have

$$T(r, F) \leq N\left( r, \frac{1}{F - B/D} \right) + N\left( r, \frac{1}{F} \right) + S(r, F)$$

$$= N\left( r, \frac{1}{G} \right) + N\left( r, \frac{1}{F} \right) + S(r, F) \tag{83}$$

$$\leq (k + 2) T(r, f) + (k + 2) T(r, g) + S(r, F).$$

Similarly,

$$T(r, G) \leq (k + 2) T(r, f) + (k + 2) T(r, g) + S(r, G). \tag{84}$$

From (73), (74), (83), and (84), we get

$$(n - 3k - 7) \left[ T(r, f) + T(r, g) \right] \leq S(r, F) + S(r, G), \tag{85}$$

which is a contradiction to $n \geq 5k + 13$.

If $D = 0$, then $F \equiv B/CG$. If $B/C = 1$, then $F \cdot G \equiv 1$; using the argument in (48) in Theorem 5 and noting that $n \geq 5k + 13$, we get a contradiction. If $B/C \neq 1$, by the condition that $F$ and $G$ share IIM, we obtain $F \neq 1$ and $F \neq B/C$. By the second fundamental theorem, we have

$$T(r, F) \leq N\left( \frac{1}{F - 1} \right) + N\left( \frac{1}{F - B/C} \right) + S(r, F) \leq S(r, F), \tag{86}$$

which is impossible.

The proof of Theorem 6 is complete.

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