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Research Article

Uniqueness of Entire Functions concerning Difference Operator

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We deal with a uniqueness question of entire functions sharing a nonzero value with their difference operators and obtain some results, which improve the results of Qi et al. (2010) and Zhang (2011).

1. Introduction and Main Results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We will use the standard notations of Nevanlinna's value distribution theory such as $T(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, and $m(r, f)$, as explained in Hayman [1], Yang [2], and Yang and Yi [3]. We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$, as $r \to \infty$ possibly outside a set of finite linear measures. For f meromorphic in $\mathbb{C}$, denote by $S(f)$ the family of all meromorphic functions $a(z)$ that satisfy $T(r, a) = o(T(r, f))$ for $r \to \infty$ outside a possible exceptional set of finite linear measure. In addition, we denote by $\rho(f)$ and $\rho_2(f)$ the order of f and the hyperorder of f [3, 4]. Moreover, we define difference operators by $\Delta_c f = f(z+c) - f(z)$ where c is a nonzero constant. If $c = 1$, we use the usual difference notation $\Delta_c f = \Delta f$.

Let f and g be two nonconstant meromorphic functions and a be a finite complex number. We say that f, g share the value a CM (counting multiplicities) if f, g have the same a-points with the same multiplicities, and we say that f, g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities. We denote by $\overline{N}_L(r, 1/(f-a))$ the counting function for a-points of both f and g about which f has larger multiplicity than g, with multiplicity not being counted. Similarly, we have the notation $\overline{N}_L(r, 1/(g-a))$. Next, we denote by $N_0(r, 1/F')$ the counting function of those zeros of F' that are not the zeros of $F(F-1)$ and denote by $N_{11}(r, 1/(f-a))$ the counting function for common simple 1-point of both f and g. In addition, we need the following three definitions.

Definition 1. Let k be a positive integer. Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. We denote by $\overline{N}_{f>k}(r, 1/(g-1))$ the reduced counting function of those 1-points of f and g such that $p > q = k$. $\overline{N}_{g>k}(r, 1/(f-1))$ is defined analogously.

Definition 2 (see [5]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \infty$, we denote by $E_k(a, f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $\leq k$ and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write that f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Definition 3. Let f be a nonconstant meromorphic function, and let p be a positive integer and $a \in C \cup \{\infty\}$. Then, by

$N_{p)}(r, 1/(f-a))$, we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p, and by $\overline{N}_{p)}(r, 1/(f-a))$, we denote the corresponding reduced counting function (ignoring multiplicities). By $N_{(p}(r, 1/(f-a))$, we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not less than p, and by $\overline{N}_{(p}(r, 1/(f-a))$, we denote the corresponding reduced counting function (ignoring multiplicities), where $N_{p)}(r, 1/(f-a))$, $\overline{N}_{p)}(r, 1/(f-a))$, $N_{(p}(r, 1/(f-a))$, and $\overline{N}_{(p}(r, 1/(f-a))$ mean $N_{p)}(r, f)$, $\overline{N}_{p)}(r, f)$, $N_{(p}(r, f)$, and $\overline{N}_{(p}(r, f)$, respectively, if $a = \infty$.

In 2010, Qi et al. [6] proved the following uniqueness theorem.

Theorem A. *Let f and g be transcendental entire functions of finite order, let c be a nonzero complex constant, and let $n \geq 6$ be an integer. If $f(z)^n f(z+c)$ and $g(z)^n g(z+c)$ share z CM, then $f = tg$ for a constant t that satisfies $t^{n+1} = 1$.*

In 2011, Zhang et al. [7] complemented the above theorem and obtained the following result.

Theorem B. *Let f and g be nonconstant entire functions of finite order, and let $n \geq 5$ be an integer. Suppose that c is a nonzero complex constant such that $\Delta_c f \not\equiv 0$ and $\Delta_c g \not\equiv 0$. If $f^n \Delta_c f$ and $g^n \Delta_c g$ share z CM, and $g(z+c)$ and $g(z)$ share 0 CM then $f = tg$, where t is a constant satisfying $t^{n+1} = 1$.*

In this paper, we complement Theorems A and B and obtain the following results which generalize the above theorems.

Theorem 4. *Let f be a transcendental entire function of finite order and $\Delta_c f \not\equiv 0$, let $a \neq 0$ be a small function with respect to f, and let c be a nonzero complex constant. Then for $n \geq 2$, $f(z)^n(f(z+c)-1)\Delta_c f - a$ has infinitely many zeros.*

Theorem 5. *Let $f(z)$ and $g(z)$ be transcendental entire functions of $\rho_2 < 1$, $n \geq 2k+7$. Suppose that c is a nonzero complex constant such that $\Delta_c f \not\equiv 0$ and $\Delta_c g \not\equiv 0$. If $[f^n \Delta_c f]^{(k)}$ and $[g^n \Delta_c g]^{(k)}$ share 1 CM, then $f = tg$ for a constant t with $t^{n+1} = 1$.*

Theorem 6. *Let f and g be transcendental entire functions of $\rho_2 < 1$, $n \geq 5k+13$. c is a nonzero complex constant such that $\Delta_c f \not\equiv 0$ and $\Delta_c g \not\equiv 0$. If $[f^n \Delta_c f]^{(k)}$ and $[g^n \Delta_c g]^{(k)}$ share 1 IM, then $f = tg$ for a constant t with $t^{n+1} = 1$.*

2. Some Lemmas

Lemma 7 (see [8]). *Let f be a nonconstant meromorphic function of finite order σ, and let c be a nonzero constant. Then, for each $\varepsilon > 0$,*

$$T(r, f(z+c)) = T(r, f(z)) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r). \quad (1)$$

Lemma 8 (see [9]). *Let f be a meromorphic function of finite order, and let $c \in \mathbb{C}$ and $\delta \in (0,1)$. Then*

$$\begin{aligned} m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) \\ = o\left(\frac{T(r,f)}{r^{\delta}}\right) = S(r,f). \end{aligned} \quad (2)$$

Lemma 9 (see [10]). *Let f_1, f_2, and f_3 be nonconstant meromorphic functions such that $f_1+f_2+f_3 = 1$. If f_1, f_2, and f_3 are linearly independent, then*

$$T(r, f_1) \leq \sum_{j=1}^{3} N_2\left(r, \frac{1}{f_j}\right) + \sum_{j=1}^{3} \overline{N}(r, f_j) + o(T(r)), \quad (3)$$

where $T(r) = \max_{1\leq j\leq 3} T(r, f_j)$, $r \notin E$, and E denote a set of positive real numbers of finite linear measure.

Lemma 10. *Let f be transcendental entire functions of finite order, let c be a nonzero complex constant, and set $F(z) = f(z)^n \Delta_c f$; then*

$$nT(r,f) + S(r,f) \leq T(r,F) \leq (n+1)T(r,f) + S(r,f). \quad (4)$$

Proof. Since

$$\begin{aligned} T(r,F) &= T(r, f(z)^n \Delta_c f) \leq nT(r,f) + T(r, \Delta_c f) \\ &\leq nT(r,f) + m(r, \Delta_c f) \leq nT(r,f) \\ &\quad + m(r,f) + S(r,f) \\ &= (n+1)T(r,f) + S(r,f), \end{aligned} \quad (5)$$

then

$$\begin{aligned} (n+1)T(r,f) &= T\left(r, f(z)^{n+1}\right) = m\left(r, f(z)^{n+1}\right) \\ &\leq m\left(r, \frac{f(z)^{n+1}}{F}\right) + m(r,F) + S(r,f) \\ &\leq m\left(r, \frac{f(z)}{\Delta_c f}\right) + m(r,F) + S(r,f) \\ &\leq T\left(r, \frac{f(z)}{\Delta_c f}\right) + T(r,F) + S(r,f) \\ &\leq T\left(r, \frac{\Delta_c f}{f(z)}\right) + T(r,F) + S(r,f) \\ &= m\left(r, \frac{\Delta_c f}{f(z)}\right) + N\left(r, \frac{\Delta_c f}{f(z)}\right) \\ &\quad + T(r,F) + S(r,f) \\ &\leq T(r,F) + N\left(r, \frac{1}{f(z)}\right) + S(r,f) \\ &\leq T(r,F) + T(r,f) + S(r,f). \end{aligned} \quad (6)$$

That is,

$$nT(r,f)+S(r,f)\le T(r,F)\le (n+1)T(r,f)+S(r,f). \tag{7}$$

□

Lemma 11 (see [11]). *Let f_1 and f_2 be two nonconstant meromorphic functions. If $c_1f_1+c_2f_2=c_3$, where c_1, c_2, and c_3 are nonzero constants, then*

$$T(r,f_1)\le \overline{N}(r,f_1)+\overline{N}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{f_2}\right)+S(r,f_1). \tag{8}$$

Lemma 12 (see [12]). *Let $f(z)$ be a nonconstant meromorphic function, and let k be a positive integer. Suppose that $f^{(k)}\not\equiv 0$; then*

$$N\left(r,\frac{1}{f^{(k)}}\right)\le N\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f). \tag{9}$$

Lemma 13 (see [13]). *Let f, g share $(1,0)$. Then*

(i) $\overline{N}_{f>1}(r,1/(g-1))\le \overline{N}(r,1/f)+\overline{N}(r,f)-N_0(r,1/f')+S(r,f)$,

(ii) $\overline{N}_{g>1}(r,1/(f-1))\le \overline{N}(r,1/g)+\overline{N}(r,g)-N_0(r,1/g')+S(r,g)$.

Lemma 14. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions. If f and g share 1 IM, then one of the following cases holds:*

(i) *$T(r,g)\le N_2(r,1/g)+N_2(r,1/f)+\overline{N}(r,1/f)+2\overline{N}(r,1/g)+S(r,f)+S(r,g)$, the same inequality holding for $T(r,f)$;*

(ii) *$f\equiv (Ag+B)/(Cg+D)$, where A, B, C, and D are finite complex numbers satisfying $AD\neq BC$.*

Proof. Let

$$\Phi(z)=\frac{f''}{f'}-2\frac{f'}{f-1}-\frac{g''}{g'}+2\frac{g'}{g-1}. \tag{10}$$

Clearly $m(r,\Phi)=S(r,f)+S(r,g)$. We consider the cases $\Phi(z)\not\equiv 0$ and $\Phi(z)\equiv 0$.

If $\Phi(z)\not\equiv 0$, then if z_0 is a common simple 1-point of f' and g', substituting their Taylor series at z_0 into (10), we see that z_0 is a zero of $\Phi(z)$. Thus, we have

$$\begin{aligned} N_{11}\left(r,\frac{1}{f-1}\right)=N_{11}\left(r,\frac{1}{g-1}\right)&\le \overline{N}\left(r,\frac{1}{\Phi}\right)\\ &\le T(r,\Phi)+O(1)\\ &\le N(r,\Phi)+S(r,f)+S(r,g). \end{aligned} \tag{11}$$

Our assumptions are that $\Phi(z)$ has poles; all are simple only at zeros of f' and g' and poles of f and g, and 1-points of f whose multiplicities are not equal to the multiplicities of the corresponding 1-points of g. Thus, we deduce from (10) that

$$\begin{aligned} N(r,\Phi)\le{}& \overline{N}_{(2}\left(r,\frac{1}{f}\right)+\overline{N}_{(2}\left(r,\frac{1}{g}\right)+N_0\left(r,\frac{1}{f'}\right)\\ &+N_0\left(r,\frac{1}{g'}\right)+\overline{N}_L\left(r,\frac{1}{f-1}\right)+\overline{N}_L\left(r,\frac{1}{g-1}\right), \end{aligned} \tag{12}$$

where $N_0(r,1/f')$ is the counting function which only counts those points such that $f'=0$, but $f(f-1)\neq 0$. By the second fundamental theorem, we have

$$\begin{aligned} T(r,g)\le{}& \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)\\ &-N_0\left(r,\frac{1}{g'}\right)+S(r,g), \end{aligned} \tag{13}$$

since

$$\begin{aligned} \overline{N}\left(r,\frac{1}{g-1}\right)={}& N_{11}\left(r,\frac{1}{g-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-1}\right)\\ &+\overline{N}_{g>1}\left(r,\frac{1}{f-1}\right). \end{aligned} \tag{14}$$

Thus, we deduce from (11)–(14) that

$$\begin{aligned} T(r,g)\le{}& \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)\\ &+\overline{N}_{(2}\left(r,\frac{1}{g}\right)+N_0\left(r,\frac{1}{f'}\right)\\ &+\overline{N}_{(2}\left(r,\frac{1}{f-1}\right)+\overline{N}_L\left(r,\frac{1}{f-1}\right)\\ &+\overline{N}_L\left(r,\frac{1}{g-1}\right)+\overline{N}_{g>1}\left(r,\frac{1}{f-1}\right)\\ &+S(r,f)+S(r,g). \end{aligned} \tag{15}$$

From the definition of $N_0(r,1/f')$, we see that

$$\begin{aligned} N_0\left(r,\frac{1}{f'}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-1}\right)+N_{(2}\left(r,\frac{1}{f}\right)&\\ -\overline{N}_{(2}\left(r,\frac{1}{f}\right)\le N\left(r,\frac{1}{f'}\right)&. \end{aligned} \tag{16}$$

The above inequality and Lemma 12 give

$$\begin{aligned} &N_0\left(r,\frac{1}{f'}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-1}\right)\\ &\quad\le N\left(r,\frac{1}{f'}\right)-N_{(2}\left(r,\frac{1}{f}\right)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)\\ &\quad\le N\left(r,\frac{1}{f}\right)-N_{(2}\left(r,\frac{1}{f}\right)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)+S(r,f)\\ &\quad\le \overline{N}\left(r,\frac{1}{f}\right)+S(r,f). \end{aligned} \tag{17}$$

Substituting (17) in (15), we get

$$\begin{aligned} T(r,g) &\le \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)+\overline{N}_{(2}\left(r,\frac{1}{g}\right) \\ &\quad+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}_L\left(r,\frac{1}{f-1}\right)+\overline{N}_L\left(r,\frac{1}{g-1}\right) \\ &\quad+\overline{N}_{g>1}\left(r,\frac{1}{f-1}\right)+S(r,f)+S(r,g) \\ &\le N_2\left(r,\frac{1}{g}\right)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right) \\ &\quad+\overline{N}_L\left(r,\frac{1}{f-1}\right)+\overline{N}_L\left(r,\frac{1}{g-1}\right) \\ &\quad+\overline{N}_{g>1}\left(r,\frac{1}{f-1}\right)+S(r,f)+S(r,g), \end{aligned} \tag{18}$$

since

$$\begin{aligned} \overline{N}_L\left(r,\frac{1}{f-1}\right) &\le N\left(r,\frac{1}{f-1}\right)-\overline{N}\left(r,\frac{1}{f-1}\right) \\ &\le N\left(r,\frac{f}{f'}\right)\le N\left(r,\frac{f'}{f}\right)+S(r,f) \\ &\le \overline{N}\left(r,\frac{1}{f}\right)+S(r,f). \end{aligned} \tag{19}$$

Similarly,

$$\overline{N}_L\left(r,\frac{1}{g-1}\right)\le \overline{N}\left(r,\frac{1}{g}\right)+S(r,g). \tag{20}$$

Combining the above inequalities, Lemma 13, and (18), we obtain

$$\begin{aligned} T(r,g) &\le N_2\left(r,\frac{1}{g}\right)+N_2\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right)+2\overline{N}\left(r,\frac{1}{g}\right) \\ &\quad-N_0\left(r,\frac{1}{g'}\right)+S(r,f)+S(r,g) \\ &\le N_2\left(r,\frac{1}{g}\right)+N_2\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right) \\ &\quad+2\overline{N}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g). \end{aligned} \tag{21}$$

Thus, we obtain (i).

If $\Phi(z)\equiv 0$, then by (10), we have

$$\frac{f''}{f'}-\frac{2f'}{f-1}\equiv\frac{g''}{g'}-\frac{2g'}{g-1}. \tag{22}$$

By integrating two sides of the above equality, we obtain

$$f\equiv\frac{Ag+B}{Cg+D}, \tag{23}$$

where A, B, C, and D are finite complex numbers satisfying $AD\ne BC$. This proves the lemma. □

Lemma 15 (see [14]). *Let $f(z)$ be a nonconstant meromorphic function, s, k be two positive integers; then*

$$\begin{aligned} N_s\left(r,\frac{1}{f^{(k)}}\right) &\le T(r,f)^{(k)}-T(r,f) \\ &\quad+N_{s+k}\left(r,\frac{1}{f}\right)+S(r,f), \\ N_s\left(r,\frac{1}{f^{(k)}}\right) &\le k\overline{N}(r,f) \\ &\quad+N_{s+k}\left(r,\frac{1}{f}\right)+S(r,f). \end{aligned} \tag{24}$$

Clearly, $\overline{N}(r,1/f^{(k)})=N_1(r,1/f^{(k)})$.

Lemma 16 (see [15]). *Let $a_0(z),a_1(z),\ldots,a_n(z),b(z)$ be polynomials such that $a_0(z)a_n(z)\not\equiv 0$; let c_j be constants and*

$$\deg\left(\sum_{\deg a_j=d}a_j\right)=d, \tag{25}$$

where $d=\max_{0\le j\le n}\{\deg a_j\}$. If $f(z)$ is a transcendental meromorphic solution of

$$\sum_{j=0}^{n}a_j(z)f(z+c_j)=b(z), \tag{26}$$

then $\rho(f)\ge 1$.

3. Proof of Theorems

3.1. Proof of Theorem 4. Let $G(z)=f(z)^n(f(z+c)-1)\Delta_c f$. Since f is a transcendental entire function of finite order, from Lemma 7, we have

$$\begin{aligned} &(n+2)T(r,(r,f(z))) \\ &\quad\le T\left(r,f(z)^{n+1}(f(z+c)-1)\right)+S(r,f) \\ &\quad\le m\left(r,f(z)^{n+1}(f(z+c)-1)\right)+S(r,f) \\ &\quad\le m\left(r,\frac{f(z)^{n+1}(f(z+c)-1)}{G}\right)+m(r,G)+S(r,f) \\ &\quad\le T(r,G)+S(r,f). \end{aligned} \tag{27}$$

By the second main theorem, we deduce that

$$\begin{aligned} T(r,G) &\le \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G-a}\right)+S(r,G) \\ &\le \overline{N}\left(r,\frac{1}{G-a}\right)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-1}\right) \\ &\quad+\overline{N}\left(r,\frac{1}{\Delta_c f}\right)+S(r,f) \end{aligned}$$

$$\begin{aligned}
&\le \overline{N}\left(r, \frac{1}{G-a}\right) + \overline{N}\left(r, \frac{1}{f}\right) + T\left(r, f(z+c) - 1\right) \\
&\quad + T\left(r, \Delta_c f\right) + S\left(r, f\right) \\
&\le \overline{N}\left(r, \frac{1}{G-a}\right) + \overline{N}\left(r, \frac{1}{f}\right) + T\left(r, f(z+c) - 1\right) \\
&\quad + m\left(r, \frac{\Delta_c f}{f} \cdot f\right) + S\left(r, f\right) \\
&\le \overline{N}\left(r, \frac{1}{G-a}\right) + \overline{N}\left(r, \frac{1}{f}\right) + T\left(r, f(z+c) - 1\right) \\
&\quad + m\left(r, \frac{\Delta_c f}{f}\right) + m\left(r, f\right) + S\left(r, f\right) \\
&\le \overline{N}\left(r, \frac{1}{G-a}\right) + 3T\left(r, f\right) + S\left(r, f\right).
\end{aligned} \tag{28}$$

According to (27) and (28), we have

$$(n-1)\, T\left(r, f\right) \le \overline{N}\left(r, \frac{1}{G-a}\right) + S\left(r, f\right). \tag{29}$$

Noting that $n \ge 2$, we get that $G - a$ has infinitely many zeros.

This completes the proof of Theorem 4.

3.2. Proof of Theorem 5. Since $[f(z)^n \Delta_c f]^{(k)}$ and $[g(z)^n \Delta_c g]^{(k)}$ share 1 CM, we have

$$\frac{[f(z)^n \Delta_c f]^{(k)} - 1}{[g(z)^n \Delta_c g]^{(k)} - 1} = e^{h(z)}, \tag{30}$$

where $h(z)$ is a polynomial. Set $F = f(z)^n \Delta_c f$, $G = g(z)^n \Delta_c g$,

$$\begin{gathered}
F_1 = F^{(k)}, \qquad F_2 = -e^{h(z)} G^{(k)}, \qquad F_3 = e^{h(z)}, \\
\text{then } F_1 + F_2 + F_3 = 1, \\
T(r) = \max_{1 \le j \le 3} T\left(r, F_j\right), \qquad S(r) = o\left(T(r)\right).
\end{gathered} \tag{31}$$

Next, we will prove that F_1, F_2, and F_3 are linearly dependent and either F_2 or F_3 is a constant.

Now, we suppose that neither F_2 nor F_3 is a constant and F_1, F_2, and F_3 are linearly independent; then by Lemma 9, we have

$$T\left(r, F_1\right) \le \sum_{j=1}^{3} N_2\left(r, \frac{1}{F_j}\right) + \sum_{j=1}^{3} \overline{N}\left(r, F_j\right) + o\left(T(r)\right). \tag{32}$$

Since F_j $(j = 1, 2, 3)$ are entire functions, by the above inequality, we get

$$T\left(r, F_1\right) \le N_2\left(r, \frac{1}{F^{(k)}}\right) + N_2\left(r, \frac{1}{G^{(k)}}\right) + o\left(T(r)\right). \tag{33}$$

From (33) and the first main theorem, we have

$$\begin{aligned}
T\left(r, \frac{1}{F^{(k)}}\right) &= T\left(r, F^{(k)}\right) + O(1) = T\left(r, F_1\right) + O(1) \\
&\le N_2\left(r, \frac{1}{F^{(k)}}\right) + N_2\left(r, \frac{1}{G^{(k)}}\right) + o\left(T(r)\right) \\
&\le N\left(r, \frac{1}{F^{(k)}}\right) \\
&\quad - \left[N_{(3}\left(r, \frac{1}{F^{(k)}}\right) - 2\overline{N}_{(3}\left(r, \frac{1}{F^{(k)}}\right)\right] \\
&\quad + N\left(r, \frac{1}{G^{(k)}}\right) \\
&\quad - \left[N_{(3}\left(r, \frac{1}{G^{(k)}}\right) - 2\overline{N}_{(3}\left(r, \frac{1}{G^{(k)}}\right)\right] \\
&\quad + o\left(T(r)\right).
\end{aligned} \tag{34}$$

Assuming that z_0 is zero of $f(z)$ (or $g(z)$) with multiplicity p, if z_0 is zero of $f(z+c)$ (or $g(z+c)$) with multiplicity $q(\ge 1)$, let $m = \min\{p, q\}$, then z_0 is a zero of $F^{(k)}$ (or $G^{(k)}$) with multiplicity $np+m-k \ge np-k \ge 3$, and if z_0 is not zero of $f(z+c)$ (or $g(z+c)$), then z_0 is a zero of $F^{(k)}$ (or $G^{(k)}$) with multiplicity $np - k \ge 3$. Therefore, we get that

$$N_{(3}\left(r, \frac{1}{F^{(k)}}\right) - 2\overline{N}_{(3}\left(r, \frac{1}{F^{(k)}}\right) \ge (n-k-2)\, N\left(r, \frac{1}{f}\right), \tag{35}$$

$$N_{(3}\left(r, \frac{1}{G^{(k)}}\right) - 2\overline{N}_{(3}\left(r, \frac{1}{G^{(k)}}\right) \ge (n-k-2)\, N\left(r, \frac{1}{g}\right), \tag{36}$$

since

$$\begin{aligned}
nm\left(r, \frac{1}{f}\right) &= m\left(r, \frac{1}{f^n}\right) = m\left(r, \frac{\Delta_c f}{F}\right) \\
&\le m\left(r, \frac{1}{F}\right) + m\left(r, \frac{\Delta_c f}{f} \cdot f\right) \\
&\le m\left(r, \frac{F^{(k)}}{F} \cdot \frac{1}{F^{(k)}}\right) + m\left(r, \frac{\Delta_c f}{f}\right) \\
&\quad + m\left(r, f\right) + S\left(r, f\right) \\
&\le m\left(r, \frac{1}{F^{(k)}}\right) + T\left(r, f\right) + S\left(r, f\right) \\
&= T\left(r, \frac{1}{F^{(k)}}\right) - N\left(r, \frac{1}{F^{(k)}}\right) + T\left(r, f\right) + S\left(r, f\right).
\end{aligned} \tag{37}$$

Therefore, from (34), (35), (36), (37), and Lemma 12,

$$\begin{aligned}
(n-1)\, T\left(r, f\right) &\le (k+2)\, N\left(r, \frac{1}{f}\right) + (k+2)\, N\left(r, \frac{1}{g}\right) \\
&\quad + T\left(r, g\right) + o\left(T(r)\right).
\end{aligned} \tag{38}$$

On the other hand, from (30), we have $G^{(k)}+e^{-h}-e^{-h}F^{(k)}=1$. Obviously, according to our assumptions, neither e^{-h} nor $e^{-h}F^{(k)}$ is a constant and F_1, F_2, and F_3 are linearly independent. Similarly, we have

$$(n-1)\,T(r,g)\le(k+2)\,N\left(r,\frac{1}{g}\right)+(k+2)\,N\left(r,\frac{1}{f}\right)+T(r,f)+o(T(r)). \tag{39}$$

From (38) and (39), we obtain that

$$[n-2k-6]\,(T(r,f)+T(r,g))\le o(T(r)), \tag{40}$$

which is a contradiction to $n\ge 2k+7$.

Therefore, F_1, F_2, and F_3 are linearly dependent, and there exist constants C_1, C_2, C_3 which are not all equal to zero such that

$$C_1F_1+C_2F_2+C_3F_3=0. \tag{41}$$

Suppose that $C_1=0$; we have $C_2F_2+C_3F_3=0$. If $C_2\ne 0$, we get $F_2=-(C_3/C_2)\,F_3$; that is, $G^{(k)}=C_3/C_2$; thus $g(z)$ is a polynomial; it is impossible. Similarly, if $C_2=0$, we also deduce a contradiction.

Suppose that $C_1\ne 0$, from (41); we know that $(C_2,C_3)\ne(0,0)$. If $C_2\ne 0$, from (41), we have

$$\left(1-\frac{C_2}{C_1}\right)F_2+\left(1-\frac{C_3}{C_1}\right)F_3=1 \tag{42}$$

and $C_1\ne C_2$, $C_1\ne C_3$. That is,

$$\left(1-\frac{C_2}{C_1}\right)G^{(k)}+\frac{1}{e^h}=1-\frac{C_3}{C_1}. \tag{43}$$

From Lemma 11, we have

$$\begin{aligned}T\left(r,G^{(k)}\right)&\le\overline{N}\left(r,\frac{1}{G^{(k)}}\right)+\overline{N}\left(r,G^{(k)}\right)+\overline{N}\left(r,e^h\right)+S(r,g)\\&=\overline{N}\left(r,\frac{1}{G^{(k)}}\right)+S(r,g)\le N\left(r,\frac{1}{G^{(k)}}\right)\\&\quad-\left[N_{(2}\left(r,\frac{1}{G^{(k)}}\right)-\overline{N}_{(2}\left(r,\frac{1}{G^{(k)}}\right)\right]+S(r,g).\end{aligned} \tag{44}$$

By the similar argument in (37), we have

$$\begin{aligned}nm\left(r,\frac{1}{g}\right)&\le T\left(r,\frac{1}{G^{(k)}}\right)-N\left(r,\frac{1}{G^{(k)}}\right)\\&\quad+T(r,g)+S(r,g).\end{aligned} \tag{45}$$

From $n\ge 2k+7>k+2$, if z_0 is zero of $g(z)$ with multiplicity p, then z_0 is a zero of $G^{(k)}$ with multiplicity $np-k\ge 2$, and we get

$$N_{(2}\left(r,\frac{1}{G^{(k)}}\right)-\overline{N}_{(2}\left(r,\frac{1}{G^{(k)}}\right)\ge(n-k-1)\,N\left(r,\frac{1}{g}\right). \tag{46}$$

According to (44), (45), and (46), we have

$$(n-1)\,T(r,g)\le(k+1)\,N\left(r,\frac{1}{g}\right)+S(r,g), \tag{47}$$

which is a contradiction to $n\ge 2k+7$.

Therefore, $C_2=0$, $C_3\ne 0$, which gives $(1-C_1/C_3)F_1+F_2=1$. Similarly, we derive a contradiction by calculation.

Hence, we deduce that either F_2 or F_3 is a constant.

Suppose $F_2=c\ne 1$; from $F_1+F_2+F_3=1$, we have $F^{(k)}+e^h=1-c$; in the same manner as above, we get a contradiction. Therefore, $c=1$; that is, $F_2=1$. Suppose $F_3=c\ne 1$; similarly as above, we get $c=1$; that is, $F_3=1$.

Therefore, we conclude that $F_2=1$ or $F_3=1$.

If $F_2=1$, since $F_1+F_2+F_3=1$, we have $F_1=-F_3=-e^{h(z)}$. That is

$$[f^n\Delta_c f]^{(k)}\cdot[g^n\Delta_c g]^{(k)}\equiv 1. \tag{48}$$

Since $n\ge 2k+7$ and f and g are transcendental entire functions with hyperorder less than one, we get that f and g have no zeros. Thus,

$$f(z)=e^{a(z)},\qquad g(z)=e^{b(z)}, \tag{49}$$

where $a(z)$, $b(z)$ are nonzero polynomials.

Substitute (49) into (48); we have

$$\left[e^{na(z)}\left(e^{a(z+c)}-e^{a(z)}\right)\right]^{(k)}\cdot\left[e^{nb(z)}\left(e^{b(z+c)}-e^{b(z)}\right)\right]^{(k)}\equiv 1. \tag{50}$$

Let $na(z)+a(z+c)=A_1$, $na(z)+a(z)=A_2$, $nb(z)+b(z+c)=B_1$, and $nb(z)+b(z)=B_2$. If $k=1$, we have

$$\left(A_1'e^{A_1}-A_2'e^{A_2}\right)\cdot\left(B_1'e^{B_1}-B_2'e^{B_2}\right)\equiv 1. \tag{51}$$

From (51), we know that $A_1'e^{A_1}-A_2'e^{A_2}=e^{A_2}(A_1'e^{A_1-A_2}-A_2')\ne 0$; If $A_1'\ne 0$, then we have $A_2'=0$; thus, A_2 must be a constant. By Lemma 16, we have $\rho(a(z))\ge 1$; thus, $\rho_2(f)\ge 1$, which is a contradiction. If $A_1'=0$, then A_1 must be a constant; similarly, we also deduce a contradiction.

If $k=2$, by calculation, we have

$$\begin{aligned}&A_1''e^{A_1}+\left(A_1'\right)^2e^{A_1}-A_2''e^{A_2}-\left(A_2'\right)^2e^{A_2}\\&\quad=e^{A_2}\left[e^{A_1-A_2}\left(A_1''+\left(A_1'\right)^2\right)-\left(A_2''+\left(A_2'\right)^2\right)\right]\ne 0.\end{aligned} \tag{52}$$

If $A_1''+(A_1')^2\ne 0$, then $A_2''+(A_2')^2=0$. If A_2 is transcendental entire, then we have

$$m\left(r,A_2'\right)=m\left(\frac{A_2''}{A_2'}\right)=S\left(r,A_2'\right), \tag{53}$$

which is a contradiction to A_2' being transcendental entire. If A_2 is a polynomial, from Lemma 16, which induces that $\rho_2(f)\ge 1$, we get a contradiction. If $A_1''+(A_1')^2=0$, similar as above, we get a contradiction. For $k\ge 3$, using the similar

Method as above, we also deduce a contradiction. Therefore, There are not transcendental entire functions $f(z)$ and $g(z)$ satisfying (48).

If $F_3 = 1$, that is, $e^{h(z)} = 1$, from (30), we get

$$[f^n \Delta_c f]^{(k)} \equiv [g^n \Delta_c g]^{(k)}. \tag{54}$$

From (54), we have

$$f^n \Delta_c f \equiv g^n \Delta_c g + p(z), \tag{55}$$

where $p(z)$ is a polynomial of degree at most $k-1$. Suppose $p(z) \not\equiv 0$; then we get

$$\frac{f^n \Delta_c f}{p(z)} = \frac{g^n \Delta_c g}{p(z)} + 1. \tag{56}$$

Therefore, from the second main theorem, we have

$$\begin{aligned}(n+1)T(r,f) &\le T\left(\frac{f^n\Delta_c f}{p(z)}\right) + S(r,f)\\ &\le \overline{N}\left(\frac{f^n\Delta_c f}{p(z)}\right) + \overline{N}\left(\frac{p(z)}{f^n\Delta_c f}\right)\\ &\quad + \overline{N}\left(\frac{p(z)}{g^n\Delta_c g}\right) + S(r,f)\\ &\le \overline{N}\left(\frac{1}{f}\right) + \overline{N}\left(\frac{1}{\Delta_c f}\right) + \overline{N}\left(\frac{1}{g}\right)\\ &\quad + \overline{N}\left(\frac{1}{\Delta_c g}\right) + S(r,f)\\ &\le 2T(r,f) + 2T(r,g) + S(r,f).\end{aligned} \tag{57}$$

Similarly, we have

$$(n+1)T(r,g) \le 2T(r,f) + 2T(r,g) + S(r,f). \tag{58}$$

Therefore,

$$\begin{aligned}&(n+1)[T(r,f)+T(r,g)]\\ &\quad \le 4[T(r,f)+T(r,g)] + S(r,f) + S(r,g),\end{aligned} \tag{59}$$

which is a contradiction to $n \ge 2k+7$. Thus, $p(z) \equiv 0$, which implies that

$$f^n \Delta_c f \equiv g^n \Delta_c g. \tag{60}$$

Let $f/g = h$; if h is not a constant, then by (60), we have

$$h^{n+1} \equiv \frac{f}{\Delta_c f} \cdot \frac{\Delta_c g}{g}. \tag{61}$$

Thus,

$$\begin{aligned}(n+1)T(r,h) &\le T\left(r, \frac{\Delta_c f}{f}\right) + T\left(r, \frac{\Delta_c g}{g}\right) + O(1)\\ &\le N\left(r, \frac{\Delta_c f}{f}\right) + N\left(r, \frac{\Delta_c g}{g}\right)\\ &\quad + S(r,f) + S(r,g)\\ &\le T(r,f) + T(r,g) + S(r,f) + S(r,g).\end{aligned} \tag{62}$$

Combining $T(r,h) = T(r,f/g) = T(r,f) + T(r,g) + O(1)$, we obtain $n(T(r,f)+T(r,g)) \le S(r,f) + S(r,g)$, which is impossible.

Therefore, h is a constant; then substituting $f = gh$ into (60), we have $h^{n+1} \equiv 1$. Hence $f(z) = tg(z)$, where t is a constant and $t^{n+1} = 1$.

The proof of Theorem 5 is complete.

3.3. Proof of Theorem 6. Let

$$\begin{aligned}F(z) &= [f(z)^n \Delta_c f]^{(k)}, \qquad G(z) = [g(z)^n \Delta_c g]^{(k)},\\ F_1(z) &= f(z)^n \Delta_c f, \qquad G_1(z) = g(z)^n \Delta_c g.\end{aligned} \tag{63}$$

Then $F(z)$ and $G(z)$ share 1 IM, and $F_1^{(k)} = F$, $G_1^{(k)} = G$. By Lemma 10, we have

$$nT(r,f) + S(r,f) \le T(r,F_1) \le (n+1)T(r,f) + S(r,f), \tag{64}$$

$$nT(r,g) + S(r,g) \le T(r,G_1) \le (n+1)T(r,g) + S(r,g). \tag{65}$$

Since f is transcendental entire, by the definition of F, we have

$$\begin{aligned}N_2\left(r, \frac{1}{F}\right) &= \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{F}\right)\\ &= N\left(r, \frac{1}{F}\right) - \left[N_{(3}\left(r, \frac{1}{F}\right) - 2\overline{N}_{(3}\left(r, \frac{1}{F}\right)\right].\end{aligned} \tag{66}$$

Using the argument in (35), we have

$$N_{(3}\left(r, \frac{1}{F}\right) - 2\overline{N}_{(3}\left(r, \frac{1}{F}\right) \ge (n-k-2)N\left(r, \frac{1}{f}\right). \tag{67}$$

It follows from Lemma 12 and (66), (67), we have

$$\begin{aligned}N_2\left(r, \frac{1}{F}\right) &\le N\left(r, \frac{1}{F}\right) - (n-k-2)N\left(r, \frac{1}{f}\right)\\ &\le N\left(r, \frac{1}{f^n\Delta_c f}\right) - (n-k-2)N\left(r, \frac{1}{f}\right)\\ &\quad + S(r,f) \le nN\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\Delta_c f}\right)\\ &\quad - (n-k-2)N\left(r, \frac{1}{f}\right) + S(r,f)\\ &\le (k+3)T(r,f) + S(r,f).\end{aligned} \tag{68}$$

From Lemma 15, we have

$$\begin{aligned}\overline{N}\left(r, \frac{1}{F}\right) &\le N_{k+1}\left(r, \frac{1}{f^n\Delta_c f}\right) + S(r,f)\\ &\le (k+1)\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\Delta_c f}\right) + S(r,f)\\ &\le (k+2)T(r,f) + S(r,f).\end{aligned} \tag{69}$$

Similarly,

$$N_2\left(r,\frac{1}{G}\right)\le(k+3)T(r,g)+S(r,g),$$
$$\overline{N}\left(r,\frac{1}{G}\right)\le(k+2)T(r,g)+S(r,f). \tag{70}$$

By Lemma 14, one of the following cases holds:

(i) $T(r,G)\le N_2(r,1/G)+N_2(r,1/F)+\overline{N}(r,1/F)+2\overline{N}(r,1/G)+S(r,F)+S(r,G)$, the same inequality holding for $T(r,F)$;

(ii) $F\equiv(AG+B)/(CG+D)$.

For case (i), we have

$$\begin{aligned}T(r,G)&\le N_2\left(r,\frac{1}{G}\right)+N_2\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F}\right)\\&\quad+2\overline{N}\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G),\\T(r,F)&\le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G}\right)\\&\quad+2\overline{N}\left(r,\frac{1}{F}\right)+S(r,F)+S(r,G).\end{aligned} \tag{71}$$

Therefore, we get

$$\begin{aligned}T(r,F)+T(r,G)&\le 2\left[N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)\right]\\&\quad+3\left[\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F}\right)\right]\\&\quad+S(r,F)+S(r,G).\end{aligned} \tag{72}$$

By (64) and Lemma 15, we have

$$\begin{aligned}nT(r,f)&\le T(r,F_1)+S(r,f)\le T(r,F)-N_2\left(r,\frac{1}{F}\right)\\&\quad+N_{k+2}\left(r,\frac{1}{F_1}\right)+S(r,f)\\&\le T(r,F)-N_2\left(r,\frac{1}{F}\right)+(k+2)\overline{N}\left(r,\frac{1}{f}\right)\\&\quad+N\left(r,\frac{1}{\Delta_c f}\right)+S(r,f)\\&\le T(r,F)-N_2\left(r,\frac{1}{F}\right)\\&\quad+(k+3)T(r,f)+S(r,f).\end{aligned} \tag{73}$$

Similarly,

$$nT(r,g)\le T(r,G)-N_2\left(r,\frac{1}{G}\right)+(k+3)T(r,g)+S(r,g). \tag{74}$$

By (70), (72), (73), and (74), we obtain

$$(n-5k-12)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g), \tag{75}$$

which is a contradiction since $n\ge 5k+13$.

For case (ii), we have

$$F\equiv\frac{AG+B}{CG+D}, \tag{76}$$

where A, B, C, and D are finite complex numbers satisfying $AD\ne BC$. Therefore, by the first fundamental theorem, $T(r,F)=T(r,G)+S(r,F)$.

Next, we consider three cases.

Case 1. $AC\ne 0$; from (76), we get

$$F-\frac{A}{C}=\frac{B-AD/C}{CG+D}. \tag{77}$$

By the second fundamental theorem and (69), we have

$$\begin{aligned}T(r,F)&\le\overline{N}\left(r,\frac{1}{F-A/C}\right)+\overline{N}\left(r,\frac{1}{F}\right)+S(r,F)\\&=\overline{N}(r,G)+(k+2)T(r,f)+S(r,F)\\&\le(k+2)T(r,f)+S(r,F).\end{aligned} \tag{78}$$

From (73), we obtain $(n-2k-5)T(r,f)\le S(r,f)$, contradicting to $n\ge 5k+13$.

Case 2. $A\ne 0$, and $C=0$. Then, $F\equiv AG+B/D$.

If $B\ne 0$, by the second fundamental theorem and (69), (70), we have

$$\begin{aligned}T(r,F)&\le\overline{N}\left(r,\frac{1}{F-B/D}\right)+\overline{N}\left(r,\frac{1}{F}\right)+S(r,F)\\&=\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F}\right)+S(r,F)\\&\le(k+2)T(r,f)+(k+2)T(r,g)+S(r,F).\end{aligned} \tag{79}$$

Similarly,

$$T(r,G)\le(k+2)T(r,f)+(k+2)T(r,g)+S(r,G). \tag{80}$$

From (73), (74), (79), and (80), we get

$$(n-3k-7)[T(r,f)+T(r,g)]\le S(r,F)+S(r,G), \tag{81}$$

which is a contradiction to $n\ge 5k+13$.

If $B=0$, then $F\equiv AG/D$. If $A/D=1$, then $F\equiv G$; that is, $[f^n\Delta_c f]^{(k)}\equiv[g^n\Delta_c g]^{(k)}$; using the argument in (54) and noting that $n\ge 5k+13$, we obtain $f(z)=tg(z)$, where t is a constant and $t^{n+1}=1$. If $A/D\ne 1$, by the condition that F and G share 1 IM, then $F\ne 1$ and $G\ne 1$. we obtain then $F\ne 1$ and $F\ne A/D$. By the second fundamental theorem, we have

$$T(r,F)\le\overline{N}\left(\frac{1}{F-1}\right)+\overline{N}\left(\frac{1}{F-A/D}\right)+S(r,F)\le S(r,F), \tag{82}$$

which is impossible.

Case 3. $A = 0$, and $C \neq 0$. Then, $F \equiv B/(CG + D)$.

If $D \neq 0$, by the second fundamental theorem and (69), (70), we have

$$\begin{aligned} T(r,F) &\leq \overline{N}\left(r, \frac{1}{F - B/D}\right) + \overline{N}\left(r, \frac{1}{F}\right) + S(r,F) \\ &= \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{F}\right) + S(r,F) \\ &\leq (k+2)\,T(r,f) + (k+2)\,T(r,g) + S(r,F). \end{aligned} \tag{83}$$

Similarly,

$$T(r,G) \leq (k+2)\,T(r,f) + (k+2)\,T(r,g) + S(r,G). \tag{84}$$

From (73), (74), (83), and (84), we get

$$(n - 3k - 7)\left[T(r,f) + T(r,g)\right] \leq S(r,F) + S(r,G), \tag{85}$$

which is a contradiction to $n \geq 5k + 13$.

If $D = 0$, then $F \equiv B/CG$. If $B/C = 1$, then $F \cdot G \equiv 1$; using the argument in (48) in Theorem 5 and noting that $n \geq 5k + 13$, we get a contradiction. If $B/C \neq 1$, by the condition that F and G share 1 IM, we obtain $F \neq 1$ and $F \neq B/C$. By the second fundamental theorem, we have

$$T(r,F) \leq \overline{N}\left(\frac{1}{F-1}\right) + \overline{N}\left(\frac{1}{F - B/C}\right) + S(r,F) \leq S(r,F), \tag{86}$$

which is impossible.

The proof of Theorem 6 is complete.

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