Research Article

Boundedness of Solutions for a Class of Sublinear Reversible Oscillators with Periodic Forcing

Tingting Zhang and Jianguo Si

School of Mathematics, Shandong University, Jinan, Shandong 250100, China

Correspondence should be addressed to Jianguo Si; sjgmath@yahoo.com.cn

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We study the boundedness of all solutions for the following differential equation

\[ \ddot{x} + f(x)\dot{x} + (B + \varepsilon e(t))|x|^{\alpha-1}x = p(t), \]

where \( f(x) \) and \( p(t) \) are odd functions, \( e(t) \) is an even function, \( e(t) \) and \( p(t) \) are smooth 1-periodic functions, \( B \) is a nonzero constant, and \( \varepsilon \) is a small parameter. A sufficient and necessary condition for the boundedness of all solutions of the above equation is established. Moreover, the existence of Aubry-Mather sets is obtained as well.

1. Introduction

It is well known that the longtime behavior for periodically forced planar systems can be very intricate. For example, there are equations having unbounded solutions but with infinitely many zeros and with nearby unbounded solutions having randomly prescribed number of zeros and also periodic solutions; see [1]. In contrast to such unbounded phenomenon Littlewood [2] suggested to study the boundedness of all the solutions of the following differential equation:

\[ \ddot{x} + g(x) = h(t) \]  

in the following two cases:

(i) superlinear case: \( g(x)/x \rightarrow +\infty \) as \( x \rightarrow \pm \infty \);

(ii) sublinear case: \( \text{sgn}(x)\cdot g(x) \rightarrow +\infty \) and \( g(x)/x \rightarrow 0 \) as \( x \rightarrow \pm \infty \). Later, one calls this subject as Littlewood boundedness problem.

The first result in superlinear case is obtained by Morris [3], who showed that all solutions of

\[ \ddot{x} + 2x^3 = e(t) \]  

are bounded, where \( e(t) \in C^0 \). Later, a series results in superlinear case were obtained by several authors, see [4–13] and references therein. However, in general, it is harder to study the Lagrange stability of sublinear systems since smoothness of sublinear term is insufficient. There are only a few works in sublinear case so far. In 1999, Küpper and You [14] proved the first result in the study of the equation

\[ \ddot{x} + |x|^{\alpha-1}x = p(t), \]  

where \( 0 < \alpha < 1 \) and \( p(t) \in C^0(T) \). Later, Liu [15] proved the same result for more general equation

\[ \ddot{x} + g(x) = e(t), \]  

where \( g(x) \in C^6 \) satisfying the sublinear condition (ii) and some inequalities, and \( e(t) \in C^5(T) \). In 2004, Ortega and Verzini [16] studied the boundedness of (4) in a special case with the variational method. In 2009, Wang [17] gave a sufficient and necessary condition for the boundedness of all solutions for sublinear equation

\[ \ddot{x} + e(t)|x|^{\alpha-1}x = p(t), \]  

where \( e(t), p(t) \in C^5(T) \).

As is widely known, there is a deep similarity between reversible and Hamiltonian dynamics. Many fundamental results of the Hamiltonian systems possess reversible counterparts. On boundedness problem for sublinear reversible systems, the first results were obtained by Li [18], later, Yang [19], in the study of a sublinear reversible systems

\[ \ddot{x} + f(x)\dot{x} + |x|^{\alpha-1}x = e(t). \]
Recently, Wang [20] gave a sufficient and necessary condition for the boundedness of all solutions of the differential equation

\[ \dot{x} + f(x)g(x) + y|x|^{\alpha-1}x = p(t) \]  

(7)

with \( 0 < \alpha < 1, y \neq 0 \).

By the discussions about the sublinear Hamiltonian equation (1.3) in [17] motivations, we will study the boundedness of all solutions for a sublinear reversible system like

\[ \ddot{x} + f(x)\dot{x} + (B + \varepsilon e(t))|x|^{\alpha-1}x = p(t), \]

(8)

where \( B \neq 0 \) and \( 0 < \alpha < 1 \). Furthermore, we also show that (8) has solutions of Mather type. The results obtained in [18–20] can be regarded as corollary of result of this paper.

**Remark 1.** Using the method of this paper we also can consider the more general equation

\[ \dot{x} + f(x)g(x) + (B + \varepsilon e(t))|x|^{\alpha-1}x = p(t) \]

(9)

provided of adding suitable conditions for \( g(x) \). For convenience, we only consider the case \( g(x) \equiv x \).

**Remark 2.** Adding the perturbation term \( \varepsilon e(t)|x|^{\alpha-1}x \) will lead to a new difficulty for estimating \( |S(\Theta T_0)|^{\alpha-1}C(\Theta T_0) \) appeared in (86). Fortunately, we can easily verify that \( \int_0^1 |S(\Theta T_0)|^{\alpha-1}C(\Theta T_0)d\theta = \int_0^1 |S(\Theta T_0)|^{\alpha-1}C(\Theta T_0)d\theta \) is bounded by a constant (see in the proof of Lemma 12).

Throughout this paper, we denote two universal positive constants without regarding their values by \( c < 1 \) and \( C \geq 1 \), and suppose that the following conditions hold:

- (A1) \( f(x) \in C^4(\mathbb{R}), p(t) \in C^3(\mathbb{T}) \) and \( e(t) \in C^3(\mathbb{T}) \), \( f(x) \) and \( p(t) \) are odd, \( e(t) \) is even, and \( e(t), p(t) \) are both \( 1 \)-periodic functions, \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \);

- (A2) there is some positive constant \( \mu \) such that the inequalities

\[ |x^{i+1}f^{(i)}(x)| \leq C|x|^{\alpha/2-\beta} \]

(10)

are satisfied for \( 0 \leq i \leq 4 \) and all \( |x| \geq \mu \), where \( 0 < \beta < \alpha/2 \).

We decompose \( e(t) \) as \( e(t) = \bar{e} + \tilde{e}(t) \), where \( \bar{e} \) is the average of \( e(t) \) and \( \bar{e} \) (t) has zero mean value. That is \( \bar{e} = \frac{1}{T} \int_0^T e(s)ds \) and \( \tilde{e}(s)ds = 0 \). If we write that \( A = B + \varepsilon \bar{e} \), then it is easy to see that \( A \) and \( B \) have the same sign when \( 0 < \varepsilon < \varepsilon^* \) with \( 0 < \varepsilon^* < |B|/\bar{e} \).

Now we state the main results of this paper.

**Theorem 3.** Assume that \( B \neq 0 \) and (A1)-(A2) hold. Then there exists an \( 0 < \varepsilon^* < \varepsilon^* \) such that for any \( 0 < \varepsilon < \varepsilon^* \), every solution of (8) is bounded if and only if \( B > 0 \).

**Theorem 4.** Under the conditions of Theorem 3, there is an \( \varepsilon_0 > 0 \) such that, for any \( \omega \in (n, n + \varepsilon_0) \), (8) has a solution \((x_\omega(t), x'_\omega(t))\) of Mather type with rotation number \( \omega \). More precisely:

(i) if \( \omega = p/q \) is rational, the solutions \((x_\omega(t + i), x'_\omega(t + i))\), \( 1 \leq i \leq q - 1 \), are periodic solutions of period \( q \); moreover, in this case

\[ \lim \min_{\omega \rightarrow \alpha \in \mathbb{R}} \left( \left| x_\omega(t) \right| + \left| x'_\omega(t) \right| \right) = +\infty; \]

(11)

(ii) if \( \omega \) is irrational, the solution \((x_\omega(t), x'_\omega(t))\) is either a usual quasi-periodic solution or a generalized one.

We recall that a solution is called generalized quasi-periodic if the closed set

\[ \left\{ [x(i), x'(i), i \in \mathbb{Z}] \right\} \]

is a Denjou minimal set.

2. Reversible Systems and Action-Angle Variables

In this section, we will assume that \( B > 0 \) and \( A > 0 \). Firstly, we consider (8) which is equivalent to the following system:

\[ \dot{x} = z + P(t), \]

\[ \dot{z} = -A|x|^{\alpha-1}x - \varepsilon e(t) |x|^{\alpha-1}x - f(x)(z + P(t)), \]

(13)

where \( P(t) = \int_0^t p(s)ds \). Then we can obtain that (13) is reversible with respect to the transformation \((x, z) \mapsto (-x, z)\) by (A1).

**Lemma 5.** There exists a \( G \)-invariant diffeomorphism \((x, y) \mapsto (x, z)\) such that (13) is transformed into the following system:

\[ \dot{x} = y + \varepsilon E(t) |x|^{\alpha-1}x + P(t), \]

\[ \dot{y} = -A|x|^{\alpha-1}x \]

\[ - \left[ \alpha E(t) |x|^{\alpha-1} + f(x) \right] \left[ y + \varepsilon E(t) |x|^{\alpha-1}x + P(t) \right], \]

(14)

where \( E(t) = -\int_0^t \bar{e}(s)ds \).

**Proof.** Introduce a transformation \( \Psi \):

\[ x = x, \quad z = y + U(x, t), \]

(15)

where \( U(x, t) \) will be determined later. Under this transformation, the system (13) is transformed into a new system as follows:

\[ \dot{x} = y + U(x, t) + P(t), \]

\[ \dot{y} = -A|x|^{\alpha-1}x - \varepsilon e(t) |x|^{\alpha-1}x \]

\[ - \left( f(x) + \frac{\partial U(x, t)}{\partial x} \right) \left[ y + U(x, t) + P(t) \right] \]

\[ - \frac{\partial U(x, t)}{\partial t}, \]

(16)
Now, we define the function $U(x, t)$ by
\[ -\varepsilon \frac{\partial}{\partial t} |x|^{\alpha-1} x - \frac{\partial U(x, t)}{\partial t} = 0. \tag{17} \]

Since $\int_0^t \hat{e}(t) dt = 0$, we can obtain $U(x, t) = \varepsilon E(t)|x|^{\alpha-1} x$. Then the new system can be expressed as in (14) by direct computation.

It is easy to know that $U(-x, -t) = U(x, t)$ by (A1), then we can obtain that the transformation $\Psi$ is a $G$-invariant diffeomorphism.

Let us consider the auxiliary system
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -Ax|x|^{\alpha-1} x, \tag{18}
\end{align*}

which is a time-independent Hamiltonian system with Hamiltonian
\[ H_0(x, y) = \frac{y^2}{2} + \frac{A}{\alpha + 1}|x|^{\alpha+1}. \tag{19} \]

It is easy to see that $H_0(x, y) > 0$, $(x, y) \in \mathbb{R}^2 \setminus \{0\}$, $H_0(0, 0) = 0$. Note that each level line $H_0(x, y) = h > 0$ is a closed orbit of system (18), hence, all the solutions of (18) are periodic with period tending to zero as $h$ tends to infinity.

Assume that $(S(t), C(t))$ is the solution of (18) with initial conditions $(S(0), C(0)) = (0, 1)$, and let $T_0 > 0$ be the minimal period. We can find that $S(t)$ and $C(t)$ satisfy

(i) $S(t) \in C^2(R), C(t) \in \mathcal{C}^1(R);$

(ii) $(S(-t), C(-t)) = (-S(t), C(t)), (S(t + T_0), C(t + T_0)) = (S(t), C(t));$

(iii) $\dot{S}(t) = C(t), \dot{C}(t) = -A|S(t)|^{\alpha-1} S(t);$ ($\alpha < 1$);

(iv) $\{1/2\}C^2(t) + (A/\alpha + 1)|S(t)|^{\alpha+1} = 1/2;$

(v) $C(T_0 t) = 0 \Leftrightarrow t \mod (1/2) = 0;$

(vi) $(S(T_0(1/2 - t)), C(T_0(1/2 - t))) = (S(T_0 t), -C(T_0 t));$

(vii) $S(T_0 t) = 0 \Leftrightarrow t \mod (1/2) = 0.$

Then we introduce the transformation
\[ \Phi: \mathbb{R}^2 \times \mathbb{T} \rightarrow \mathbb{R}^2 \setminus \{0\}, \]
\[ (\rho, \varphi) \mapsto (x, y), \tag{20} \]

which is
\begin{align*}
x &= \rho^\beta S(\varphi T_0), \\
y &= \rho^{1-b} C(\varphi T_0), \tag{21}
\end{align*}

where $b = 2/(3 + \alpha)$. It is easy to see that $1/2 < b < 2/3$ by $0 < \alpha < 1$. Since $(S(-t), C(-t)) = (-S(t), C(t))$, this transformation is invariant with respect to the involutions $(\rho, \varphi) \mapsto (\rho, -\varphi)$ and $(x, y) \mapsto (-x, y)$, and we can find that the mapping $\Phi$ is a generalized canonical transformation by (iv). In fact, $\frac{\partial (x, y)}{\partial (\rho, \varphi)} = \frac{\partial h_0}{\partial \rho} = \frac{1}{T_0} \cdot \rho^{-b},$ $(23)$

where $d = ((1 - b) T_0)^{-1}$.

Under the transformation $\Phi$, the system (18) is transformed into the simpler form
\begin{align*}
\frac{\rho}{\rho} &= -\frac{\partial h_0}{\partial \rho} = 0, \quad \varphi = \frac{\partial h_0}{\partial \varphi} = \frac{1}{T_0} \cdot \rho^{-b}, \tag{23}
\end{align*}

where $h_0(\rho) = ((2 - 2b) T_0)^{-1} \cdot \rho^{2(1-b)}$.

The original system (13) is transformed into the system
\begin{align*}
\frac{d\rho}{dt} &= l_1(\rho, \varphi) + l_2(\rho, \varphi, t) + \varepsilon l_3(\rho, \varphi, t) + \gamma l_4(\rho, \varphi, t), \tag{24}
\end{align*}

where
\begin{align*}
l_1(\rho, \varphi) &= -d_T P f (\rho^2 S(\varphi T_0)) C^2(\varphi T_0)
&= -d_T P f (\rho^2 S(\varphi T_0)) C^2(\varphi T_0), \tag{21}
\end{align*}

$\rho = \rho^2 P f (\rho^2 S(\varphi T_0)) C(\varphi T_0)$.

\[ d_T = (1/2) \rho^2 S(\varphi T_0) C(\varphi T_0), \tag{21} \]

$\rho = \rho^2 P f (\rho^2 S(\varphi T_0)) C(\varphi T_0)$.

\[ d_T = (1/2) \rho^2 S(\varphi T_0) C(\varphi T_0), \tag{21} \]

$\rho = \rho^2 P f (\rho^2 S(\varphi T_0)) C(\varphi T_0)$.
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\[ h_1(\rho, \varphi) = dbf(\rho^b S(\varphi T_0)) C(\varphi T_0) S(\varphi T_0) = dx_\rho f(x) y, \]
\[ h_2(\rho, \varphi, t) = dbe^{1-2b} f(\rho^b S(\varphi T_0)) S(\varphi T_0)^{a+1} E(t) \]
\[ + adb^2 \rho^{3-b} |S(\varphi T_0)|^{2a} E^2(t) \]
\[ + d (1-b) \rho^b C(\varphi T_0) P(t) \]
\[ + d b^{b-1} f(\rho^b S(\varphi T_0)) S(\varphi T_0) P(t) \]
\[ + adb^{b-1} S(\varphi T_0)^{a+1} S(\varphi T_0) E(t) P(t) \]
\[ = dx_\rho |x|^{-1} x f(x) E(t) + adb^2 x_\rho |x|^{2b-2} x E^2(t) \]
\[ + dy_\rho P(t) + dx_\rho f(x) P(t) + adx_\rho |x|^{-1} E(t) P(t), \]
\[ h_3(\rho, \varphi, t) = d (1-b + ab) \rho^{2-b} |S(\varphi T_0)|^{a-c} S(\varphi T_0) C(\varphi T_0) E(t) \]
\[ = dy_\rho |x|^{a-2} x E(t) + adx_\rho |x|^{a-1} E(t). \]  

(25)

Let
\[ L_2(\rho, \varphi, t) = l_2(\rho, \varphi, t) + el_3(\rho, \varphi, t) \]
\[ + \alpha T_0 |S(\varphi T_0)|^{a+1} C(\varphi T_0) el_4(\rho, \varphi, t), \]
\[ H_2(\rho, \varphi, t) = h_2(\rho, \varphi, t) + eh_3(\rho, \varphi, t). \]  

(26)

Clearly, \( x \) is odd in \( \varphi \) and \( y \) is even in \( \varphi \) by the definitions of \( S(t) \) and \( C(t) \). Thus, by the evenness of \( P(t) \) and the oddness of \( f(x) \) and \( E(t) \) we have
\[ l_1(\rho, -\varphi) = -l_1(\rho, \varphi), \quad l_2(\rho, -\varphi, -t) = -l_2(\rho, \varphi, t), \]
\[ h_1(\rho, -\varphi) = h_1(\rho, \varphi), \quad h_2(\rho, -\varphi, -t) = h_2(\rho, \varphi, t). \]  

(27)

This implies that system (24) is reversible with respect to the involutions \((\rho, \varphi) \mapsto (\rho, -\varphi)\).

Lemma 6. For \( 0 \leq k + m \leq 4 \), the following inequalities hold:

1. \( |\partial^k / \partial p^\rho l_1(\rho, \varphi)| \leq C\rho^{-k+2-\gamma-(5/2)\beta} \rho^a \rho^{k+1}, \)
2. \( |\partial^{k+m} / \partial p^\rho \partial t^m l_2(\rho, \varphi, t)| \leq C\rho^{-k+1}, \)
3. \( |\partial^{k+m} / \partial p^\rho \partial t^m l_3(\rho, \varphi, t)| \leq C\rho^{-k+1-3-\beta}, \)
4. \( |\partial^{k+m} / \partial p^\rho \partial t^m l_4(\rho, \varphi, t)| \leq C\rho^{-k+1-3+\beta}, \)
5. \( |\partial^k / \partial p^\rho h_3(\rho, \varphi)| \leq C\rho^{-k+1-\gamma-(5/2)\beta}, \)
6. \( |\partial^{k+m} / \partial p^\rho \partial t^m h_3(\rho, \varphi, t)| \leq C\rho^{-k+1+\tau}, \)
7. \( |\partial^{k+m} / \partial p^\rho \partial t^m h_4(\rho, \varphi, t)| \leq C\rho^{-k+2}, \)

where \( \gamma = \beta b, a = \max(3-(9/2)b-\gamma, 1-b), \) and \( \tau = \max(3-6b, -b). \)

Proof. (1) It is easy to know that \( (\partial^k / \partial p^\rho l_1(\rho, \varphi) \) is a sum of the form
\[ \frac{\partial^i x_\rho}{\partial p^\rho} \frac{\partial^j y}{\partial p^\rho} \quad i + 2 + 3 = k, \]  

where \( 0 \leq i, j, n \leq k. \) Meanwhile, \( \partial^i (x_\rho) / \partial p^\rho \) is a sum of the form
\[ f^{(s)}(x_\rho) \frac{\partial^i x_\rho}{\partial p^\rho} \frac{\partial^j y}{\partial p^\rho}, \quad 0 \leq s \leq i, \]  

(29)

Hence, we obtain
\[ \left| \frac{\partial^k}{\partial p^\rho} l_1(\rho, \varphi) \right| \leq C \rho^{-i} |x| \cdot \rho^{-i} |f(x) \cdot \rho^{i-1} y| \]
\[ \leq C \rho^{k-1} |x| \cdot |f(x) \cdot \rho^{i-1} y| \leq C \rho^{k-1} |x|^{\alpha-\beta} |y| \]
\[ \leq C \rho^{k-1-\gamma-(5/2)\beta} \]  

(30)

by the assumptions on \( f(x) \) and the definitions of \( x(\rho, \varphi) \) and \( y(\rho, \varphi) \).

(2) From the expression of \( l_2(\rho, \varphi, t) \), we have
\[ \left| \frac{\partial^{k+m}}{\partial p^\rho \partial t^m} \left( -dx_\rho \frac{\partial^k}{\partial p^\rho} \left( |x|^{a-1} \varphi(\rho, \varphi, t) \right) \right) \right| \]
\[ \leq C \left| \frac{\partial^k}{\partial p^\rho} \left( -dx_\rho \frac{\partial^k}{\partial p^\rho} \right) \right| \left( |x|^m \varphi(\rho, \varphi, t) \right) \]
\[ \leq C \rho^{k-1-\gamma-\beta} \]  

(31)

We can find that
\[ \left| \frac{\partial^{k+m}}{\partial p^\rho \partial t^m} l_2(\rho, \varphi, t) \right| \leq C \rho^{k+a}, \]  

(32)

where \( a = \max(3-(9b)/2-\gamma, 1-b). \)
(3) From the expression of $l_3(\rho, \phi, t)$, we have
\[ \frac{\partial^{k+m} (dy_\rho | x'|^{\alpha-1} x E(t))}{\partial \rho^k \partial t^m} \leq C \frac{\partial^k (dy_\rho | x'|^{\alpha-1} x)}{\partial \rho^k} \left| \frac{d^m (E(t))}{dt^m} \right| \leq C \rho^{-k^3-4b}. \]  
(4) From the expression of $l_4(\rho, \phi, t)$, we can obtain that
\[ \frac{\partial^{k+m} (\partial \rho | x'|^{\alpha-1} x E(1))}{\partial \rho^k \partial t^m} \leq C \frac{\partial^k (\partial \rho | x'|^{\alpha-1} x)}{\partial \rho^k} \left| \frac{d^m (E(1))}{dt^m} \right| \leq C \rho^{-k^3-4b}. \]
(5) From the definition of $h_1(\rho, \phi)$, we have
\[ \frac{\partial^k h_1(\rho, \phi)}{\partial \rho^k} \leq C \rho^{-1-\gamma} \left| x \cdot f(x) \cdot \phi \right| \leq C \rho^{-k+1} |x|^{\alpha/2-\beta} \left| \phi \right| \leq C \rho^{-k+1-\gamma-(5/2)b}. \]
(6) From the definition of $h_2(\rho, \phi, t)$, we can obtain
\[ \frac{\partial^{k+m} (a d x_\rho | x'|^{\alpha-1} x f(x) E(t))}{\partial \rho^k \partial t^m} \leq C \frac{\partial^k (a d x_\rho | x'|^{\alpha-1} x f(x))}{\partial \rho^k} \left| \frac{d^m (E(t))}{dt^m} \right| \leq C \rho^{-k+3-4b}. \]

Hence, we can know that
\[ \frac{\partial^{k+m} \partial \rho^k t^m h_2(\rho, \phi, t)}{\partial \rho^k \partial t^m} \leq C \rho^{-k^3-4b}. \]

(7) From the expression of $h_3(\rho, \phi, t)$, we have
\[ \frac{\partial^{k+m} h_3(\rho, \phi, t)}{\partial \rho^k \partial t^m} \leq C \frac{\partial^k (h_3(\rho, \phi, t))}{\partial \rho^k} \left| \frac{d^m (E(t))}{dt^m} \right| \leq C \rho^{-k^3-4b}. \]

For $\lambda_0 > 0$, we define the domain
\[ \mathcal{A}_{\lambda_0} = \left\{ (\lambda, \phi, t) : \lambda \geq \lambda_0, (\phi, t) \in \mathbb{T} \right\}. \]

Lemma 7. There exists a $G$-invariant diffeomorphism $\Psi_1$: $\rho = I + U_1 (I, \theta), \phi = \theta$ such that $\mathcal{A}_I \subset \Psi_1(\mathcal{A}_I) \subset \mathcal{A}_I$ for some $I < I_0 < I_1$. Under this transformation, (24) is transformed into the system
\[
\begin{align*}
\frac{dI}{dt} & = \bar{T}_1 (I, \theta) + \bar{T}_2 (I, \theta, t) + \bar{I}_3 (I, \theta, t) \\
\frac{d\theta}{dt} & = \bar{h}_0 (I, \theta) + \bar{\theta}_1 (I, \theta) + \bar{\theta}_2 (I, \theta, t) + \alpha \bar{T}_0 |S(\theta T_0)|^{\alpha-1} C(\theta T_0) \bar{\theta}_3 (I, \theta, t).
\end{align*}
\]
where

\[
\tilde{I}_1(I, \theta) = \frac{\partial V_1(\rho, \varphi)}{\partial \rho} \cdot l_1(\rho, \varphi) + \frac{\partial V_1(\rho, \varphi)}{\partial \varphi} \cdot h_1(\rho, \varphi),
\]

\[
\tilde{I}_2(I, \theta, t) = l_2(\rho, \varphi, t)
\]

+ \frac{\partial V_1(\rho, \varphi)}{\partial \rho} \cdot (l_2(\rho, \varphi, t) + e l_3(\rho, \varphi, t))
\]

+ \frac{\partial V_1(\rho, \varphi)}{\partial \varphi} \cdot (h_2(\rho, \varphi, t) + \varepsilon h_3(\rho, \varphi, t))
\]

+ \varepsilon(l_3(\rho, \varphi, t) - l_3(I, \theta, t)),
\]

\[
\tilde{I}_3(I, \theta, t) = l_3(I, \theta, t),
\]

\[
\tilde{I}_4(I, \theta, t) = l_4(\rho, \varphi, t) + \frac{\partial V_1(\rho, \varphi)}{\partial \rho} \cdot l_4(\rho, \varphi, t),
\]

\[
\tilde{I}_5(I, \theta) = h_0'(\rho) - h_0'(I) + h_1(\rho, \varphi),
\]

\[
\tilde{I}_6(I, \theta, t) = h_2(\rho, \varphi, t) + \varepsilon(h_3(\rho, \varphi, t) - h_3(I, \theta, t)),
\]

\[
\tilde{I}_7(I, \theta, t) = h_3(I, \theta, t),
\]

\[
\text{with}
\]

\[
V_1(\rho, \varphi) = -\int_0^\varphi l_1(\rho, s) \frac{h_0'(\rho)}{h_0'(\rho)} \, ds.
\]

**Proof.** Define a transformation \( \Phi_1 \) by

\[
\Phi_1 : I = \rho + V_1(\rho, \varphi), \quad \theta = \varphi.
\]

By

\[
l_1(\rho, -\varphi) = -l_1(\rho, \varphi),
\]

\[
\left| \frac{\partial^k l_1(\rho, \varphi)}{\partial \rho^k} \right| \leq C \rho^{-k+1-\gamma-b/2}, \quad 0 \leq k \leq 4,
\]

we get

\[
V_1(\rho, \varphi) = V_1(\rho, \varphi),
\]

\[
\left| \frac{\partial^k}{\partial \rho^k} V_1(\rho, \varphi) \right| \leq C \rho^{-k+1-\gamma-b/2}.
\]

Let \( \Psi_1 = \Phi_1^{-1} : \rho = I + U_1(I, \theta), \varphi = \theta \). The system (24) is transformed into (41).

**Lemma 8.** For \( I \) large enough, the following conclusions hold:

(i) \(|\partial^k U_1(I, \theta) / \partial I^k| \leq CI^{-k+1-\gamma-b/2},
\]

(ii) \( U_1(I, -\theta) = U_1(I, \theta). \)

**Proof.** In view of

\[
I = \rho + V_1(\rho, \varphi), \quad \rho = I + U_1(I, \theta),
\]

we obtain

\[
U_1(I, \theta) = -V_1(I + U_1(I, \theta), \theta).
\]

By \(|\partial^k / \partial I^k V_1(\rho, \varphi)| \leq C \rho^{-k+1-\gamma-b/2}, \) we have \(|\partial / \partial \rho V_1(\rho, \varphi)| \leq C \rho^{-\gamma-b/2} \leq 1/2 \) for \( \rho \) large enough. Hence, \( U_1 \) is uniquely determined by the contraction mapping principle. Moreover, \( U_1(\cdot, \cdot) \in C^\infty(\mathbb{R}_+), \) for some \( I_0 > 0, \) as a consequence of the implicit function theorem and

\[
I^{-1-\gamma-b/2} |U_1(I, \theta)| \leq C.
\]

Above all, if \( k = 1, \) from (47) and (49), we get

\[
\left| \frac{\partial U_1}{\partial I} \right| = \left| \frac{\partial V_1}{\partial \rho} \right| \left| \frac{\partial \rho}{\partial I} \right| \leq \sum_{n=0}^{\infty} (C \rho^{-1+1-\gamma-b/2})^n \leq C \cdot \rho^{-1+1-\gamma-b/2}
\]

\[
= C \cdot I^{-1+1-\gamma-b/2} \sum_{n=0}^{\infty} (1 + \frac{U_1}{I})^{-1+1-\gamma-b/2} \leq C \cdot I^{-1+1-\gamma-b/2}.
\]

We note that

\[
\frac{\partial^k U_1(I, \theta)}{\partial I^k} = \frac{\partial^k V_1(I + U_1(I, \theta), \theta)}{\partial I^k},
\]

and the right side hand is sum of the term

\[
\frac{\partial^k V_1}{\partial \rho^k} \cdot \frac{\partial^k (I + U_1)}{\partial I^k} \cdots \frac{\partial^k (I + U_1)}{\partial I^k},
\]

where \( 1 \leq s \leq k, k_1 + \cdots + k_s = k, k_i \geq 1 \) (for \( 1 \leq i \leq s \)). The highest order term in \( U_1 \) is the one with \( s = 1, \) namely, \( (\partial V_1 / \partial \rho) \cdot (\partial^k (I + U_1) / \partial I^k). \) We move the part \( (\partial V_1 / \partial \rho) \cdot (\partial^k (I + U_1) / \partial I^k) \) to the left hand side of (52). Since \(|\partial / \partial \rho V_1(\rho, \varphi)| \leq 1/2 \) for \( \rho \) large enough, this also provides immediately a bound on \( \partial^k U_1(I, \theta) / \partial I^k. \) The rest part \(|\partial^k V_1 / \partial \rho^k \cdot (\partial^k (I + U_1) / \partial I^k)| \leq CI^{-k+1-\gamma-b/2} \).

Now, we proceed inductively by assuming that for \( j \leq k - 1 \) the estimates

\[
\left| \frac{\partial^j U_1(I, \theta)}{\partial I^j} \right| \leq CI^{-j+1-\gamma-b/2}
\]

hold and we wish to conclude that the same estimate holds for \( j = k. \)

Indeed, if \( s \geq 2, \) we have

\[
\left| \frac{\partial^s V_1}{\partial \rho^s} \cdot \frac{\partial^k (I + U_1)}{\partial I^k} \cdots \frac{\partial^k (I + U_1)}{\partial I^k} \right| \leq C \cdot (I + U_1)^{-s+1-\gamma-b/2} \cdot I^{-k+1} \cdots I^{-k+1} \leq C \cdot I^{-k+1-\gamma-b/2}
\]

by

\[
\left| \frac{\partial^j (I + U_1(I, \theta))}{\partial I^j} \right| \leq CI^{-j+1}, \quad 1 \leq j \leq k - 1.
\]

This proves (i) of Lemma 8.
Now we check (ii). In fact, since
\[ U_1(I, \theta) = - V_1(I + U_1(I, \theta), \theta), \]
we have
\[ |U_1(I, \theta) - U_1(I, -\theta)| \leq \sup_{t \in [0, T]} \left| \frac{\partial V_1(I, \theta)}{\partial \theta} \right| \left| U_1(I, \theta) - U_1(I, -\theta) \right|. \]  
(58)

From (47), we have \( |(\partial/\partial \rho)V_1(\rho, \phi)| \leq 1/2 \) for \( I \geq I_0 \) sufficiently large and therefore we obtain \( U_1(I, \theta) = U_1(I, -\theta). \)

By the estimates in Lemma 6, we can prove the following inequalities.

**Lemma 9.** For \( 0 \leq k + m \leq 4 \), the following inequalities hold:

1. \( |\frac{\partial}{\partial \rho} \frac{\partial}{\partial I} \tilde{h}_1(I, \theta)| \leq CI^{-k-3-2b} \).
2. \( |\frac{\partial^{k+m} \frac{\partial}{\partial \rho} \frac{\partial}{\partial t} \tilde{h}_1(I, \theta, t)| \leq CI^{-k+a} \).
3. \( |\frac{\partial^{k+m} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \phi} \tilde{h}_1(I, \theta, t)| \leq CI^{-k+3-4b} \).
4. \( |\frac{\partial^{k+m} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} \tilde{h}_1(I, \theta, t)| \leq CI^{-k+3-4b} \).
5. \( |\frac{\partial}{\partial \rho} \frac{\partial}{\partial t} \tilde{h}_1(I, \theta)| \leq CI^{-k+1-\gamma-(9/2)b} \).
6. \( |\frac{\partial^{k+m} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \phi} \tilde{h}_2(I, \theta, t)| \leq CI^{-k+7} \).
7. \( |\frac{\partial^{k+m} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} \tilde{h}_2(I, \theta, t)| \leq CI^{-k+2-4b} \).

Proof. (1) From the estimates (1) and (5) of Lemmas 6 and 8, it follows that
\[
|\frac{\partial}{\partial \rho} \frac{\partial}{\partial t} \tilde{h}_1(I, \theta)| \\
\leq C \sum_{l_1, l_2, l_3 = k} |\frac{\partial^{l_1+l_2} V_1(\rho, \phi)}{\partial \rho^{l_1} \partial \rho^{l_2}}| + C \sum_{l_1, l_2, l_3 = k} |\frac{\partial^{l_1+l_2} h_1(\rho, \phi)}{\partial \rho^{l_1} \partial \rho^{l_2}}| \\
\leq C \sum_{l_1, l_2, l_3 = k} \left( \sum_{l_1 = k, l_2, l_3 = k} |\frac{\partial^{l_1+l_2} V_1(\rho, \phi)}{\partial \rho^{l_1} \partial \rho^{l_2}}| \right) \\
\times |\frac{\partial^l l_1 l_2}{\partial \rho^{l_1} \partial \rho^{l_2}}| \\
+ C \sum_{l_1, l_2, l_3 = k} \left( \sum_{l_1 = k, l_2, l_3 = k} |\frac{\partial^{l_1+l_2} h_1(\rho, \phi)}{\partial \rho^{l_1} \partial \rho^{l_2}}| \right) \\
\times |\frac{\partial^l l_1 l_2}{\partial \rho^{l_1} \partial \rho^{l_2}}| \\
\leq C \rho^{-k+2-2y+3b} \leq CI^{-k+2-2y+3b}. \]  
(59)

(2) Since
\[
\tilde{I}_2(I, \theta, t) = I_2(\rho, \phi, t) \\
+ \frac{\partial V_1(\rho, \phi)}{\partial \rho} \cdot \left( \tilde{I}_2(\rho, \phi, t, \phi) + \epsilon \tilde{I}_2(\rho, \phi, t) \right) \\
+ \frac{\partial V_1(\rho, \phi)}{\partial \phi} \cdot \left( I_2(\rho, \phi, t) + \epsilon h_2(\rho, \phi, t) \right) \\
+ \epsilon (I_2(\rho, \phi, t) - I_2(I, \theta, t)),
\]
we can prove that
\[
|\frac{\partial^{k+m} \frac{\partial}{\partial \rho} \frac{\partial}{\partial t} \tilde{I}_2(\rho, \phi, t)| \leq CI^{-k+a}, \]  
(60)

Their proofs are similar to the proofs in (1).

Next, we check the last part of \( \tilde{I}_2(I, \theta, t) \). We get
\[
\left| \frac{\partial^{k+m} \frac{\partial}{\partial \rho} \frac{\partial}{\partial t} \tilde{I}_2(\rho, \phi, t) \right| \\
= \left| \frac{\partial^{k+m} \frac{\partial}{\partial \rho} \frac{\partial}{\partial t} \tilde{I}_2(\rho, \phi, t)|_0 \cdot \tilde{U}_1(I, \theta, t) \right| \\
\leq \int_0^1 \sum_{l_1 = k} \left| \frac{\partial^{l_1+m} \frac{\partial}{\partial \rho} \frac{\partial}{\partial t} \tilde{I}_2(\rho, \phi, t)|_0 \cdot \frac{\partial}{\partial \rho} \frac{\partial}{\partial t} \tilde{U}_1(I, \theta, t) \right| ds \\
\leq CI^{-k+3-\gamma-(9/2)b} \leq CI^{-k+a},
\]
(62)

by the estimate in Lemma 6 and the definition of \( a \).

(3) It is clearly by (3) in Lemma 6.

(4) It is clearly by (4) in Lemmas 6 and 8.

(5) We have that
\[
\tilde{h}_1(I, \theta) = h'_0(\rho) - h'_0(I) + h_1(\rho, \phi),
\]
\[
\left| \frac{\partial^{k+m} \frac{\partial}{\partial \rho} \frac{\partial}{\partial t} \tilde{h}_1(I, \theta)|_0 \cdot \frac{\partial}{\partial \rho} \frac{\partial}{\partial t} \tilde{U}_1(I, \theta, t) \right| \\
\leq \int_0^1 \left| \frac{\partial^{l_1+m} \frac{\partial}{\partial \rho} \frac{\partial}{\partial t} \tilde{U}_1(I, \theta, t)|_0 \cdot \frac{\partial}{\partial \rho} \frac{\partial}{\partial t} \tilde{U}_1(I, \theta, t) \right| ds \]
\[
\leq CI^{-k+1-\gamma-(9/2)b} \leq CI^{-k+1-5b/2}.
\]  
(63)
From the last inequalities and (5) in Lemma 6, we obtain
\[ \left| \frac{\partial^k}{\partial t^k} \tilde{h}_1 (I, \theta) \right| \leq CI^{-k+1-\gamma-(5/2)b}. \] (64)

(6) Since
\[ \tilde{h}_2 (I, \theta, t) = h_2 (\rho, \varphi, t) + \varepsilon \left( h_3 (\rho, \varphi, t) - h_3 (I, \theta, t) \right), \]
\[ \left| \frac{\partial^{k+m}}{\partial t^k \partial \theta^m} h_2 (\rho, \varphi, t) \right| \leq CI^{-k+\tau}, \] (65)
we just have to prove that
\[ \left| \frac{\partial^{k+m}}{\partial t^k \partial \theta^m} (h_3 (\rho, \varphi, t) - h_3 (I, \theta, t)) \right| \leq CI^{-k+\tau}. \] (66)

In fact,
\[ \left| \frac{\partial^{k+m}}{\partial t^k \partial \theta^m} (h_3 (\rho, \varphi, t) - h_3 (I, \theta, t)) \right| \]
\[ \leq \left| \frac{\partial^{k+m}}{\partial t^k \partial \theta^m} \left( \int_0^1 \frac{\partial h_3}{\partial \theta} (I + sU_1 (I, \theta), \theta) \cdot U_1 (I, \theta) \, ds \right) \right| \]
\[ \leq \left| \int_0^1 \sum_{i+j=k} \left| \frac{\partial^{i+m}}{\partial t^i \partial \theta^m} (\frac{\partial h_3 (I + sU_1, \theta, t)}{\partial \theta}) \right| \left| \frac{\partial^{i+m}}{\partial t^i \partial \theta^m} U_1 \right| \, ds \right| \]
\[ \leq CI^{-k+2-\gamma-(9/2)b} \leq CI^{-k+\tau}, \] (67)
so we have proved (6).

(7) We have
\[ \left| \frac{\partial^{k+m}}{\partial t^k \partial \theta^m} \tilde{h}_3 (I, \theta, t) \right| \leq CI^{-k+2-4b}, \] (68)
by (7) in Lemma 6.

\[ \eta_3 (I, \theta, t) \]
\[ = - \left( \tilde{h}_3 (I, \theta, t) \right) \times \left( \left( h_0 (I) + \tilde{h}_1 (I, \theta) \right) \right) \times \left( \left( h_0 (I) + \tilde{h}_1 (I, \theta) + \tilde{h}_2 (I, \theta, t) + \varepsilon \tilde{h}_3 (I, \theta, t) \right) \right)^{-1}, \]

3. The Proof of Boundedness

In this section, all the solutions of (8) which are bounded will be proved via the KAM theory for reversible systems developed by Sevryuk [21] or Moser [22, 23] if \( B > 0 \).

We define the functions \( \eta_0, \eta_1, \eta_2, \eta_3, \xi_1, \xi_2, \) and \( \xi_3 \) as
\[ \eta_0 (I) = \frac{1}{h_0 (I)}, \]
\[ \eta_1 (I, \theta) = - \frac{\tilde{h}_1 (I, \theta)}{h_0 (I) \left( h_0 (I) + \tilde{h}_1 (I, \theta) \right)}, \]
\[ \eta_2 (I, \theta, t) \]
\[ = - \left( \tilde{h}_2 (I, \theta, t) \right) \times \left( \left( h_0 (I) + \tilde{h}_1 (I, \theta) \right) \right) \times \left( \left( h_0 (I) + \tilde{h}_1 (I, \theta) + \tilde{h}_2 (I, \theta, t) + \varepsilon \tilde{h}_3 (I, \theta, t) \right) \right)^{-1}, \]
\[ \eta_3 (I, \theta, t) \]
\[ = - \left( \tilde{h}_3 (I, \theta, t) \right) \times \left( \left( h_0 (I) + \tilde{h}_1 (I, \theta) + \tilde{h}_2 (I, \theta, t) + \varepsilon \tilde{h}_3 (I, \theta, t) \right) \right)^{-1}. \]

Then system (41) is equivalent to the following system:
\[ \frac{dt}{d\theta} = \eta_0 (I) + \eta_1 (I, \theta) + \eta_2 (I, \theta, t) + \varepsilon \eta_3 (I, \theta, t), \]
\[ \frac{dI}{d\theta} = \xi_1 (I, \theta, t) + \varepsilon \xi_2 (I, \theta, t) \]
\[ + \alpha T_0 |\theta T_0|^{\gamma-1} C (\theta T_0) \varepsilon \xi_3 (I, \theta, t). \] (70)

In addition, one can verify that system (70) is reversible with respect to involution \( G : (t, I) \mapsto (-t, I) \).

Then some estimates on the functions \( \eta_i (i = 0, 1, 2, 3) \) and \( \xi_i (i = 1, 2, 3) \) are given.

**Lemma 10.** The following inequalities hold:

1. \( CI^{2b-1} \leq |\eta_0 (I)| \leq CI^{2b-1}, \)
2. \( |(\partial^k / \partial t^k) \eta_1 (I, \theta)| \leq CI^{k+1-\gamma-3b/2}, \)
3. \( |(\partial^k / \partial t^k \partial \theta^m) \eta_2 (I, \theta, t)| \leq CI^{k+m+4b-2}, \)
4. \( |(\partial^k / \partial t^k \partial \theta^m) \eta_3 (I, \theta, t)| \leq CI^{k}, \)
5. \( |(\partial^k / \partial t^k \partial \theta^m) \xi_1 (I, \theta, t)| \leq CI^{k+m+2b-1}, \)
6. \( |(\partial^k / \partial t^k \partial \theta^m) \xi_2 (I, \theta, t)| \leq CI^{k+2-2b}, \)
7. \( |(\partial^k / \partial t^k \partial \theta^m) \xi_3 (I, \theta, t)| \leq CI^{k+2-2b}, \)

**Proof.** (1) It is clear.
(2) Note that \( 1 - 2b > 1 - \gamma - 2b/5 \), and
\[ \left| \tilde{h}_1 (I, \theta) \right| \leq CI^{1-\gamma-(5/2)b}, \] (71)
it follows that
\[
\left\| h_0'(I) + \tilde{h}_1(I, \theta) \right\| \geq \left\| h_0'(I) \right\| - \left\| \tilde{h}_1(I, \theta) \right\| \\
\geq \frac{1}{t_0} I^{1-2b} - CI^{-\gamma - (5/2)b} \\
\geq c I^{1-2b}
\]
as \( I \gg 1 \).

Moreover, we also have
\[
\left\| \frac{\partial}{\partial t} \left( h_0'(I) + \tilde{h}_1(I, \theta) \right) \right\| \leq CI^{1-1-2b} + CI^{1-1-\gamma (2/5)b} \\
\leq CI^{1-2b}.
\]
(73)

So
\[
\left\| \frac{\partial^i}{\partial I^i} \left( \frac{1}{h_0'(I) + \tilde{h}_1(I, \theta)} \right) \right\| \leq C \sum_{l_1 + \cdots + l_k = i} \left\| \frac{(-1)^i s!}{(h_0'(I) + \tilde{h}_1(I, \theta))^{i+1}} \right\| \\
\times \left\| \frac{\partial^i}{\partial I^i} \left( h_0'(I) + \tilde{h}_1(I, \theta) \right) \right\| \\
\leq C \sum_{l_1 + \cdots + l_k = i} I^{(2b-1)(i+1)} I^{-i-1-2b} \\
\leq CI^{1-2b}.
\]
(74)

From (72) and (74), it is easy to see that
\[
\left\| \frac{\partial^k}{\partial I^k} \tilde{h}_1(I, \theta) \right\| \leq C \sum_{i_1 + i_2 + i_3 = k} \left\| \frac{\partial^i}{\partial I^i} \tilde{h}_1(I, \theta) \right\| \\
\leq C \sum_{i_1 + i_2 + i_3 = k} \left\| \frac{\partial^i}{\partial I^i} \tilde{h}_1(I, \theta) \right\| \\
\times \left\| \frac{\partial^j}{\partial I^j} \left( 1 \right) \right\| \\
\leq C \sum_{i_1 + i_2 + i_3 = k} \left\| \frac{\partial^i}{\partial I^i} \tilde{h}_1(I, \theta) \right\| \\
\leq CI^{k-1-\gamma (3/2)b}.
\]
(75)

(3) We have
\[
\left\| \frac{\partial^{k+m}}{\partial t^m \partial I^{k}} \tilde{h}_2(I, \theta, t) \right\| \leq CI^{-k+r},
\]
(76)

By (72), \( 1 - 2b > \tau (\tau = \max(3 - 6b, -b)) \) and \( 1 - 2b > 2 - 4b \), we have
\[
\left\| h_0'(I) + \tilde{h}_1(I, \theta) + \tilde{h}_2(I, \theta, t) + \tilde{h}_3(I, \theta, t) \right\| \\
\geq \left\| h_0'(I) + \tilde{h}_1(I, \theta) \right\| - \left\| \tilde{h}_2(I, \theta, t) + \tilde{h}_3(I, \theta, t) \right\| \\
\geq \left\| h_0'(I) + \tilde{h}_1(I, \theta) \right\| - \left\| \tilde{h}_2(I, \theta, t) \right\| - \varepsilon \left\| \tilde{h}_3(I, \theta, t) \right\| \\
\geq c I^{1-2b} - CI^{-\gamma} - Ce^{-2b} \geq c I^{1-2b},
\]
(77)

for \( I \gg 1 \).

Let \( h_0'(I) + \tilde{h}_1(I, \theta) + \tilde{h}_2(I, \theta, t) + \tilde{h}_3(I, \theta, t) = H(I, \theta, t) \).

We find that
\[
\left\| \frac{\partial}{\partial t} \left( \frac{1}{H(I, \theta, t)} \right) \right\| \\
\leq C \sum_{i_1 + i_2 + i_3 = i} \left\| \frac{(-1)^i s!}{H(I, \theta, t)^{i+1}} \right\| \\
\times \left\| \frac{\partial^i}{\partial I^i} (H(I, \theta, t)) \right\| \\
\leq C \sum_{i_1 + i_2 + i_3 = i} I^{(2b-1)(i+1)} e^{-i(1-2b)} \leq CI^{(2b-1)(i+1)},
\]
(78)

so
\[
\left\| \frac{\partial^{k}}{\partial t^k \partial I^k} \left( \frac{1}{h_0'(I) + \tilde{h}_1(I, \theta) + \tilde{h}_2(I, \theta, t) + \tilde{h}_3(I, \theta, t)} \right) \right\| \\
\leq C \left\| \frac{\partial^k}{\partial I^k} \left( \sum_{i_1 + i_2 + i_3 = k} \frac{(-1)^i s!}{H(I, \theta, t)^{i+1}} \frac{\partial^i}{\partial I^i} (H(I, \theta, t)) \right) \right\| \\
\leq C \sum_{i_1 + i_2 + i_3 = k} \sum_{j_1 + j_2 + j_3 = i} \left\| \frac{\partial^j}{\partial I^j} \left( 1 \right) \right\| \\
\leq C \sum_{i_1 + i_2 + i_3 = k} I^{(2b-1)(i+1)} \Gamma^{(i_1 + i_2 + i_3)(1-2b)} \leq CI^{-k+r (2b-1)}.
\]
(79)

When \( m = 0 \), the proof of (3) is similar to the proof of (2).
When \( m > 0 \), then
\[
\left| \frac{\partial^{k+m}}{\partial I^k \partial \eta^m} \xi_2 (I, \theta, t) \right| \\
\leq C \left| \frac{\partial^m}{\partial I^m} \left( \sum_{k_1+k_2=k} \frac{\partial^k \tilde{I}_1 (I, \theta, t)}{\partial I^k} \cdot \frac{\partial^m \eta (I, \theta, t)}{\partial \eta^m} \right) \right|
\leq C \left| \frac{\partial^m}{\partial I^m} \left( \sum_{l_1+k_2=k} \frac{\partial^{l_1+2} \xi_2 (I, \theta, t)}{\partial I^{l_1} \partial \eta^{2m}} \right) \right|
\leq C \left| \frac{\partial^m}{\partial I^m} \left( \sum_{l_1+k_2=k} \frac{\partial^{l_1} \tilde{I}_1 (I, \theta, t)}{\partial I^{l_1}} \cdot \frac{\partial^2 \xi_2 (I, \theta, t)}{\partial \eta^2 \partial \eta^m} \right) \right|
\leq C \left| \frac{\partial^m}{\partial I^m} \left( \sum_{l_1+k_2=k} \frac{\partial^{l_1+2} \xi_2 (I, \theta, t)}{\partial I^{l_1} \partial \eta^{2m}} \right) \right|
\leq CI^{-k_1+3-4b-2} \cdot I^{-k_2+2b-1}
\leq CI^{-k_1+2-2b}.
\] (83)

(6) By using the estimates on the functions \( \tilde{T}_1 \) and \( \eta_i \) \((i = 0, 1, 2, 3)\), it follows that
\[
\left| \frac{\partial^{k+m}}{\partial I^k \partial \eta^m} \xi_2 (I, \theta, t) \right| \\
\leq C \left| \frac{\partial^m}{\partial I^m} \left( \sum_{k_1+k_2=k} \frac{\partial^k \tilde{I}_1 (I, \theta, t)}{\partial I^k} \cdot \frac{\partial^m \eta (I, \theta, t)}{\partial \eta^m} \right) \right|
\leq C \left| \frac{\partial^m}{\partial I^m} \left( \sum_{l_1+k_2=k} \frac{\partial^{l_1+2} \xi_2 (I, \theta, t)}{\partial I^{l_1} \partial \eta^{2m}} \right) \right|
\leq C \left| \frac{\partial^m}{\partial I^m} \left( \sum_{l_1+k_2=k} \frac{\partial^{l_1} \tilde{I}_1 (I, \theta, t)}{\partial I^{l_1}} \cdot \frac{\partial^2 \xi_2 (I, \theta, t)}{\partial \eta^2 \partial \eta^m} \right) \right|
\leq C \left| \frac{\partial^m}{\partial I^m} \left( \sum_{l_1+k_2=k} \frac{\partial^{l_1+2} \xi_2 (I, \theta, t)}{\partial I^{l_1} \partial \eta^{2m}} \right) \right|
\leq CI^{k_1+2-2b}.
\] (84)

Let \( t = s, \theta = \theta, r = \eta_0 (I) \) and
\[
F_0 (r, \theta) = \eta_3 (I (r), \theta),
F_1 (r, \theta, t) = \eta_2 (I (r), \theta, t) + \varepsilon \eta_3 (I (r), \theta, t),
F_2 (r, \theta, t) = \eta'_0 (I (r)) \cdot (\xi_1 (I (r), \theta, t) + \varepsilon \xi_2 (I (r), \theta, t)),
F_3 (r, \theta, t) = \eta'_0 (I (r)) \cdot \xi_3 (I (r), \theta, t),
\] (85)

where \( I (r) \) is the inverse function of \( r = \eta_0 (I) \).

Then system (70) is transformed into the following form:
\[
\frac{dr}{d\theta} = r + F_0 (r, \theta, t) + F_1 (r, \theta, t),
\] (86)

Moreover, one can verify that system (86) is reversible with respect to involution \( G : (t, r) \mapsto (-t, r) \).
It is easy to see that \( I \gg 1 \) if and only if \( r \gg 1 \), and the solutions of system (86) do exist on \( 0 \leq \theta \leq 1 \) when \( r(0) = r_0 \gg 1 \).

By using the estimates on \( \eta_1 \) and \( \xi_1 \) (\( i = 1, 2, 3 \)) in Lemma 10, the following inequalities can be proved.

**Lemma 11.** For \( 0 < k + m \leq 4 \) and \( r \gg 1 \), the following inequalities hold:

1. \( |(\partial^k \partial^p) F_0 (r, \theta)| \leq C r^{-k+(1+\gamma-3b/2)/(2b-1)} \),
2. \( |(\partial^{k+m} \partial^p) F_0 (r, \theta)| \leq C (r^{-k+(r+4b-2)/(2b-1)} + \varepsilon) \),
3. \( |(\partial^{k+m} \partial^p) F_0 (r, \theta)| \leq C (r^{-k+(r+4b-3)/(2b-1)} + \varepsilon) \),
4. \( |(\partial^{k+m} \partial^p) F_0 (r, \theta)| \leq C r^{-k} \).

**Proof.** Above all, we know that \( r = \eta_0 (I) = T_0 I^{2b-1} \), so we can get \( I = ((1/T_0) r_0)^{1/(2b-1)} \). Then we have

\[
\left| \frac{d^2 I}{dr^2} \right| \leq C r^{-j+1/(2b-1)},
\]
\[
\left| \frac{d^2 \eta_0 (I (r))}{dr^2} \right| \leq C \left| \frac{d^2 I (r^{-1/(2b-1)})}{dr^2} \right| \leq C r^{-j+(2b-2)/(2b-1)}. \quad (87)
\]

(1) We have that

\[
\left| \frac{\partial^k \partial^p F_0 (r, \theta)}{\partial r^k} \right|
\leq \sum_{k_1+\cdots+k_j=k} \left| \frac{\partial^i \eta_1 (I, \theta)}{\partial I^i} \right| \left| \frac{\partial^j \eta_1 (I, \theta)}{\partial r^j} \right| \left| \frac{\partial^k \eta_1 (I, \theta)}{\partial r^k} \right|
\leq C r^{-s-1+\gamma-(3/2)b} r^{-k+(1/(2b-1))} \eta_1 (r^{-1/(2b-1)}) \leq C r^{-s-(1/(2b-1)) + \gamma-3b/2} r^{-k+(1/(2b-1))} \eta_1 (r^{-1/(2b-1)}) \leq C r^{-k+(1+\gamma-3b/2)/(2b-1)} \eta_1 (r^{-1/(2b-1)}). \quad (88)
\]

(2) We have that

\[
\left| \frac{\partial^{k+m} F_0 (r, \theta, t)}{\partial r^k \partial t^m} \right|
\leq C \sum_{j_1+\cdots+j_k=j} \left| \frac{\partial^{i+m} \eta_2 (I, \theta, t)}{\partial I^i \partial r^j} \right| \left| \frac{\partial^j \eta_1 (I, \theta, t)}{\partial r^j} \right| \left| \frac{\partial^k \eta_1 (I, \theta, t)}{\partial r^k} \right|
+ C \varepsilon \sum_{j_1+\cdots+j_k=j} \left| \frac{\partial^{i+m} \eta_2 (I, \theta, t)}{\partial I^i \partial r^j} \right| \left| \frac{\partial^j \eta_1 (I, \theta, t)}{\partial r^j} \right| \left| \frac{\partial^k \eta_1 (I, \theta, t)}{\partial r^k} \right|
\leq C \left[ r^{-k+(r+4b-2)/(2b-1)} + \varepsilon r^{-k} \right]. \quad (89)
\]

(3) We have that

\[
\left| \frac{\partial^{k+m} F_0 (r, \theta, t)}{\partial r^k \partial t^m} \right|
\leq \sum_{k_1+k_2=k} \frac{d^{k_1} \eta_0 (I (r))}{dr^{k_1}} \left( \frac{\partial^{k_1+m} \xi_2 (I (r), \theta, t)}{\partial r^{k_1} \partial t^m} \right) + \varepsilon \frac{\partial^{k_1+m} \xi_2 (I (r), \theta, t)}{\partial r^{k_1} \partial t^m}
\leq C \sum_{k_1+k_2=k} r^{-k+(2b-2)/(2b-1)} \eta_1 (r^{-1/(2b-1)})(\cdots + r^{-k+(1/(2b-1))} \eta_1 (r^{-1/(2b-1)})) \leq C r^{-k+(2b-2)/(2b-1)} \eta_1 (r^{-1/(2b-1)})) \leq C r^{-k+(2b-2)/(2b-1)} \eta_1 (r^{-1/(2b-1)})) \leq C r^{-k} \eta_1 (r^{-1/(2b-1)})). \quad (90)
\]

(4) We have that

\[
\left| \frac{\partial^{k+m} F_0 (r, \theta, t)}{\partial r^k \partial t^m} \right|
\leq C \left[ r^{-k+4b-3/(2b-1)} + \varepsilon r^{-k} \right]. \quad (91)
\]

**Lemma 12.** The time 1 map \( \Phi^1 \) of the flow \( \Psi^\theta \) of the system (86) is of the form

\[
\Phi^1 : r_1 = r + Q_2 (r, t), \quad t_1 = t + \bar{\omega} (r) + Q_1 (r, t), \quad (92)
\]

where \( \bar{\omega} (r) = r + \int_0^1 F_0 (r, \theta) d\theta \). And there exists a \( \mu_0 > 0 \) such that, for \( 0 < k + m \leq 4 \), sufficiently large \( r \) and sufficiently small \( \varepsilon \),

\[
\left| \frac{\partial^{k+m} Q_i (r, t)}{\partial r^k \partial t^m} \right| \leq C r^{-\mu_0 + \varepsilon}, \quad i = 1, 2 \quad (93)
\]
hold. Moreover, the map $\Phi^1$ is reversible with respect to the involution $G : (t, r) \mapsto (-t, r)$.

**Proof.** Since

$$t(\theta) = t + r \theta + D_1(r, t, \theta), \quad r(\theta) = r + D_2(r, t, \theta),$$

(95)

then we get $\int_0^1 \alpha T_0 |S(\theta T_0)|^{-\alpha} |C(\theta T_0)| d\theta$ is bounded.

Let $\alpha T_0 |S(\theta T_0)|^{-\alpha} |C(\theta T_0)| = S_1(\theta)$. Set $(r(\theta), t(\theta)) = \Phi^\theta(r, t)$ with $\Phi^0 = id$ for the flow:

$$t(\theta) = t + r \theta + D_1(r, t, \theta), \quad r(\theta) = r + D_2(r, t, \theta).$$

(95)

Since

$$\Phi^\theta = \Phi^0 + \int_0^\theta X \cdot \Phi^n d\nu,$$

(96)

where $X$ denotes the vector field of the system (86), we have

$$t(\theta) = t + \int_0^\theta [r(\nu) + F_0(r(\nu), \nu) + F_1(r(\nu), \nu, t(\nu))] d\nu$$

$$= t + r \theta + \int_0^\theta [D_2(r, t, \nu) + F_0(r + D_2, \nu) + F_1(r + D_2, v, t + rv + D_1)] d\nu$$

$$= t + r \theta + D_1(r, t, \theta),$$

$$r(\theta) = r + \int_0^\theta [F_2(r(\nu), \nu, t(\nu)) + S_1(\nu) F_3(r(\nu), \nu, t(\nu))] d\nu$$

$$= r + D_2(r, t, \theta),$$

(97)

which is equivalent to the following equations for $D_1$ and $D_2$:

$$D_1(r, t, \theta) = \int_0^\theta [D_2(r, t, \nu) + F_0(r + D_2, \nu) + F_1(r + D_2, v, t + rv + D_1)] d\nu,$$

$$D_2(r, t, \theta) = \int_0^\theta [F_2(r + D_2, \nu, t + rv + D_1) + S_1(\nu) F_3(r + D_2, \nu, t + rv + D_1)] d\nu.$$ 

(98)

Let $D(r, t, \theta) = (D_1(r, t, \theta), D_2(r, t, \theta))$, $|D_1(r, t, \theta)| = \sup_{\|\alpha\| \leq 3/2} |D_1(\alpha(r, t, \theta))|$. Define $|D| = |D_1|/3 + 2|D_2|/3$, and $T(D) = (T_1(D), T_2(D))$, where

$$T_1(D) = \int_0^\theta [D_2(r, t, \nu) + F_0(r + D_2, \nu) + F_1(r + D_2, v, t + rv + D_1)] d\nu,$$

$$T_2(D) = \int_0^\theta [F_2(r + D_2, \nu, t + rv + D_1) + S_1(\nu) F_3(r + D_2, \nu, t + rv + D_1)] d\nu.$$ 

(99)
Next, we will prove that $T$ is a contraction map. From the definition of $T(D)$, we have

$$|T_1 D - T_1 \overline{D}|$$

$$= \left| \int_0^8 \left[ D_2 - \overline{D}_2 + F_0 (r + D_2, v) - F_0 (r + \overline{D}_2) + F_1 (r + D_2, v, t + rv + D_1) - F_1 (r + \overline{D}_2, v, t + rv + D_1) \right] dv \right|$$

$$\leq |D_2 - \overline{D}_2|$$

$$+ \int_0^1 \frac{\partial F_0 (r + s (D_2 - \overline{D}_2), v)}{\partial r} \cdot |D_2 - \overline{D}_2| ds$$

$$+ \int_0^1 \frac{\partial F_1 (r + s (D_2 - \overline{D}_2), v, t + rv + D_1)}{\partial r} \cdot |D_2 - \overline{D}_2| ds$$

$$+ \int_0^1 \frac{\partial F_2 (r + \overline{D}_2, v, t + rv + s \left(D_1 - \overline{D}_1 \right))}{\partial r} \cdot |D_2 - \overline{D}_2| ds$$

$$+ \int_0^1 |S_1 (v)| dv$$

$$+ \int_0^1 \left[ F_0 (r + D_2, v, t + rv + s \left(D_1 - \overline{D}_1 \right)) \right]$$

$$\cdot \int_0^1 \frac{\partial F_0 (r + s D_2, v)}{\partial r} \cdot D_2 ds$$

$$+ \int_0^1 \frac{\partial F_1 (r + D_2, v, t + rv + D_1)}{\partial r} \cdot D_2 ds$$

$$+ \int_0^1 \frac{\partial F_2 (r + D_2, v, t + rv + D_1)}{\partial r} \cdot D_2 ds$$

$$+ \int_0^1 \frac{\partial F_1 (r + D_2, v, t + rv + D_1)}{\partial r} \cdot D_2 ds$$

$$\leq \frac{3}{20} |D_2 - \overline{D}_2| + \frac{1}{8} |D_1 - \overline{D}_1|,$$

(100)

by Lemma II and the boundedness of $\int_0^1 |S_1 (v)| dv$. Then we have

$$\|T (D) - T (\overline{D})\|$$

$$= \frac{1}{3} |T_1 (D) - T_1 (\overline{D})| + \frac{2}{3} |T_2 (D) - T_2 (\overline{D})|$$

$$\leq \frac{1}{3} \times \left( \frac{6}{5} |D_2 - \overline{D}_2| + \frac{1}{4} |D_1 - \overline{D}_1| \right)$$

$$+ \frac{2}{3} \times \left( \frac{3}{20} |D_2 - \overline{D}_2| + \frac{1}{8} |D_1 - \overline{D}_1| \right)$$

$$= \frac{1}{6} |D_1 - \overline{D}_1| + \frac{1}{2} |D_2 - \overline{D}_2|$$

$$\leq \frac{3}{4} \times \left( \frac{1}{5} |D_1 - \overline{D}_1| + \frac{2}{3} |D_2 - \overline{D}_2| \right)$$

$$\leq \frac{3}{4} \|D - \overline{D}\|,$$

by the definition of the norm $\| \cdot \|$.

Using the contraction principle, one verifies easily that for $r \geq r_0$, (98) has a unique solution in the space $\{|D_1| \leq 1, |D_2| \leq 1\}$. Moreover, $D_1$ and $D_2$ are smooth.

Next, we will estimate $Q_1 (r, t)$ and $Q_2 (r, t)$ as follows:

$$Q_1 (r, t) = D_1 (r, t, 1) - \int_0^1 F_0 (r, v) dv$$

$$= \int_0^1 \left[ D_2 (r, t, v) + \int_0^1 \frac{\partial F_0 (r + s D_2, v)}{\partial r} \cdot D_2 ds \right.$$

$$\left. + \int_0^1 \frac{\partial F_1 (r + D_2, v, t + rv + D_1)}{\partial r} \cdot D_2 ds \right. + \int_0^1 \frac{\partial F_2 (r + D_2, v, t + rv + D_1)}{\partial r} \cdot D_2 ds$$

$$+ F_1 (r + D_2, v, t + rv + D_1) \right] dv,$$

(101)

$$Q_2 (r, t) = D_2 (r, t, 1)$$

$$= \int_0^1 \left[ F_2 (r + D_2, v, t + rv + D_1) \right.$$}

$$\left. + \int_0^1 |S_1 (v)| dv \right. + \int_0^1 \frac{\partial F_0 (r + s D_2, v)}{\partial r} \cdot D_2 ds$$

$$+ \int_0^1 \frac{\partial F_1 (r + D_2, v, t + rv + D_1)}{\partial r} \cdot D_2 ds$$

$$+ \int_0^1 \frac{\partial F_2 (r + D_2, v, t + rv + D_1)}{\partial r} \cdot D_2 ds$$

$$\leq \frac{3}{4} |D_2 - \overline{D}_2| + \frac{1}{2} |D_1 - \overline{D}_1|$$

$$\leq \frac{3}{4} \|D - \overline{D}\|,$$

(102)
In order to prove (93), we just need to prove that
\[
\frac{\partial^{k+m}}{\partial r^k \partial t^m} D_i(r, t, \theta) \leq C r^{-\mu_0} + C \varepsilon, \quad i = 1, 2, \quad \text{(103)}
\]
hold for \(k + m \leq 4\).

1. When \(k + m = 0\),
\[
|D_2(r, t, \theta)| \\
\leq \int_0^\theta (|F_2(r + D_2, v, t + rv + D_1)| + |S_1(\theta)| |F_3(r + D_2, v, t + rv + D_1)|) \, dv \\
\leq C(r^{-\mu_0} + \varepsilon) + \int_0^1 |S_1(\theta)| \, dv \cdot (C \varepsilon) \leq C(r^{-\mu_0} + \varepsilon),
\]
\[
|D_1(r, t, \theta)| \\
\leq |D_2(r, t, \theta)| \\
+ \int_0^\theta (|F_0(r + D_2, v)| + |F_1(r + D_2, v, t + rv + D_1)|) \, dv,
\]
\[
\leq |D_2(r, t, \theta)| + \int_0^\theta C(r^{-\mu_0} + C \varepsilon) \, dv \\
\leq |D_2(r, t, \theta)| + C(r^{-\mu_0} + \varepsilon) \leq C(r^{-\mu_0} + \varepsilon), \quad \text{(104)}
\]
where \(\mu_0 = \min\{(1 + 3 - 2\beta)/(2\beta - 1), (2 - 4\beta - \tau)/(2\beta - 1), (3 - 4\beta - a)/(2\beta - 1)\}\).

2. When \(m = 0\) and \(k \neq 0\), we check the case when \(k = 1\) firstly
\[
\left| \frac{\partial D_2(r, t, \theta)}{\partial r} \right| \\
\leq \int_0^\theta \left| \frac{\partial F_2(r + D_2, v, t + rv + D_1)}{\partial r} \right| \, dv \\
+ \int_0^\theta \left| \frac{\partial F_2(r + D_2, v, t + rv + D_1)}{\partial t} \right| \, dv \\
+ \int_0^1 |S_1(\theta)| \, dv \cdot \int_0^\theta \left| \frac{\partial F_3(r + D_2, v, t + rv + D_1)}{\partial r} \right| \, dv \\
+ \int_0^1 |S_1(\theta)| \, dv \cdot \int_0^\theta \left| \frac{\partial F_3(r + D_2, v, t + rv + D_1)}{\partial t} \right| \, dv \\
\leq C r^{-1}(r^{-\mu_0} + \varepsilon) \cdot \left( 1 + \left| \frac{\partial D_2(r, t, \theta)}{\partial r} \right| \right) \\
+ C(r^{-\mu_0} + \varepsilon) \cdot \left( 1 + \left| \frac{\partial D_1(r, t, \theta)}{\partial r} \right| \right),
\]
\[
\left| \frac{\partial D_1(r, t, \theta)}{\partial r} \right| \\
\leq \left| \frac{\partial D_2(r, t, \theta)}{\partial r} \right| \\
+ \int_0^\theta \left| \frac{\partial F_0(r + D_2, v, t + rv + D_1)}{\partial r} \right| \, dv \\
+ \int_0^\theta \left| \frac{\partial F_1(r + D_2, v, t + rv + D_1)}{\partial r} \right| \, dv \\
+ \int_0^\theta \left| \frac{\partial F_1(r + D_2, v, t + rv + D_1)}{\partial t} \right| \, dv \\
\leq C(r^{-\mu_0} + \varepsilon) \cdot \left( 1 + \left| \frac{\partial D_2(r, t, \theta)}{\partial r} \right| \right).
\]

Hence,
\[
\left| \frac{\partial D_1(r, t, \theta)}{\partial r} \right| \leq C(r^{-\mu_0} + \varepsilon), \quad \left| \frac{\partial D_2(r, t, \theta)}{\partial r} \right| \leq C(r^{-\mu_0} + \varepsilon). \quad \text{(105)}
\]

Now, we proceed inductively by assuming that for \(j < k - 1\) the estimates
\[
\left| \frac{\partial^j D_1(r, t, \theta)}{\partial r^j} \right| \leq C(r^{-\mu_0} + \varepsilon), \quad \left| \frac{\partial^j D_2(r, t, \theta)}{\partial r^j} \right| \leq C(r^{-\mu_0} + \varepsilon), \quad \text{(106)}
\]
hold and we wish to conclude that the same estimate holds for \(j = k\)
\[
\left| \frac{\partial^k D_2(r, t, \theta)}{\partial r^k} \right| \leq \int_0^\theta \left| \frac{\partial F_2(r + D_2, v, t + rv + D_1)}{\partial r} \right| \, dv \\
+ \int_0^\theta \left| \frac{\partial F_2(r + D_2, v, t + rv + D_1)}{\partial t} \right| \, dv \\
+ \int_0^1 |S_1(\theta)| \, dv \cdot \int_0^\theta \left| \frac{\partial F_3(r + D_2, v, t + rv + D_1)}{\partial r} \right| \, dv \\
+ \int_0^1 |S_1(\theta)| \, dv \cdot \int_0^\theta \left| \frac{\partial F_3(r + D_2, v, t + rv + D_1)}{\partial t} \right| \, dv \\
\leq C r^{-1}(r^{-\mu_0} + \varepsilon) \cdot \left( 1 + \left| \frac{\partial D_2(r, t, \theta)}{\partial r} \right| \right) \\
+ C(r^{-\mu_0} + \varepsilon) \cdot \left( 1 + \left| \frac{\partial D_1(r, t, \theta)}{\partial r} \right| \right),
\]
\[
\left| \frac{\partial^k D_1(r, t, \theta)}{\partial r^k} \right| \leq \sum_{k_1 + k_2 = k} \left| \frac{\partial^{k_1} (r + D_2)}{\partial r^{k_1}} \right| \cdot \left| \frac{\partial^{k_2} (r + D_2)}{\partial r^{k_2}} \right| \cdot \left| \frac{\partial^j (r + D_1)}{\partial r^j} \right|.
\]
\[
+ \int_0^1 |S_1(v)| \, dv
\]
\[
\cdot \left[ \int_0^\theta \frac{\partial F_2(r + D_2, v, t + rv + D_1)}{\partial r} \right] \cdot \left| \frac{\partial^k D_2(r, t, v)}{\partial r^k} \right| \, dv
\]
\[\ + \int_0^1 |S_1(v)| \, dv\]
\[
\cdot \left[ \int_0^\theta \frac{\partial F_3(r + D_2, v, t + rv + D_1)}{\partial t} \right] \cdot \left| \frac{\partial^k D_1(r, t, v)}{\partial r^k} \right| \, dv
\]
\[\ + C(r^{-\mu_0} + \varepsilon)\]
\[
\sum_{k_1, k_2, k_3, k_1 + \ldots + k_3 = k} \left| \frac{\partial^i (r + D_1)}{\partial r^i} \right| \ldots \left| \frac{\partial^i (r + D_1)}{\partial r^i} \right|
\]
\[
\times \left| \frac{\partial^i (r + D_1)}{\partial r^i} \right| \ldots \left| \frac{\partial^i (r + D_1)}{\partial r^i} \right|
\]
\\leq Cr^{-1} (r^{-\mu_0} + \varepsilon) \cdot \left| \frac{\partial^k D_2(r, t, v)}{\partial r^k} \right|
\[\ + C(r^{-\mu_0} + \varepsilon) \cdot \left| \frac{\partial^k D_1(r, t, v)}{\partial r^k} \right| + C(r^{-\mu_0} + \varepsilon),\]

where \(s + v \leq 2\). Hence,
\[
\frac{\partial^k D_1(r, t, \theta)}{\partial r^k} \leq C(r^{-\mu_0} + \varepsilon),
\]
\[
\frac{\partial^k D_2(r, t, \theta)}{\partial r^k} \leq C(r^{-\mu_0} + \varepsilon).
\]

(3) We can prove that
\[
\frac{\partial^m D_1(r, t, \theta)}{\partial t^m} \leq C(r^{-\mu_0} + \varepsilon),
\]
\[
\frac{\partial^m D_2(r, t, \theta)}{\partial t^m} \leq C(r^{-\mu_0} + \varepsilon)
\]
similarly to (2) when \(m \neq 0\).

(4) we have that
\[
\left| \frac{\partial^k D_2(r, t, \theta)}{\partial r^k} \right|
\]
\[
\leq \left[ \int_0^\theta \frac{\partial F_2(r + D_2, v, t + rv + D_1)}{\partial r} \right] \cdot \left| \frac{\partial^k D_2(r, t, \theta)}{\partial r^k} \right| \, dv
\]
\[
\cdot \left[ \int_0^\theta \frac{\partial F_3(r + D_2, v, t + rv + D_1)}{\partial t} \right] \cdot \left| \frac{\partial^k D_1(r, t, v)}{\partial r^k} \right| \, dv
\]
\[\ + C(r^{-\mu_0} + \varepsilon) \cdot \sum_{i \in A, i = k} \left| \frac{\partial^i (r + D_1)}{\partial r^i} \right| \ldots \left| \frac{\partial^i (r + D_1)}{\partial r^i} \right|
\]
\[
\times \left| \frac{\partial^i (r + D_1)}{\partial r^i} \right| \ldots \left| \frac{\partial^i (r + D_1)}{\partial r^i} \right|
\]
\[
\leq C(r^{-\mu_0} + \varepsilon) \cdot \left| \frac{\partial^k D_2(r, t, \theta)}{\partial r^k} \right|
\]
\[\ + C(r^{-\mu_0} + \varepsilon) \cdot \left| \frac{\partial^k D_1(r, t, \theta)}{\partial r^k} \right| + C(r^{-\mu_0} + \varepsilon),\]

(108)
Thus, the system (115) can be written in the form
\[
\frac{dp}{dt} = -\frac{1}{2(1-b)}\bar{g}'(\varphi) p^{2(1-b)} + O(\varepsilon p^{2(1-b)}),
\]
\[
\frac{dp}{dt} = \rho^{1-2b} \tilde{g}(\varphi) + O(\varepsilon \rho^{1-2b}).
\]
From the equality
\[
\frac{1}{2} C^2(t) + \frac{A}{\alpha+1} |S(t)|^{\alpha+1} = \frac{1}{2} \quad \forall t \in \mathbb{R},
\]
it follows that
\[
0 \leq |S(T_0)|^{\alpha+1} \leq -\frac{\alpha + 1}{2A}.
\]
Hence, the function \( \bar{g}(\varphi) \) is \( C^1 \), 1-periodic and change the sign. Since \( |S(T_0 - \varphi T_0)| = |S(\varphi T_0)| \) for any \( \varphi \in [0, 1] \), there exists \( \varphi_1 \in (0, 1/2) \) such that
\[
|S(T_0 - \varphi_1 T_0)|^{\alpha+1} = |S(\varphi_1 T_0)|^{\alpha+1} = -\frac{\alpha + 1}{4A}.
\]
That is, \( \bar{g}(\varphi_1) = \bar{g}(1 - \varphi_1) = 0 \). In view of
\[
S(T_0 - \varphi T_0) = -S(\varphi T_0), \quad C(T_0 - \varphi T_0) = C(\varphi T_0),
\]
we find
\[
\bar{g}'(\varphi_1) \cdot \bar{g}'(1 - \varphi_1) = -(\alpha + 1)^2 (2b\delta A T_0)^2 |S(\varphi_1 T_0)|^{2(\alpha+1)} S^2(\varphi_1 T_0) C^2(\varphi_1 T_0) < 0.
\]
Hence, we obtain that \( \bar{g}'(\varphi) \) or \( \bar{g}'(1 - \varphi) \) is negative. This proves that there exists a \( \varphi^* \) such that \( \bar{g}(\varphi^*) = 0 \) and \( \bar{g}'(\varphi^*) < 0 \). Therefore, there are \( \varphi > 0 \) and \( \delta_0 > 0 \) such that \( \bar{g}(\varphi) < -\delta_0 \)
for \( \varphi \in [\varphi^* - v, \varphi^* + v] \) and \( \bar{g}(\varphi) > 0 \) for \( \varphi \in (\varphi^* - v, \varphi^*) \), \( \bar{g}(\varphi) < 0 \) for \( \varphi \in (\varphi^*, \varphi^* + v) \). Let
\[
\mathcal{K}_{f,v} = \{ (\rho, \varphi) \in \mathbb{R}^+ \times \mathbb{T} : \rho > J, \varphi \in [\varphi^* - v, \varphi^* + v] \}.
\]
Then, if \( J \) is sufficiently large, on the set \( \mathcal{K}_{f,v} \), we have
\[
-\frac{1}{2(1-b)}\bar{g}'(\varphi) p^{2(1-b)} + O(\varepsilon p^{2(1-b)}) > \frac{\delta_0}{2} \cdot \rho^{2(1-b)},
\]
\[
\rho^{1-2b} \bar{g}(\varphi) + O(\varepsilon \rho^{1-2b}) > 0,
\]
for \( \rho \geq J, \varphi \in [\varphi^* - v, \varphi^* - \frac{v}{2}] \),
\[
\rho^{1-2b} \bar{g}(\varphi) + O(\varepsilon \rho^{1-2b}) < 0,
\]
for \( \rho \geq J, \varphi \in [\varphi^* + \frac{v}{2}, \varphi^* + v] \).
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From (117) and (124) we obtain, for \( t \geq 0 \),
\[
\rho(t, \rho_0, \varphi_0) = \rho_0 + \int_0^t \left( -\frac{1}{2} (1-b) \tilde{g}'(\varphi) \rho^{2(1-b)}(t) + O(\rho^{2(1-b)}) \right) dt
\]
\[
> \rho_0 + \int_0^t \delta_0 \cdot \rho^{2(1-b)}(t) dt \geq \rho_0 > J.
\]

(126)

Moreover, for \( \rho(t, \rho_0, \varphi_0) > J \) and \( \varphi(t, \rho_0, \varphi_0) \in [\varphi^*-v, \varphi^*-v/2] \cup [\varphi^*+v/2, \varphi^*+v] \), we have
\[
\rho^{1-2b} \tilde{g}(\varphi) + O(\rho^{1-2b})
\]
\[
= \rho^{1-2b} \tilde{g}(\varphi) (\varphi - \varphi^*) + O(\rho^{1-2b})
\]
\[
> -\frac{\delta_0}{2} (\varphi - \varphi^*) \rho^{1-2b}.
\]

(127)

From (126) and (127), it follows that any solution \( \rho(t, \rho_0, \varphi_0) \) of (115) with the initial condition \( \rho(0, \rho_0, \varphi_0) = (\rho_0, \varphi_0) \in \mathcal{H}_{J,\nu} \) always stays in \( \mathcal{H}_{J,\nu} \) and satisfies \( \rho(t, \rho_0, \varphi_0) > \delta t + \rho(0) \) with \( \delta = \delta_0^{3-2b}/2 \), for all \( t \geq 0 \).

The proof of Theorem 3 is completed.

5. The Proof of Theorem 4

In this section, we will prove Theorem 4 by using the abstract result on the existence of quasi-periodic solutions proved in [24] in the context Aubry-Mather theory for reversible systems. We only need to show that the Poincaré map (92) has the monotone property; that is,
\[
\frac{\partial F_0}{\partial r}(r, t) > 0.
\]

(128)

We can get that
\[
\left| \frac{\partial F_0}{\partial r}(r, \theta) \right| \leq C r^{1-(1+\gamma-3b/2)/(2b-1)}
\]

(129)

by Lemma II, and
\[
\left| \frac{\partial Q_2}{\partial r}(r, t) \right| \leq r^{-\rho_0} + \varepsilon
\]

(130)

by Lemma 12. Then we have
\[
\frac{\partial F_0}{\partial r}(r, t) = 1 + \int_0^t \frac{\partial F_0}{\partial r}(r, \theta) + \frac{\partial Q_2}{\partial r}(r, \theta) \rightarrow c_0, \quad \text{as } r \rightarrow +\infty,
\]

(131)

where \( c_0 \geq 1 - \varepsilon \). Therefore, we have
\[
\frac{\partial F_0}{\partial r}(r, t) > 0
\]

(132)

as \( r \gg 1 \) and \( \varepsilon \ll 1 \). This proves the validity of (128).

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References


