Research Article

\(\triangle\)-Convergence Problems for Asymptotically Nonexpansive Mappings in CAT(0) Spaces

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New \(\triangle\)-convergence theorems of iterative sequences for asymptotically nonexpansive mappings in CAT(0) spaces are obtained. Consider an asymptotically nonexpansive self-mapping \(T\) of a closed convex subset \(C\) of a CAT(0) space \(X\). Consider the iteration process \(\{x_n\}\), where \(x_0\in C\) is arbitrary and

\[x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)Tx_n, \quad (3)\]

\(\alpha_n\) are nonexpansive and every nonexpansive mapping is asymptotically nonexpansive with sequence \(\{k_n\} \in (0,1)\). It is shown that under certain appropriate conditions on \(\alpha_n\), \(\beta_n\), \(\{x_n\}\) \(\triangle\)-converges to a fixed point of \(T\).

1. Introduction and Preliminaries

Let \(C\) be a nonempty subset of a metric space \((X,d)\). A mapping \(T : C \to C\) is a contraction if there exists \(k \in [0,1)\) such that for all \(x, y \in C\), we have \(d(Tx, Ty) < kd(x, y)\). It is said to be nonexpansive if for all \(x, y \in C\), we have \(d(Tx, Ty) \leq d(x, y)\). \(T\) is said to be asymptotically nonexpansive if there exists a sequence \(\{k_n\} \in [1,\infty)\) with \(k_n \to 1\) such that \(d(T^n x, T^n y) \leq k_n d(x, y)\) for all integers \(n \geq 1\) and all \(x, y \in C\). Clearly, every contraction mapping is nonexpansive and every nonexpansive mapping is asymptotically nonexpansive with sequence \(k_n = 1\), for all \(n \geq 1\). There are, however, asymptotically nonexpansive mappings which are not nonexpansive (see, e.g., [1]). As a generalization of the class of nonexpansive mappings, the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] in 1972 and has been studied by several authors (see, e.g., [3–5]). Goebel and Kirk proved that if \(C\) is a nonempty closed convex and bounded subset of a uniformly convex Banach space (more general than a Hilbert space, i.e., CAT(0) space), then every asymptotically nonexpansive self-mapping of \(C\) has a fixed point. The weak and strong convergence problems to fixed points of nonexpansive and asymptotically nonexpansive mappings have been studied by many authors.

We will denote by \(F(T)\) the set of fixed points of \(T\). In 1967, Halpern [6] introduced an explicit iterative scheme for a nonexpansive mapping \(T\) on a subset \(C\) of a Hilbert space by taking any point \(u, x_1 \in C\) and defined the iterative sequence \(\{x_n\}\) by

\[x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \text{for} \ n \geq 1, \quad (1)\]

where \(\alpha_n \in [0,1]\). He pointed out that under certain appropriate conditions on \(\alpha_n\), \(\{x_n\}\) converges strongly to a fixed point of \(T\). In 1994, Tan and Xu [7] introduced the following iterative scheme for asymptotically nonexpansive mapping on uniformly convex Banach space:

\[x_0 \in C,\]

\[x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad \text{for} \ n \geq 0, \quad (2)\]

\[y_n = y_n x_n + (1 - y_n)Tx_n, \quad \text{for} \ n \geq 0,\]

where \(\{\alpha_n\}, \{\gamma_n\} \subseteq (0,1)\). They proved that under certain appropriate conditions on \(\alpha_n, y_n, \{x_n\}\) converges weakly to a fixed point of \(T\).

In 2012, we [8] studied the viscosity approximation methods for nonexpansive mappings on CAT(0) space. For a contraction \(f \in C\), consider the iteration process \(\{x_n\}\), where \(x_0 \in C\) is arbitrary and

\[x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)Tx_n, \quad (3)\]
for $n \geq 1$, where $\{\alpha_n\} \subset (0, 1)$. We proved that under certain appropriate conditions on $\alpha_n, \{x_n\}$ converges strongly to a fixed point of $T$ which solves some variational inequality.

The purpose of this paper is to study the iterative scheme defined as follows: consider an asymptotically nonexpansive self-mapping $T$ of a closed convex subset $C$ of a CAT(0) space $X$ with coefficient $k_n$, consider the iteration process $\{x_n\}$, where $x_0 \in C$ is arbitrary and

\[ x_{n+1} = \alpha_n x_n \ominus (1 - \alpha_n) T^n y_n, \]

or

\[ y_n = \beta_n x_n \ominus (1 - \beta_n) T^n x_n, \]

for $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1).$ We show that $\{x_n\}$ $\triangle$-converges to a fixed point of $T$ under certain appropriate conditions on $\alpha_n, \beta_n$, and $k_n$.

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

**Lemma 1.** Let $X$ be a CAT(0) space. Then, one has the following:

(i) (see [9, Lemma 2.4]) for each $x, y, z \in X$ and $t \in [0, 1]$, one has

\[ d\left( (1 - t)x \oplus ty, z \right) \leq (1 - t)d(x, z) + td(y, z), \]

(ii) (see [10]) for each $x, y, z \in X$ and $t, s \in [0, 1]$ one has

\[ d\left( (1 - t)x \oplus ty, (1 - s)x \ominus sy \right) \leq |t - s|d(x, y), \]

(iii) (see [5, Lemma 3]) for each $x, y, z \in X$ and $t \in [0, 1]$, one has

\[ d\left( (1 - t)z \ominus tx, (1 - t)z \ominus ty \right) \leq td(x, y), \]

(iv) (see [9]) for each $x, y, z \in X$ and $t \in [0, 1]$, one has

\[ \left| t d^2(x, z) + (1 - t)d^2(y, z) - t(1 - t)d^2(x, y) \right| \leq \left| t d^2(x, z) + (1 - t)d^2(y, z) - t(1 - t)d^2(x, y) \right|, \]

Let $X$ be a complete CAT(0) space and let $\{x_n\}$ be a bounded sequence in a complete $X$ and for $x \in X$ set

\[ r(x, [x_n]) = \limsup_{n \to \infty} d(x, x_n). \]

The asymptotic radius $r([x_n])$ of $[x_n]$ is given by

\[ r([x_n]) = \inf \{ r(x, [x_n]) : x \in X \}, \]

and the asymptotic center $A([x_n])$ of $[x_n]$ is the set

\[ A([x_n]) = \{ x \in X : r(x, [x_n]) = r([x_n]) \}. \]

It is known (see, e.g., [11, Proposition 7]) that in a CAT(0) space, $A([x_n])$ consists of exactly one point.

A sequence $\{x_n\}$ in $X$ is said to $\Delta$-converge to $x \in X$ if $x$ is the unique asymptotic center of $[u_n]$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta$-$\lim x_n = x$ and call $x$ the $\Delta$-limit of $\{x_n\}$.

**Lemma 2.** Assume that $X$ is a CAT(0) space. Then, one has the following:

(i) (see [12]) every bounded sequence in $X$ has a $\Delta$-convergent subsequence;

(ii) (see [13]) if $K$ is a closed convex subset of $X$ and $T : K \to X$ is an asymptotically nonexpansive mapping, then the conditions $\{x_n\}$ $\Delta$-converge to $x$ and $d(x_n, T(x_n)) \to 0$, imply $x \in K$ and $x \in F(T)$.

**Lemma 3** (see [14, 15]). Let $\{a_n\}, \{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying the following condition:

\[ a_{n+1} \leq (1 + b_n) a_n + c_n, \forall n \geq n_0, \]

where $n_0$ is some nonnegative integer, $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$. Then the limit $\lim_{n \to \infty} a_n$ exists.

### 2. $\Delta$-Convergence of the Iteration Sequences

In this section, we will study the $\Delta$-convergence of the iteration sequence for asymptotically nonexpansive mappings in CAT(0) spaces.

Suppose that $X$ be a CAT(0) space, $C$ a closed convex subset of $X$, and $T : C \to C$ an asymptotically nonexpansive mapping with coefficient $k_n$. Firstly, we consider the iteration process:

\[ x_0 \in C, \]

\[ x_{n+1} = \alpha_n x_n \ominus (1 - \alpha_n) T^n y_n, \quad n \geq 0, \tag{14} \]

\[ y_n = \beta_n x_n \ominus (1 - \beta_n) T^n x_n, \quad n \geq 0, \tag{15} \]

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ and $k_n$ satisfy the following.

(i) There exist positive integers $n_0, n_1$, and $\delta > 0, 0 < b < \min\{1, 1/L\}$, where $L = \sup_n k_n$, such that

\[ 0 < \delta < \alpha_n < 1 - \delta, \quad n \geq n_0, \]

\[ 0 < 1 - \beta_n < b, \quad n \geq n_1, \]

(ii) Consider $\sum_{n=1}^{\infty} (k_n^2 - 1) < \infty$.

We will prove that $\{x_n\}$ $\Delta$-converges to a fixed point of $T$.

**Lemma 4.** Let $X$ be a CAT(0) space, $C$ a closed convex subset of $X$, $T : C \to C$ an asymptotically nonexpansive mapping with coefficient $k_n$, and $\sum_{n=1}^{\infty} (k_n^2 - 1) < \infty$. If $F(T) \neq \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n x_n \ominus (1 - \alpha_n) T^n y_n$, $y_n = \beta_n x_n \ominus (1 - \beta_n) T^n x_n$, $n \geq 0$. Then the limit $\lim_{n \to \infty} d(x_n, p)$ exists for all $p \in F(T)$. 

Proof. Taking $p \in F(T)$, we have

$$
d(x_{n+1}, p) = d(\alpha_n x_n \oplus (1 - \alpha_n) T^n y_n, p)
\leq \alpha_n d(x_n, p) + (1 - \alpha_n) d(T^n y_n, p)
\leq \alpha_n d(x_n, p) + (1 - \alpha_n) d(y_n, p)
\leq \alpha_n d(x_n, p)
+ (1 - \alpha_n) k_n \{\beta_n d(x_n, p)
+ (1 - \beta_n) d(T^n x_n, p)\}
\leq \alpha_n d(x_n, p) + (1 - \alpha_n) k_n \{\beta_n d(x_n, p)
+ (1 - \beta_n) d(T^n x_n, p)\}
= [1 + (1 - \alpha_n) (k_n - 1)
\times [k_n (1 - \beta_n) + 1]] d(x_n, p)
\leq [1 + (k_n^2 - 1)] d(x_n, p).
\tag{16}
$$

By Lemma 3, we can get that $\lim_{n \to \infty} d(x_n, p)$ exists. \qed

Remark 5. The above lemma implies that $\{x_n\}$ is bounded and so is the sequence $\{T^n x_n\}$. Moreover, let $L = \sup_n k_n$, then we have

$$
d(T^n x_n, p) \leq k_n d(x_n, p) \leq L d(x_n, p),
\quad d(y_n, p) \leq \beta_n d(x_n, p) + (1 - \beta_n) d(T^n x_n, p)
\leq L d(x_n, p),
\quad d(T^n y_n, p) \leq k_n d(y_n, p) \leq L^2 d(x_n, p).
\tag{17}
$$

It follows that the sequences $\{T^n x_n\}$, $\{y_n\}$, $\{T^n y_n\}$ are bounded.

Proposition 6. Let $X$ be a CAT(0) space, $C$ a closed convex subset of $X$, and $T : C \to C$ an asymptotically nonexpansive mapping with coefficient $k_n$. If $F(T) \neq \emptyset$, $\{\alpha_n\}$, $\{\beta_n\} \subseteq (0, 1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) T^n y_n$, $y_n = \beta_n x_n \oplus (1 - \beta_n) T^n x_n, n \geq 0$. Then under the hypotheses (i) and (ii), one can get that $\lim_{n \to \infty} d(x_n, T^n y_n) = 0$.

Proof. By the assumption, $F(T)$ is nonempty. Take $p \in F(T)$, by Lemma I(iv), we have

$$
d^2(x_{n+1}, p) = d^2(\alpha_n x_n \oplus (1 - \alpha_n) T^n y_n, p)
\leq \alpha_n d^2(x_n, p) + (1 - \alpha_n) d^2(T^n y_n, p)
\leq \alpha_n d^2(x_n, p) + (1 - \alpha_n) d^2(x_n, T^n y_n)
\leq d^2(x_n, p) + (1 - \alpha_n) \{d^2(T^n y_n, p)
+ (1 - \alpha_n) d^2(T^n x_n, p)\}
= d^2(y_n, p) + (1 - \alpha_n) \{d^2(T^n y_n, p)
- \alpha_n (1 - \alpha_n) d^2(x_n, T^n y_n)\}.
\tag{18}
$$

which implies that

$$
d^2(y_n, p) - d^2(x_n, p) \leq (1 - \beta_n) \{d^2(T^n x_n, p) - d^2(x_n, p)\}
\leq (1 - \beta_n) (k_n^2 - 1) d^2(x_n, p).
\tag{19}
$$

Therefore, we have

$$
d^2(x_{n+1}, p) \leq d^2(x_n, p) + (1 - \alpha_n) (k_n^2 - 1) d^2(y_n, p)
+ (1 - \alpha_n) (1 - \beta_n) (k_n^2 - 1) d^2(x_n, p)
- \alpha_n (1 - \alpha_n) d^2(x_n, T^n y_n).
\tag{20}
$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded and $0 < \delta < \alpha_n < 1 - \delta$ for all $n \geq n_0$, we have

$$
\sum_{n=1}^{\infty} \delta^2 d^2(x_n, T^n y_n) < \infty,
\tag{22}
$$

which implies that

$$
\lim_{n \to \infty} d^2(x_n, T^n y_n) = 0.
\tag{23}
$$

Theorem 7. Let $X$ be a CAT(0) space, $C$ a closed convex subset of $X$, and $T : C \to C$ an asymptotically nonexpansive mapping with coefficient $k_n$. If $F(T) \neq \emptyset$, $\{\alpha_n\}$, $\{\beta_n\} \subseteq (0, 1)$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) T^n y_n$, $y_n = \beta_n x_n \oplus (1 - \beta_n) T^n x_n, n \geq 0$. Then under the hypotheses (i) and (ii), one can get that $\{x_n\} \triangle$-converges to a fix point of $T$. \qed
Proof. We first show that \( \lim_{n \to \infty} d(x_n, T^n x_n) = 0 \). Indeed
\[
d(x_n, y_n) = d(x_n, \beta_n x_n \oplus (1 - \beta_n) T^n x_n)
\leq (1 - \beta_n) d(x_n, T^n x_n)
\leq (1 - \beta_n) [d(x_n, T^n y_n) + d(T^n y_n, T^n x_n)]
\leq (1 - \beta_n) [d(x_n, T^n y_n) + L d(y_n, x_n)];
\]

it follows that
\[
[1 - L (1 - \beta_n)] d(x_n, y_n) \leq (1 - \beta_n) d(x_n, T^n y_n).
\]

By Proposition 6, we get that \( \lim_{n \to \infty} d(x_n, T^n x_n) = 0 \).

We will prove that \( \lim_{n \to \infty} d(x_n, T^n x_n) = 0 \).

Indeed we have
\[
d(y_n, T^n x_n) = d(\beta_n x_n \oplus (1 - \beta_n) T^n x_n, T^n x_n)
\leq \beta_n d(x_n, T^n x_n) \to 0.
\]

By Proposition 6, we get that \( \lim_{n \to \infty} d(x_n, T^n x_n) = 0 \).

We claim that \( \lim_{n \to \infty} d(x_n, T^n x_n) = 0 \).

Indeed we have
\[
d(y_n, T^n x_n) = d(\beta_n x_n \oplus (1 - \beta_n) T^n x_n, T^n x_n)
\leq \beta_n d(x_n, T^n x_n) \to 0.
\]

By Proposition 6, we get that \( \lim_{n \to \infty} d(x_n, T^n x_n) = 0 \).

We claim that \( \lim_{n \to \infty} d(x_n, T^n x_n) = 0 \).

Indeed we have
\[
d(y_n, T^n x_n) = d(\beta_n x_n \oplus (1 - \beta_n) T^n x_n, T^n x_n)
\leq \beta_n d(x_n, T^n x_n) \to 0.
\]

Thus,
\[
d(x_n, T^n x_n) \leq d(x_n, T^n x_n) + d(T^n x_n, T^n x_n)
\leq d(x_n, T^n x_n) + L d(T^n x_n, x_n) \to 0.
\]

Since \( \{x_n\} \) is bounded, we may assume that \( \{x_n\} \) \( \Delta \)-converges to a point \( \hat{x} \). By Lemma 2, we have \( \hat{x} \in F(T) \).

Next we will consider another iteration process:
\[
x_{n+1} = \alpha_n T^n x_n \oplus (1 - \alpha_n) y_n, \quad n \geq 0,
\]
\[
y_n = \beta_n x_n \oplus (1 - \beta_n) T^n x_n, \quad n \geq 0,
\]

where \( \{\alpha_n\}, \{\beta_n\} \subseteq (0, 1) \), and \( k_n \) satisfy the following

(H1) There exist positive integers \( n_0 \) and \( \delta > 0 \), such that
\[
0 < \delta < \alpha_n < 1 - \delta, \quad n \geq n_0;
\]
\[
1 - \beta_n \to 0;
\]

(H2) \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \).

We will prove that \( \{x_n\} \) also \( \Delta \)-converges to a fixed point of \( T \).

Lemma 8. Let \( X \) be a CAT(0) space, \( C \) a closed convex subset of \( X \), \( T : C \to C \) an asymptotically nonexpansive mapping with coefficient \( k_n \), and \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \). If \( F(T) \neq \emptyset \), \( \{\alpha_n\}, \{\beta_n\} \subseteq (0, 1) \). Let \( x_0 \in C, \{x_n\} \) be generated by \( x_{n+1} = \alpha_n T^n x_n \oplus (1 - \alpha_n) y_n, \; y_n = \beta_n x_n \oplus (1 - \beta_n) T^n x_n \), \( n \geq 0 \). Then the limit \( \lim_{n \to \infty} d(x_n, p) \) exists for all \( p \in F(T) \).

Proof. Taking \( p \in F(T) \), we have
\[
d(x_{n+1}, p) = d(\alpha_n T^n x_n \oplus (1 - \alpha_n) y_n, p)
\leq \alpha_n k_n d(x_n, p) + (1 - \alpha_n) d(y_n, p)
\leq \alpha_n k_n d(x_n, p)
+ (1 - \alpha_n) [\beta_n d(x_n, p) + (1 - \beta_n) d(T^n x_n, p)]
\leq \alpha_n k_n d(x_n, p)
+ (1 - \alpha_n) [\beta_n d(x_n, p) + (1 - \beta_n) k_n d(x_n, p)]
= [1 + (k_n - 1) (1 - \beta_n)] d(x_n, p).
\]

By Lemma 3, we can get that \( \lim_{n \to \infty} d(x_n, p) \) exists.

Next, we will prove \( \lim_{n \to \infty} d(T^n x_n, y_n) = 0 \).

Proposition 9. Let \( X \) be a CAT(0) space, \( C \) a closed convex subset of \( X \), and \( T : C \to C \) an asymptotically nonexpansive mapping with coefficient \( k_n \). If \( F(T) \neq \emptyset \), \( \{\alpha_n\}, \{\beta_n\} \subseteq (0, 1) \). Let \( x_0 \in C, \{x_n\} \) be generated by \( x_{n+1} = \alpha_n T^n x_n \oplus (1 - \alpha_n) y_n, \; y_n = \beta_n x_n \oplus (1 - \beta_n) T^n x_n \), \( n \geq 0 \). Then under the hypotheses (H1) and (H2), one can get that \( \lim_{n \to \infty} d(T^n x_n, y_n) = 0 \).
Proof. By the assumption, \( F(T) \) is nonempty. Take \( p \in F(T) \), let \( L = \sup k_n \), then we have
\[
\begin{align*}
  d(T^{n}x_n, p) &\leq k_n d(x_n, p) \leq L d(x_n, p), \\
  d(y_n, p) &\leq \beta_n d(x_n, p) + (1 - \beta_n) d(T^{n}x_n, p) \\
  &\leq L d(x_n, p), \\
  d(T^{n}y_n, p) &\leq k_n d(y_n, p) \leq L^2 d(x_n, p).
\end{align*}
\]
(32)
It follows that the sequences \( \{x_n\}, \{T^{n}x_n\}, \{y_n\}, \{T^{n}y_n\} \) are bounded.

By Lemma 1, we have
\[
\begin{align*}
  d^2(x_{n+1}, p) &\leq d^2(x_n, p) + (1 - \alpha_n) \beta_n d^2(x_n, y_n) \\
 &\leq d^2(x_n, p) + (1 - \alpha_n) \beta_n \left[ d^2(x_n, y_n) + d^2(y_n, p) - d^2(x_n, p) \right] \\
  &\leq d^2(x_n, p) + \alpha_n (k_n^2 - 1) d^2(x_n, p) \\
  &\leq \alpha_n (1 - \alpha_n) d^2(T^{n}x_n, y_n).
\end{align*}
\]
(33)

Similar to the proof of Proposition 6, we can get
\[
\begin{align*}
  d^2(y_{n+1}, p) - d^2(x_n, p) &\leq 2 \beta_n (k_n^2 - 1) d^2(x_n, p).
\end{align*}
\]
(34)

Therefore, we have
\[
\begin{align*}
  d^2(x_{n+1}, p) &\leq d^2(x_n, p) + (1 - \alpha_n) \beta_n (k_n^2 - 1) d^2(x_n, p) \\
  &\leq (1 - \alpha_n) \beta_n (k_n^2 - 1) d^2(x_n, p) \\
  &\leq (1 - \alpha_n) \beta_n (k_n^2 - 1) d^2(T^{n}x_n, y_n).
\end{align*}
\]
(35)

By the conditions (H1) and (H2), we have \( \sum_{n=1}^{\infty} (k_n^2 - 1) < \infty \) and
\[
\sum_{n=1}^{\infty} d^2(T^{n}x_n, y_n) < \infty,
\]
(37)
which implies that
\[
\lim_{n \to \infty} d^2(T^{n}x_n, y_n) = 0.
\]
(38)

\textbf{Theorem 10.} Let \( X \) be a CAT(0) space, \( C \) a closed convex subset of \( X \), and \( T: C \to C \) an asymptotically nonexpansive mapping with coefficient \( k_n \). If \( F(T) \neq \emptyset \), \( \{x_n\}, \{\beta_n\} \subseteq (0, 1) \).

Let \( x_0 \in C \), \( \{x_n\} \) be generated by \( x_{n+1} = \alpha_n T^{n}x_n + (1 - \alpha_n) y_n \), \( y_n = \beta_n x_n + (1 - \beta_n) T^{n}x_n, n \geq 0 \). Then under the hypotheses (H1) and (H2), one can get that \( \{x_n\} \) \( \Delta \)-converges to a fixed point of \( T \).

\textbf{Proof.} We first show that \( \lim_{n \to \infty} d(x_n, T^{n}x_n) = 0 \).
Indeed, by Lemma 1, and \( \beta_n \to 1 \), we can get
\[
\begin{align*}
  d(x_n, y_n) &= d(x_n, \beta_n x_n + (1 - \beta_n) T^{n}x_n) \\
  &\leq (1 - \beta_n) d(x_n, T^{n}x_n) \to 0.
\end{align*}
\]
(39)

And then,
\[
\begin{align*}
  d(x_n, T^{n}x_n) &\leq d(x_n, y_n) + d(y_n, T^{n}x_n).
\end{align*}
\]
(40)

By Proposition 9, we obtain that \( \lim_{n \to \infty} d(x_n, T^{n}x_n) = 0 \).

We claim that \( \lim_{n \to \infty} d(x_n, T^{n}x_n) = 0 \). Indeed we have
\[
\begin{align*}
  d(x_{n+1}, x_n) &= d(\alpha_n T^{n}x_n + (1 - \alpha_n) y_n, x_n) \\
  &\leq \alpha_n d(T^{n}x_n, x_n + (1 - \alpha_n) d(x_n, y_n) \to 0. \\
  d(x_n, T^{n-1}x_n) &\leq \alpha_n d(T^{n-1}x_{n-1} + (1 - \alpha_n) y_{n-1}, T^{n-1}x_n) \\
  &\leq \alpha_n \left[ d(x_{n-1}, T^{n-1}x_{n-1}) + (1 - \alpha_n) d(y_{n-1}, T^{n-1}x_{n-1}) \right] \\
  &\leq \alpha_n (1 - \alpha_n) d(x_{n-1}, x_n) \\
  &\to 0.
\end{align*}
\]
(41)

Thus,
\[
\begin{align*}
  d(x_n, T^{n}x_n) &\leq d(x_n, T^{n-1}x_{n-1}) + d(T^{n-1}x_{n-1}, T^{n}x_n) \\
  &\leq d(x_n, T^{n-1}x_{n-1}) + L d(T^{n-1}x_{n-1}, x_n) \\
  &\to 0.
\end{align*}
\]
(42)

Since \( \{x_n\} \) is bounded, we may assume that \( \{x_n\} \) \( \Delta \)-converges to a point \( \bar{x} \).

By Lemma 2, we have \( \bar{x} \in F(T) \).

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\textbf{References}


