Research Article

Oscillation for Higher Order Dynamic Equations on Time Scales

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We investigate the oscillation of the following higher order dynamic equation:

\[ a_n(t) \left[ \left( a_{n-1}(t) \left( \cdots \left( a_1(t) \Delta x(t) \right) \cdots \right) \right) \Delta \right] ^\alpha \Delta + p(t) x^\beta(t) = 0, \quad (E) \]

on some time scale \( T \), where \( n \geq 2 \), \( a_k(t) \) (1 \( \leq k \leq n \)) and \( p(t) \) are positive rd-continuous functions on \( T \) and \( \alpha, \beta \) are the quotient of odd positive integers. We give sufficient conditions under which every solution of this equation is either oscillatory or tends to zero.

1. Introduction

In this paper, we investigate the oscillation of the following higher order dynamic equation:

\[ \left\{ a_n(t) \left[ \left( a_{n-1}(t) \left( \cdots \left( a_1(t) x^\Delta(t) \right) \cdots \right) \right) \Delta \right] ^\alpha \Delta \right\} + p(t) x^\beta(t) = 0, \quad (E) \]

on some time scale \( T \), where \( n \geq 2 \), \( a_k(t) \) (1 \( \leq k \leq n \)) and \( p(t) \) are positive rd-continuous functions on \( T \) and \( \alpha, \beta \) are the quotient of odd positive integers. Write

\[ S_k(t,x(t)) = \begin{cases} x(t), & \text{if } k = 0, \\ a_k(t) S_{k-1}(t,x(t)), & \text{if } 1 \leq k \leq n-1, \\ a_n(t) \left[ S_{n-1}(t,x(t)) \right]^\alpha, & \text{if } k = n, \end{cases} \]

then \( E \) reduces to the following equation:

\[ S_n(t,x(t)) + p(t) x^\beta(t) = 0. \quad (2) \]

Since we are interested in the oscillatory behavior of solutions near infinity, we assume that \( \sup T = \infty \) and \( t_0 \in T \) is a constant. For any \( a \in T \), we define the time scale interval \([a,\infty)_T = \{ t \in T : t \geq a \} \). By a solution of (2), we mean a nontrivial real-valued function \( x(t) \in C^{[1]}_T(T_x, \infty), T_x = t_0 \), which has the property that \( S_k(t,x(t)) \in C^{[1]}_T[T_x, \infty) \) for \( 0 \leq k \leq n \) and satisfies (2) on \([T_x, \infty) \), where \( C^{[1]}_T \) is the space of differentiable functions whose derivative is rd-continuous.

The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution \( x(t) \) of (2) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory.

The theory of time scale, which has recently received a lot of attention, was introduced by Hilger’s landmark paper [1], a rapidly expanding body of the literature that has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus, where a time scale is an nonempty closed subset of the real numbers, and the cases when this time scale is equal to the real numbers or to the integers represent the classical theories of differential or of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [2]). The new theory of the so-called “dynamic equations” not only unifies the theories of differential equations and difference equations, but also extends these classical cases to cases “in between,” for example, to the so-called \( q \)-difference equations when \( T = \{ 1, q, q^2, \ldots, q^k, \ldots \} \), which has important applications in quantum theory (see [3]). In this work, knowledge and understanding of time scales and time scale notation are assumed; for an excellent introduction to the calculus on time
Abstract and Applied Analysis

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to the papers [5–20].

Recently, Erbe et al. in [21–23] considered the third-order dynamic equations

\[
\begin{aligned}
&\left( a(t) \left[ r(t) x^{\Delta}(t) \right]^{\Delta} \right)^{\Delta} + p(t) f(x(t)) = 0, \\
&x^{\Delta\Delta\Delta}(t) + p(t) x(t) = 0,
\end{aligned}
\]

respectively, and established some sufficient conditions for oscillation.

Hassan [24] studied the third-order dynamic equations

\[
\begin{aligned}
&\left( a(t) \left[ r(t) x^{\Delta}(t) \right]^{\Delta} \right)^{\Delta} + f(t, x(\tau(t))) = 0,
\end{aligned}
\]

and obtained some oscillation criteria, which improved and extended the results that have been established in [21–23].

2. Main Results

In this section, we investigate the oscillation of (2). To do this, we need the following lemmas.

**Lemma 1** (see [25]). Assume that

\[
\int_{t_0}^{\infty} \left[ \frac{1}{a_i(s)} \right]^{1/\alpha} \Delta s = \int_t^{\infty} \frac{\Delta s}{a_i(s)} = \infty \quad \forall 1 \leq i \leq n - 1,
\]

and \(1 \leq m \leq n\). Then,

(1) \( \liminf_{t \to \infty} S_m(t, x(t)) > 0 \) implies \( \lim_{t \to \infty} S_m(t, x(t)) = \infty \) for \(0 \leq i \leq m - 1\);

(2) \( \limsup_{t \to \infty} S_m(t, x(t)) < 0 \) implies \( \lim_{t \to \infty} S_m(t, x(t)) = -\infty \) for \(0 \leq i \leq m - 1\).

**Lemma 2** (see [25]). Assume that (5) holds. If \( S_n(t, x(t)) < 0 \) and \( x(t) > 0 \) for \( t \geq t_0 \), then there exists an integer \( 0 \leq m \leq n \) with \( m + n \) even such that

(1) \( (-1)^{m+n} S_n(t, x(t)) > 0 \) for \( t \geq t_0 \) and \( m \leq i \leq n\);

(2) if \( m > 1 \), then there exists \( T \geq t_0 \) such that \( S_n(t, x(t)) > 0 \) for \( 0 \leq i \leq m - 1 \) and \( t \geq T\).

**Remark 3.** Let \( a_n(t) = \cdots = a_1(t) = 1 \), and let \( T \) be the set of integers. Then, Lemmas 1 and 2 are Lemma 1.8.10 and Theorem 1.8.11 of [26], respectively.

**Lemma 4.** Assume that (5) holds. Furthermore, suppose that

\[
\int_{t_0}^{\infty} \frac{1}{a_{n-1}(u)} \left\{ \int_u^{\infty} \frac{1}{a_n(s)} \int_s^{\infty} p(v) \Delta v \right\}^{1/\alpha} \Delta u = \infty.
\]

If \( x(t) \) is an eventually positive solution of (2), then there exists \( T \geq t_0 \) sufficiently large such that

(1) \( S_n(t, x(t)) < 0 \) for \( t \geq T \);

(2) either \( S_i(t, x(t)) > 0 \) for \( t \geq T \) and \(0 \leq i \leq n \) or \( \lim_{t \to \infty} x(t) = 0 \).

**Proof.** Pick \( t_1 \geq t_0 \) so that \( x(t) > 0 \) on \([t_1, \infty)_T\). It follows from (2) that

\[
S_n(t, x(t)) = -p(t) x^\beta(t) < 0 \quad \text{for} \ t \geq t_1.
\]

By Lemma 2, we see that there exists an integer \( 0 \leq m \leq n \) with \( m + n \) even such that \((-1)^{m+n} S_n(t, x(t)) > 0 \) for \( t \geq t_1 \) and \( m \leq i \leq n \), and \( x(t) \) is eventually monotone.

We claim that \( \lim_{t \to \infty} x(t) \neq 0 \) implies \( m = n \). If not, then \( S_{n-1}(t, x(t)) < 0(t \geq t_1) \) and \( S_{n-2}(t, x(t)) > 0(t \geq t_1) \), and there exist \( t_2 \geq t_1 \) and a constant \( c > 0 \) such that \( x(t) \geq c \) on \([t_2, \infty)_T\). Integrating (2) from \( t \) to \( \infty \), we get that for \( t \geq t_2 \)

\[
-a_n(t) \left[ S_{n-1}(t, x(t)) \right]^\alpha = -S_n(t, x(t)) \leq -c^\beta T \int_{t}^{\infty} p(v) \Delta v.
\]

Thus,

\[
S_{n-1}(t, x(t)) \leq -c^\beta
\]

\[
\times \int_{t}^{\infty} \frac{1}{a_n(s)} \int_s^{\infty} p(v) \Delta v \right\}^{1/\alpha} \Delta s \quad \text{for} \ t \geq t_2.
\]

Again, integrating the above inequality from \( t_2 \) into \( t \), we obtain that for \( t \geq t_2 \)

\[
S_{n-2}(t, x(t)) \leq S_{n-2}(t_2, x(t_2)) - c^\beta \int_{t_2}^{t} \frac{1}{a_{n-1}(u)} \left\{ \int_u^{\infty} \frac{1}{a_n(s)} \right\}^{1/\alpha} \Delta s \Delta u.
\]

It follows from (6) that \( \lim_{t \to \infty} S_{n-2}(t, x(t)) = -\infty \), which is a contradiction to \( S_{n-2}(t, x(t)) = 0 \). The proof is completed.

**Lemma 5.** Assume that \( x(t) \) is an eventually positive solution of (2) such that \( S_n(t, x(t)) < 0 \) for \( t \geq T \) and \( S_i(t, x(t)) > 0 \) for \( t \geq T \) and \(0 \leq i \leq n \). Then,

\[
S_i(t, x(t)) \geq S_n^{1/\alpha}(t, x(t)) B_{n-1}(t, T)
\]

\[
\text{for} \ 0 \leq i \leq n - 1, \quad t \geq T,
\]

and there exist \( T_1 > T \) and a constant \( c > 0 \) such that

\[
x(t) \leq c B_1(t, T) \quad \text{for} \ t \geq T_1,
\]
where
\[
B_i(t, T) = \int_T^t \left[ \frac{1}{a_i(s)} \right]^{1/\alpha} \Delta s, \quad \text{if } i = n, \\
\int_T^t B_{i+1}(s, T) \Delta s, \quad \text{if } 1 \leq i \leq n - 1.
\] (13)

Proof. Since \( S_n^1(t, x(t)) < 0 \) (\( t \geq T \)), it follows that \( S_n(t, x(t)) \) is strictly decreasing on \([T, \infty)_T\). Then, for \( t \geq T \),
\[
S_{n-1}(t, x(t)) = S_{n-1}(T, x(T)) - S_{n-1}(t, x(T)) \\
= \int_T^t \left[ \frac{S_{n-1}(s, x(s))}{a_n(s)} \right]^{1/\alpha} \Delta s \\
\geq S_n^{1/\alpha}(t, x(t)) B_n(t, T)
\]
\[
S_{n-2}(t, x(t)) = S_{n-2}(T, x(T)) - S_{n-2}(t, x(T)) \\
= \int_T^t \left[ \frac{S_{n-2}(s, x(s))}{a_{n-1}(s)} \right]^{1/\alpha} \Delta s + S_{n-1}(T, x(T)) \\
\leq S_{n-1}(T, x(T)) + S_n^{1/\alpha}(T, x(T)) B_n(t, T).
\] (14)

On the other hand, we have that for \( t \geq T \),
\[
S_{n-1}(t, x(T)) = \int_T^t \left[ \frac{S_n(s, x(s))}{a_n(s)} \right]^{1/\alpha} \Delta s + S_{n-1}(T, x(T)) \\
\leq S_{n-1}(T, x(T)) + S_n^{1/\alpha}(T, x(T)) B_n(t, T).
\] (15)

Thus, there exist \( T_1 > T \) and a constant \( b_1 > 0 \) such that
\[
S_{n-1}(t, x(t)) \leq b_1 B_n(t, T) \quad \text{for } t \geq T_1.
\] (16)

Again,
\[
S_{n-2}(t, x(t)) = S_{n-2}(T_1, x(T_1)) + \int_{T_1}^t \left[ \frac{S_{n-2}(s, x(s))}{a_{n-1}(s)} \right]^{1/\alpha} \Delta s \\
\leq S_{n-2}(T_1, x(T_1)) + b_1 \int_T^{T_1} B_n(s, T) \Delta s
\]
\[
\leq S_{n-2}(T_1, x(T_1)) + b_1 \int_T^{T_1} B_n(s, T) \Delta s \\
\quad \text{for } t \geq T_1.
\] (17)

Thus, there exists a constant \( b_2 > 0 \) such that
\[
S_{n-2}(t, x(t)) \leq b_2 B_{n-1}(t, T) \quad \text{for } t \geq T_1.
\] (18)

Again,
\[
S_{n-3}(t, x(t)) = S_{n-3}(T_1, x(T_1)) + \int_{T_1}^t \left[ \frac{S_{n-3}(s, x(s))}{a_{n-2}(s)} \right]^{1/\alpha} \Delta s \\
\leq S_{n-3}(T_1, x(T_1)) + b_2 \int_T^{T_1} B_{n-1}(s, T) \Delta s
\]
\[
\leq S_{n-3}(T_1, x(T_1)) + b_2 \int_T^{T_1} B_{n-1}(s, T) \Delta s \\
\quad \text{for } t \geq T_1.
\] (19)

Thus, there exists a constant \( b_3 > 0 \) such that
\[
S_{n-3}(t, x(t)) \leq b_3 \int_T^{T_1} B_{n-1}(s, T) \Delta s
\]
\[
= b_3 B_{n-2}(t, T) \quad \text{for } t \geq T_1.
\] (20)

The rest of the proof is by induction. The proof is completed. ☐

**Lemma 6** (see [2]). Let \( f : \mathbb{R} \to \mathbb{R} \) be continuously differentiable and suppose that \( g : T \to \mathbb{R} \) is delta differentiable. Then, \( f \circ g \) is delta differentiable and the formula
\[
(f \circ g)^\Delta (t) = \int_0^1 f'(\theta g(t) + (1 - \theta) g^\Delta(t)) d\theta.
\] (21)

**Lemma 7** (see [27]). If \( A, B \) are nonnegative and \( \lambda > 1 \), then
\[
A^\Delta - \lambda AB^{\lambda-1} + (\lambda - 1) B^\lambda \geq 0.
\] (22)

Now, we state and prove our main results.

**Theorem 8.** Suppose that (5) and (6) hold. If there exists a positive nondecreasing delta differentiable function \( \theta \) such that for all sufficiently large \( T \in [t_0, \infty)_T \) and for any positive constants \( c_1, c_2 \), there is a \( T_1 > T \) such that
\[
\limsup_{t \to \infty} \int_{T_1}^t \left[ \frac{\theta(s - \Delta s)}{B^\alpha(s, T) \delta_1(t, T, c_1, c_2)} \right] \Delta s = \infty,
\] (23)

where
\[
\delta_1(t, T, c_1, c_2) = \begin{cases} 
 c_1, & \text{if } \alpha < \beta, \\
 1, & \text{if } \alpha = \beta, \\
 c_2 B_t^{\alpha-\beta}(t, T), & \text{if } \alpha > \beta,
\end{cases}
\] (24)

and \( B_t(t, T) \) is as in Lemma 5. Then, every solution of (2) is either oscillatory or tends to zero.

Proof. Assume that (2) has a nonoscillatory solution \( x(t) \) on \([t_0, \infty)_T\). Then, without loss of generality, there is a \( t_2 \geq t_0 \), sufficiently large, such that \( x(t) > 0 \) for \( t \geq t_2 \). Therefore, we get from Lemma 4 that there exists \( t_2 \geq t_1 \) such that
\[
(i) \ S_n^2(t, x(t)) < 0 \quad \text{for } t \geq t_2;
\]
\[
(ii) \text{either } S_i(t, x(t)) > 0 \quad \text{for } t \geq t_2 \text{ and } 0 \leq i \leq n \text{ or } \lim_{t \to \infty} \delta(t) = 0.
\]
Let $S_i(t, x(t)) > 0$ for $t \geq t_2$ and $0 \leq i \leq n$. Consider
\[
 w(t) = \theta(t) \frac{S_n(t, x(t))}{x^\beta(t)} \quad \text{for } t \geq t_2.
\]
(25)

It follows from Lemma 6 that
\[
 \left( x^\beta \right)^\Delta (t) = \beta x^\Delta (t) \int_0^1 (h x(t) + (1 - h) x(t))^\beta - 1 \, dh > 0
\]
for $t \geq t_2$.
(26)

Then,
\[
 w^\Delta (t) \leq \frac{\theta^\Delta (t)}{B^\alpha_1(t, t_2)} x^{\alpha - \beta} (t) - \theta (t) p (t) \quad \text{for } t \geq t_2.
\]
(28)

Now, we consider the following three cases.

Case 1. If $\alpha = \beta$, then
\[
 x^{\alpha - \beta} (t) = 1 \quad \text{for } t \geq t_2.
\]
(29)

Case 2. If $\alpha > \beta$, then it follows from (12) that there exist $t_3 > t_2$ and a constant $c_0 > 0$ such that
\[
 x^{\alpha - \beta} (t) \leq c_0 B^{\alpha - \beta}_1 (t, t_2) \quad \text{for } t \geq t_3.
\]
(30)

Case 3. If $\alpha < \beta$, then
\[
 x(t) \geq x(t_2) \quad \text{for } t \geq t_2.
\]
(31)

Thus,
\[
 x^{\alpha - \beta} (t) \leq c_1 = x^{\alpha - \beta} (t_2) \quad \text{for } t \geq t_3.
\]
(32)

From (27)–(32), we obtain
\[
 w^\Delta (t) \leq \frac{\theta^\Delta (t)}{B^\alpha_1(t, t_2)} \delta_1 \left( t, t_2, c_1, c_2 \right) - \theta (t) p (t) \quad \text{for } t \geq t_3.
\]
(33)

Integrating the above inequality from $t_3$ into $t$, we have
\[
 \int_{t_3}^{t} \left[ \theta (s) p (s) - \frac{\theta^\Delta (s)}{B^\alpha_1(s, t_2)} \delta_1 \left( s, t_2, c_1, c_2 \right) \right] \Delta s \leq w (t) < \infty,
\]
(34)

which gives a contradiction to (23). The proof is completed.

\begin{theorem}
 Suppose that (5) and (6) hold. If there exists a positive nondecreasing delta differentiable function $\theta$ such that for all sufficiently large $T \in [t_0, \infty)_T$ and for any positive constants $c_1, c_2$, there is a $T_1 > T$ such that
\[
 \limsup_{t \to \infty} \int_{T_1}^{t} \left[ \theta (s) p (s) - \frac{\alpha / \beta \theta^\alpha (s) \left( \delta (s) (B^\alpha_1(s, t) \theta (s) \delta_2 (s, T, c_1, c_2) \right) \Delta s}{(\alpha + 1) \delta^\alpha_1 (B_2 (s, T) \theta (s) \delta_2 (s, T, c_1, c_2))} \right] \Delta s
\]
\[
 = \infty,
\]
(35)

where
\[
 \delta_2 (t, T, c_1, c_2) = \begin{cases}
 c_1 & \text{if } \alpha < \beta, \\
 1, & \text{if } \alpha = \beta, \\
 c_2 B_1^\beta / (\delta (\sigma) T), & \text{if } \alpha > \beta,
\end{cases}
\]
(36)

and $B_1(t, T), B_2(t, T)$ are as in Lemma 5. Then, every solution of (2) is either oscillatory or tends to zero.

\begin{proof}
 Assume that (2) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_T$. Then, without loss of generality, there is a $t_1 \geq t_0$, sufficiently large, such that $x(t) > 0$ for $t \geq t_1$. Therefore, we get from Lemma 4 that there exists $t_2 \geq t_1$ such that
\begin{enumerate}
  \item[(i)] $S_{\alpha}^\mu (t, x(t)) < 0$ for $t \geq t_2$;
  \item[(ii)] either $S_i(t, x(t)) > 0$ for $t \geq t_2$ and $0 \leq i \leq n$ or $\lim_{t \to \infty} x(t) = 0$.
\end{enumerate}
Let $S_i(t, x(t)) > 0$ for $t \geq t_2$ and $0 \leq i \leq n$. Note that
\[
 \left( x^\beta \right)^\Delta = \beta x^\Delta \int_0^1 (h x(t) + (1 - h) x(t))^\beta - 1 \, dh
\]
\[
 = \beta x^\Delta \int_0^1 \frac{(h x(t) + (1 - h) x(t))^\beta}{h x(t) + (1 - h) x(t)} \, dh
\]
\[
 \geq \beta x^\Delta \frac{x^\beta}{x^\sigma}.
\]
(37)

From (11), we have
\[
 \frac{\left( x^\beta \right)^\Delta}{x^\beta} \geq \beta x^\Delta \frac{x^\beta}{x^\sigma} \geq \beta \frac{S_{\alpha}^\mu (s, x) B_2 (s, t_2)}{a_1 x^\sigma}
\]
\[
 \geq \beta \frac{S_{\alpha}^\mu (s, x)^{1/\alpha} B_2 (s, t_2)}{a_1 x^\sigma}
\]
\[
 \geq \beta \frac{\left( \frac{S_{\alpha}^\mu (s, x)}{a_1 (\theta^\alpha) \left( \delta (s) (B^\alpha_1(s, T) \theta (s) \delta_2 (s, T, c_1, c_2) \right) \Delta s} \right)^{1/\alpha} B_2 (s, t_2)}{a_1 x^\sigma}.
\]
(38)
Then it follows from (27) that for \( t \geq t_2 \),
\[
\begin{align*}
\omega^\Delta &= \left[ \frac{\partial}{\partial x} \right]^\Delta S^\alpha_n (\cdot, x) + \frac{\partial}{\partial x} \sigma^\alpha_n (\cdot, x) \\
&= \left[ \frac{\partial^\Delta}{(x^\beta)_{\alpha}} - \frac{\theta (x^\beta)^{\Delta}}{x^\beta (x^\beta)_{\alpha}} \right] S^\alpha_n (\cdot, x) - \partial p \\
&\leq \frac{\theta^\Delta}{\theta^\beta} w^\Delta - \frac{\partial B_2 (\cdot, t_2) \theta \left( u^\alpha \right)^{1+(1/\alpha)} - \theta p}{a_1},
\end{align*}
\]
(39)

Now, we consider the following three cases.

**Case 1.** If \( \alpha = \beta \), then
\[
(x^\alpha)^{(\beta/\alpha) - 1} (t) = 1 \quad \text{for } t \geq t_2.
\]
(40)

**Case 2.** If \( \alpha > \beta \), then it follows from (12) that there exist \( t_3 > t_2 \) and a constant \( c \) such that
\[
x(t) \leq c B_1 (t_3, t_2) \quad \text{for } t \geq t_3.
\]
(41)

Thus,
\[
(x^\alpha)^{(\beta/\alpha) - 1} (t) \geq c_2 B_1^{(\beta/\alpha) - 1} (\sigma (t), t_2),
\]
with \( c_2 = e^{(\beta/\alpha) - 1} \).
(42)

**Case 3.** If \( \alpha < \beta \), then
\[
x(t) \geq x(t_2) \quad \text{for } t \geq t_2.
\]
(43)

Thus,
\[
(x^\alpha)^{(\beta/\alpha) - 1} (t) \geq c_1 = x^{(\beta/\alpha) - 1} (t_2).
\]
(44)

From (39)–(44), we obtain that for \( t \geq t_3 \),
\[
\begin{align*}
\omega^\Delta &\leq w^\sigma \theta^\alpha - \frac{\partial B_2 (\cdot, t_2) \theta \delta_2 (\cdot, t_2, c_1, c_2)}{a_1} \\
&= -\frac{\beta B_2 (\cdot, t_2) \theta \delta_2 (\cdot, t_2, c_1, c_2)}{a_1} \\
&\quad \left\{ \frac{w^\alpha}{(\theta^\alpha)^{1+(1/\alpha)}} - \frac{w^\alpha}{\theta^\beta B_2 (\cdot, t_2) \theta \delta_2 (\cdot, t_2, c_1, c_2)} \right\} - \theta p.
\end{align*}
\]
(45)

Let
\[
A = \frac{w^\sigma}{\theta^\alpha}, \quad B = \left[ \frac{\alpha a_1 \theta^\Delta}{(\alpha + 1) \beta B_2 (\cdot, t_2) \theta \delta_2 (\cdot, t_2, c_1, c_2)} \right]^\alpha,
\]
with \( \lambda = 1 + 1/\alpha \). By Lemma 7, we have
\[
\omega^\Delta \leq \frac{(\alpha/\beta)^\sigma (\theta^\Delta)^{\alpha + 1} a_1^\alpha}{(\alpha + 1)^{\alpha + 1} (B_2 (\cdot, t_2) \theta \delta_2 (\cdot, t_2, c_1, c_2))^\alpha} - \theta p.
\]
(47)

Integrating the above inequality from \( t_3 \) into \( t \), it follows that
\[
\int_{t_3}^{t} \left[ \frac{\theta (s) p(s)}{(\alpha/\beta)^\sigma (\theta^\Delta)^{\alpha + 1} a_1^\alpha}{(\alpha + 1)^{\alpha + 1} (B_2 (s, t_2) \theta \delta_2 (s, t_2, c_1, c_2))^\alpha} \right] \Delta s \\
\leq w(t_3) < \infty,
\]
which gives a contradiction to (35). The proof is completed.

**Remark 10.** The trick used in the proofs of Theorems 8 and 9 is from [16].

**Theorem 11.** Suppose that (5) and (6) hold. If for all sufficiently large \( T \in [t_0, \infty) \),
\[
\int_{t}^{\infty} p(u) \left[ \int_{u}^{a} \frac{\Delta s}{a_1 (s)} \right]^\beta \Delta u = \infty,
\]
(49)
then every solution of (2) is either oscillatory or tends to zero.

**Proof.** Assume that (2) has a nonoscillatory solution \( x(t) \) on \([t_0, \infty) \). Then, without loss of generality, there is a \( t_1 \geq t_0 \), sufficiently large, such that \( x(t) > 0 \) for \( t \geq t_1 \). Therefore, we get from Lemma 4 that there exists \( t_2 \geq t_1 \) such that
\[
(i) \quad S^\alpha_n (t, x(t)) < 0 \quad \text{for } t \geq t_2;
\]
\[
(ii) \quad \text{either } S_i (t, x(t)) > 0 \quad \text{for } t \geq t_2 \quad \text{and } 0 \leq i \leq n \quad \text{or} \quad \lim_{t \to \infty} x(t) = 0.
\]

Let \( S_i (t, x(t)) > 0 \) for \( t \geq t_2 \) and \( 0 \leq i \leq n \). Then, for \( t \geq t_2 \),
\[
x(t) = x(t_2) + \int_{t_2}^{t} \frac{S_i (s, x(s))}{a_1 (s)} \Delta s
\]
(50)
\[
\geq S_i (t, x(t_2)) \int_{t_2}^{t} \frac{\Delta s}{a_1 (s)}.
\]

It follows from (2) that
\[
-S^\Delta_n (t, x(t)) = p(t) \left[ S_i (t_2, x(t_2)) \int_{t_2}^{t} \frac{\Delta s}{a_1 (s)} \right]^\beta.
\]
(51)

Integrating the above inequality from \( t_3 \) into \( \infty \), we have
\[
S^\Delta_n (t_2, x(t_2)) \geq S^\Delta_i (t_2, x(t_2)) \int_{t_2}^{\infty} p(t) \left[ \int_{t_2}^{t} \frac{\Delta s}{a_1 (s)} \right]^\beta \Delta u,
\]
(52)
which gives a contradiction to (49). The proof is completed.

**Theorem 12.** Suppose that (5) and (6) hold. If for all sufficiently large \( T \in [t_0, \infty) \),
\[
\limsup_{t \to \infty} \frac{P(t)}{\Delta s} > 1,
\]
(53)
where
\[\delta_3(t,T,c_1,c_2) = \begin{cases} c_1, & \text{if } \alpha < \beta, \\ 1, & \text{if } \alpha = \beta, \\ c_2 B_1^{\beta-\alpha}(t,T), & \text{if } \alpha > \beta, \end{cases}\]

and \(B_1(t,T)\) is as in Lemma 5, then every solution of (2) is either oscillatory or tends to zero.

**Proof.** Assume that (2) has a nonoscillatory solution \(x(t)\) on \([t_0, \infty)\). Then, without loss of generality, there is \(t_1 \geq t_0\), sufficiently large, such that \(x(t) > 0\) for \(t \geq t_1\). Therefore, we get from Lemma 4 that there exists \(t_2 \geq t_1\) such that
(i) \(S^3_n(t,x(t)) < 0\) for \(t \geq t_2\);
(ii) either \(S_2(t,x(t)) > 0\) for \(t \geq t_2\) and \(0 \leq i \leq n\) or \(\lim_{t \to \infty} x(t) = 0\).

Let \(S_1(t,x(t)) > 0\) for \(t \geq t_2\) and \(0 \leq i \leq n\). Then, it follows from (2) and (11) that for \(t \geq t_2\),
\[\int_{t_2}^\infty p(s) x^\alpha(s) \Delta s \leq S_n(t,x(t)) \leq \left[\frac{x(t)}{B_1(t,t_2)}\right]^\alpha.\]

Using the fact that \(x(t)\) is strictly increasing on \([t_2, \infty)_T\), we obtain
\[x^\beta(t) \int_{t_2}^\infty p(s) \Delta s \leq \left[\frac{x(t)}{B_1(t,t_2)}\right]^\alpha.\]

Thus,
\[B_1^\alpha(t,t_2) x^{\beta-\alpha}(t) \int_{t_2}^\infty p(s) \Delta s \leq 1.\]

Now, we consider the following three cases.

**Case 1.** If \(\alpha = \beta\), then
\[x^{\beta-\alpha}(t) = 1 \quad \text{for } t \geq t_2.\]

**Case 2.** If \(\alpha > \beta\), then it follows from (12) that there exist \(t_3 > t_2\) and a constant \(c\) such that
\[x(t) \leq c B_1(t,t_2) \quad \text{for } t \geq t_3.\]

Thus,
\[x^{\beta-\alpha}(t) \geq c \beta B_1^{\beta-\alpha}(t,t_2),\]
with \(c_2 = c^{\beta-\alpha}\).

**Case 3.** If \(\alpha < \beta\), then
\[x(t) \geq x(t_2) \quad \text{for } t \geq t_2.\]

Thus,
\[x^{\beta-\alpha}(t) \geq c_2 B_1^{\beta-\alpha}(t,t_2),\]

which gives a contradiction to (53). The proof is completed.

**3. Examples**

In this section, we give some examples to illustrate our main results.

**Example 1.** Consider the following higher order dynamic equation:
\[S^3_n(t,x(t)) + \gamma x^\beta(t) = 0,\]

on an arbitrary time scale \(T\) with sup \(T = \infty\), where \(n \geq 2\), \(\alpha, \beta\) and \(S_n(t,x(t)) (0 \leq k \leq n)\) are as in (2) with \(a_k(t) = t^{\alpha-1}, a_{k-1}(t) = \cdots = a_1(t) = t, \) and \(\gamma > -1\). Then, every solution of (64) is either oscillatory or tends to zero.

**Proof.** Note that
\[\int_{t_0}^\infty \frac{1}{a_n(s)} \frac{1}{s^{\alpha-1}} \Delta s = \int_{t_0}^\infty \frac{1}{s^{\alpha-1}} \Delta s = \infty,\]

\[\int_{t_0}^\infty \frac{\Delta s}{s} = \int_{t_0}^\infty s^{\gamma} \Delta s = \infty,\]
by Example 5.60 in [4]. Pick \(t_1 > t_0\) such that
\[\int_{t_0}^{t_1} \frac{1}{a_{n-1}(u)} \left\{ \int_{t}^{t_1} \frac{1}{a_n(s)} \Delta s \right\} \Delta u > 0.\]

Then,
\[\int_{t_0}^{t_1} \frac{1}{a_{n-1}(u)} \left\{ \int_{t}^{\infty} \frac{1}{a_n(s)} \Delta s \right\} \Delta u \geq \left[ \int_{t_1}^{\infty} p(v) \Delta v \right]^{1/\alpha} \Delta u \]
\[\times \int_{t_0}^{t_1} \frac{1}{a_{n-1}(u)} \left( \int_{t}^{t_1} \frac{1}{a_n(s)} \Delta s \right) \Delta u = \infty.\]
Let $T \in [t_0, \infty)$, sufficiently large, and $u_1 > T$ such that
\[ \int_T^{u_1} (1/a_1(s)) \Delta s > 1, \]
then
\[ \int_T^\infty p(u) \left( \int_T^u \frac{1}{a_1(s)} \Delta s \right)^\beta \Delta u \]
\[ \geq \int_T^\infty p(u) \left( \int_T^1 \frac{1}{a_1(s)} \Delta s \right)^\beta \Delta u \]
(68)
\[ \geq \int_T^\infty p(u) \Delta u = \infty. \]

Thus, conditions (5), (6), and (23) are satisfied. By Theorem 8, every solution of (69) is either oscillatory or tends to zero.

**Example 2.** Consider the following higher order dynamic equation:
\[ S_n^\alpha (t, x(t)) + \frac{1}{t^{1+n}} x^\beta (t) = 0, \]
(69)
on an arbitrary time scale $T$ with $\sup T = \infty$, where $n \geq 2$, $S_n(t, x(t))$ $(0 \leq k \leq n)$ are as in (2) with $a_n(t) = 1$, $a_{n-1}(t) = t^{1/\alpha}$, $a_{n-2}(t) = \cdots = a_1(t) = t$, $0 < \gamma < \min[\alpha, \beta]$, and $\alpha, \beta$ are the quotient of odd positive integers with $\alpha \geq 1$. Then, every solution of (69) is either oscillatory or tends to zero.

**Proof.** Note that
\[ \int_{t_i}^\infty \frac{1}{a_1(s)} \Delta s = \int_{t_i}^\infty \Delta s = \infty, \]
(70)
\[ \int_{t_i}^\infty \frac{1}{a_{n-1}(s)} \Delta s = \int_{t_i}^\infty \frac{1}{s^{1/\alpha}} \Delta s = \infty, \]
(71)
\[ \int_{t_i}^\infty \frac{1}{a_i(s)} \Delta s = \int_{t_i}^\infty \Delta s = \infty \quad \text{for } 1 \leq i \leq n-2. \]

Pick $t_1 > t_0$ such that $\int_{t_i}^1 (\Delta u/\gamma u)^{1/\alpha} > 0$, then
\[ \int_{t_i}^\infty \frac{1}{a_{n-1}(u)} \left( \int_u^\infty \frac{1}{a_n(s)} \Delta s \right)^{1/\alpha} \Delta u \]
\[ = \int_{t_i}^\infty \frac{1}{u^{1/\alpha}} \left( \int_u^\infty \frac{1}{v^{1/\alpha}} \Delta v \right)^{1/\alpha} \Delta u \]
\[ \geq \frac{1}{\gamma} \int_{t_i}^\infty \frac{1}{u^{1/\alpha}} \left( \int_u^\infty \frac{1}{v^{1/\alpha}} \Delta v \right)^{1/\alpha} \Delta u \]
\[ = \frac{1}{\gamma} \int_{t_i}^\infty \frac{1}{u^{1/\alpha}} \left( \int_u^\infty \frac{1}{v^{1/\alpha}} \Delta v \right)^{1/\alpha} \Delta u \]
(72)
\[ \geq \frac{1}{\gamma} \int_{t_i}^1 \frac{1}{u^{1/\alpha}} \left( \int_u^1 \frac{1}{v^{1/\alpha}} \Delta v \right)^{1/\alpha} \Delta u \]
\[ = \frac{1}{\gamma} \int_{t_i}^1 \frac{1}{u^{1/\alpha}} \left( \int_u^1 \frac{1}{v^{1/\alpha}} \Delta v \right) \Delta u \]
\[ = \frac{1}{\gamma} \int_{t_i}^1 \frac{1}{u^{1/\alpha}} \left( \int_u^1 \frac{1}{v^{1/\alpha}} \Delta v \right) \Delta u \]
\[ = \infty. \]

Let $M = \max[c_1, c_2]$ with $c_1, c_2$ being positive constants, $\rho = \min[\alpha, \beta]$, and $\gamma < \tau < \min[1, \beta]$. Pick $T_1 > T > 0$ such that
\[ \frac{1}{t^{\rho}} \geq \frac{2}{t^{\tau}} \geq \frac{2M}{(1/2)^{n+1/\alpha}} (t - 2^{n-1}T)^\rho \quad \text{for } t \geq T_1. \]

Let $\theta(t) = t$, then
\[ B_1(t, T) \]
\[ = \int_T^t \frac{1}{a_1(u_1)} \]
\[ \times \left[ \int_T^u \frac{1}{a_2(u_2)} \right] \]
\[ \times \left[ \cdots \left[ \int_T^u \frac{1}{a_{u_{n-1}}(u_{n-1})} \right] \right] \]
\[ \int_T^u \Delta u_{n-1} \cdots \Delta u_2 \Delta u_1 \]
\[ \geq \left( \frac{1}{2} \right)^{n+1/\alpha} (t - 2^{n-1}T), \]
\[ \int_{T_1}^{t_1} \left[ \frac{1}{T} \right] \left[ \frac{\theta^\Delta (s)}{B_1^\Delta (s, T)} \delta_1 (s, T) \right] \Delta s \]
\[ = \int_{T_1}^{t_1} \left[ \frac{1}{T} - \frac{1}{B_1^\Delta (s, T)} \delta_1 (s, T, c_1, c_2) \right] \Delta s \]
\[ \geq \int_{T_1}^{t_1} \left[ \frac{2}{T^\gamma} \left( \frac{1}{2} \right)^{n+1/\alpha} (t - 2^{n-1}T)^\rho \right] \Delta s \]
\[ \geq \int_{T_1}^{t_1} \frac{1}{T^\gamma} \Delta s. \]
(73)
Thus,
\[ \limsup_{t \to \infty} \int_{T_1}^{t_1} \left[ \frac{\theta^\Delta (s)}{B_1^\Delta (s, T)} \delta_1 (s, T, c_1, c_2) \right] \Delta s = \infty. \]
(74)
So conditions (5), (6), and (23) are satisfied. Then, by Theorem 8, every solution of (69) is either oscillatory or tends to zero.\qed
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References