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Research Article

Central Configurations for Newtonian $N + 2p + 1$-Body Problems

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We show the existence of spatial central configurations for the $N + 2p + 1$-body problems. In the $N + 2p + 1$-body problems, N bodies are at the vertices of a regular N-gon T ; $2p$ bodies are symmetric with respect to the center of T, and located on the straight line which is perpendicular to the regular N-gon T and passes through the center of T; the $N + 2p + 1$th is located at the the center of T. The masses located on the vertices of the regular N-gon are assumed to be equal; the masses located on the same line and symmetric with respect to the center of T are equal.

1. Introduction and Main Results

The Newtonian n-body problems [1–3] concern with the motions of n particles with masses $m_j \in R^+$ and positions $q_j \in R^3$ $(j = 1, 2, \ldots, n)$, and the motion is governed by Newton's second law and the Universal law:

$$m_j \ddot{q}_j = \frac{\partial U(q)}{\partial q_j}, \tag{1}$$

where $q = (q_1, q_2, \ldots, q_n)$ and with Newtonian potential:

$$U(q) = \sum_{1 \leqslant j < k \leqslant n} \frac{m_j m_k}{|q_j - q_k|}. \tag{2}$$

Consider the space

$$X = \left\{ q = (q_1, q_2, \ldots, q_n) \in R^{3n} : \sum_{j=1}^{N} m_j q_j = 0 \right\}, \tag{3}$$

that is, suppose that the center of mass is fixed at the origin of the space. Because the potential is singular when two particles have the same position, it is natural to assume that the configuration avoids the collision set $\Delta = \{q = (q_1, \ldots, q_N) : q_j = q_k \text{ for some } k \neq j\}$. The set $X \setminus \Delta$ is called the configuration space.

Definition 1 (see [2, 3]). A configuration $q = (q_1, q_2, \ldots, q_n) \in X \setminus \Delta$ is called a central configuration if there exists a constant λ such that

$$\sum_{j=1, j \neq k}^{n} \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k, \quad 1 \leqslant k \leqslant n. \tag{4}$$

The value of constant λ in (4) is uniquely determined by

$$\lambda = \frac{U}{I}, \tag{5}$$

where

$$I = \sum_{k=1}^{n} m_k |q_k|^2. \tag{6}$$

Since the general solution of the n-body problem cannot be given, great importance has been attached to search for particular solutions from the very beginning. A homographic solution is a configuration which is preserved for all time. Central configurations and homographic solutions are linked by the Laplace theorem [3]. Collapse orbits and parabolic orbits have relations with the central configurations [2, 4–6], so finding central configurations becomes very important. The main general open problem for the central configurations

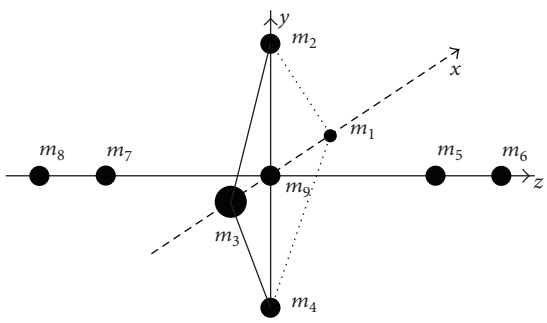

Figure 1

is due to Wintner [3] and Smale [7]: is the number of central configurations finite for any choice of positive masses $m_1, \ldots, m_n$? Hampton and Moeckel [8] have proved this conjecture for any given four positive masses.

For 5-body problems, Hampton [9] provided a new family of planar central configurations, called stacked central configurations. A stacked central configuration is one that has some proper subset of three or more points forming a central configuration. Ouyang et al. [10] studied pyramidal central configurations for Newtonian $N+1$-body problems; Zhang and Zhou [11] considered double pyramidal central configurations for Newtonian $N+2$-body problems; Mello and Fernandes [12] analyzed new classes of spatial central configurations for the $N+3$-body problem. Llibre and Mello studied triple and quadruple nested central configurations for the planar n-body problem. There are many papers studying central configuration problems such as [13–22].

Based on the above works, we study stacked central configuration for Newtonian $N+2p+1$-body problems. In the $N+2p+1$-body problems, N bodies are at the vertices of a regular N-gon T, and $2p$ bodies are symmetrically located on the same straight line which is perpendicular to T and passes through the center of T; the $N+2p+1$th body is located at the center of T. The masses located on the vertices of the regular N-gon are equal; the masses located on the line and symmetric with respect to the center of T are equal. (see Figure 1 for $N=4$ and $p=2$).

In this paper we will prove the following result.

Theorem 2. *For $N+2p+1$-body problem in R^3 where $N \geq 2$ and $p \geq 1$, there is at least one central configuration such that N bodies are at the vertices of a regular N-gon T, and $2p$ bodies are symmetric with respect to the center of the regular N-gon T, and located on a line which is perpendicular to the regular N-gon T; the $N+2p+1$th body is located at the center of T. The masses at the vertices of T are equal and the masses symmetric with respect to the center of T are equal.*

2. The Proof of Theorem 2

Our approach to Theorem 2 is inspired by the of arguments of Corbera et al. in [23].

2.1. Equations for the Central Configurations of $N+2p$-Body Problems. To begin, we take a coordinate system which simplifies the analysis. The particles have positions given by $q_j = (\cos\alpha_j, \sin\alpha_j, 0)$, where $\alpha_j = ((j-1)/N)2\pi$, $j = 1, \ldots, N$; $q_{N+j} = (0,0,r_j)$, $q_{N+j+p} = (0,0,-r_j)$, where $j = 1, \ldots, p$; $q_{N+2p+1} = (0,0,0)$.

The masses are given by $m_1 = m_2 = \cdots = m_N = 1$, $m_{N+j} = m_{N+j+p} = M_j$, where $j = 1, \ldots, p$, $m_{N+2p+1} = M_0$.

Notice that $(q_1, \ldots, q_N, q_{N+1}, \ldots, q_{N+2p}, q_{N+2p+1})$ is a central configuration if and only if

$$\sum_{j=1, j\neq k}^{N+2p+1} \frac{m_j m_k}{\left|q_j - q_k\right|^3}\left(q_j - q_k\right) = -\lambda m_k q_k, \quad 1 \leqslant k \leqslant N+2p. \tag{7}$$

By the symmetries of the system, (7) is equivalent to the following equations:

$$\sum_{j=1, j\neq k}^{N+2p+1} \frac{m_j m_k}{\left|q_j - q_k\right|^3}\left(q_j - q_k\right) = -\lambda m_k q_k, \quad k = 1, N+1, \ldots, N+p, \tag{8}$$

that is,

$$\begin{aligned} -\lambda(1,0,0) = & -\beta(1,0,0) - (1,0,0) M_0 \\ & - \sum_{i=1}^{p} \frac{2}{\left|1+r_i^2\right|^{3/2}} M_i (1,0,0), \end{aligned} \tag{9}$$

where $\beta = (1/4)\sum_{j=1}^{N-1} \csc(\pi j/N)$

$$\begin{aligned} -\lambda(0,0,r_1) = & -\frac{N}{\left|1+r_1^2\right|^{3/2}}(0,0,r_1) - \left(0,0,\frac{1}{r_1^2}\right) M_0 \\ & - \frac{1}{\left|2r_1\right|^2} M_1 (0,0,1) \\ & + \sum_{i=2}^{p}\left[\frac{1}{\left|r_i - r_1\right|^2} - \frac{1}{\left|r_i + r_1\right|^2}\right] M_1 (0,0,1), \\ -\lambda(0,0,r_j) = & -\frac{N}{\left|1+r_j^2\right|^{3/2}}(0,0,r_j) - \left(0,0,\frac{1}{r_j^2}\right) M_0 \\ & - \sum_{1\leq i<j}\left(\frac{1}{\left|r_i - r_j\right|^2} + \frac{1}{\left|r_i + r_j\right|^2}\right) M_i (0,0,1) \\ & - \frac{1}{\left|2r_j\right|^2} M_j (0,0,1) \\ & + \sum_{j<i\leq p}\left(\frac{1}{\left|r_i - r_j\right|^2} - \frac{1}{\left|r_i + r_j\right|^2}\right) M_i (0,0,1), \\ & j = 2, \ldots, p-1 \end{aligned} \tag{10}$$

$$-\lambda\left(0,0,r_p\right) = -\frac{N}{\left|1+r_p^2\right|^{3/2}}\left(0,0,r_p\right) - \left(0,0,\frac{1}{r_p^2}\right)M_0 - \sum_{1\le i<p}\left(\frac{1}{\left|r_i-r_p\right|^2}+\frac{1}{\left|r_i+r_p\right|^2}\right)M_i\left(0,0,1\right) - \frac{1}{\left|2r_p\right|^2}M_p\left(0,0,1\right). \tag{11}$$

In order to simplify the equations, we defined

$a_{0,j} = -2/|1+r_j^2|^{3/2}$, $a_{j,j} = -1/4r_j^3$, where $j = 1,\ldots,p$;

$a_{i,j} = (1/|r_i - r_j|^2 r_i - 1/|r_i + r_j|^2 r_i)$, when $i < j$;

$a_{i,j} = -(1/|r_i - r_j|^2 r_i + 1/|r_i + r_j|^2 r_i)$, when $i > j$;

$a_{0,0} = a_{i,0} = 1$ when $i = 1,\ldots,p$;

$b_0 = \beta + M_0$, $b_i = N/|1+r_i^2|^{3/2} + M_0/r_i^3$, where $i = 1,\ldots,p$.

Equations (9)–(11) can be written as a linear system of the form $AX = b$ given by

$$\begin{pmatrix} 1 & a_{0,1} & a_{0,2} & a_{0,3} & \cdots & a_{0,p} \\ 1 & a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,p} \\ 1 & a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,p} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_{p,1} & a_{p,2} & a_{p,3} & \cdots & a_{p,p} \end{pmatrix}\begin{pmatrix} \lambda \\ M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_P \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_p \end{pmatrix}. \tag{12}$$

The column vector is given by the variables $X = (\lambda, M_1, M_2,\ldots,M_p)^T$. Since the a_{ij} is function of $r_1, r_2,\ldots,r_p$, we write the coefficient matrix as $A_p(r_1, r_2,\ldots,r_p)$.

2.2. For $p = 1$. We need the next lemma.

Lemma 3 (see [12]). *Assuming $m_1 = \cdots = m_N = 1$, $m_{N+1} = m_{N+2} = M_1$, there is a nonempty interval $I \subset R$, $M_0(r_1)$ and $M_1(r_1)$, such that for each $r_1 \in I$, $(q_1,\ldots,q_N,q_{N+1},q_{N+2},q_{N+3})$ forms a central configuration of the $N+2+1$-body problem.*

For $p = 1$, system (12) becomes

$$\begin{pmatrix} 1 & a_{0,1} \\ 1 & a_{1,1} \end{pmatrix}\begin{pmatrix} \lambda \\ M_1 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}, \tag{13}$$

$$\left|A_1\left(r_1\right)\right| = a_{1,1} - a_{0,1} = -\frac{1}{4r_1^3} + \frac{2}{\left|1+r_1^2\right|^{3/2}}. \tag{14}$$

If we consider $|A_1(r_1)|$ as a function of r_1, then $|A_1(r_1)|$ is an analytic function and nonconstant. By Lemma 3, there exists a $\check{r}_1 \in I$ such that (13) has a unique solution (λ, M_1) satisfying $\lambda > 0$ and $M_1 > 0$.

2.3. For All $p>1$. The proof for $p \ge 1$ is done by induction. We claim that there exists $0 < r_1 < r_2 < \cdots < r_p$ such that system (12) has a unique solution $\lambda = \lambda(r_1,\ldots,r_p) > 0$, $M_i = M_i(r_1,\ldots,r_p) > 0$ for $i = 1,\ldots,p$. We have seen that the claim is true for $p = 1$. We assume the claim is true for $p-1$ and we will prove it for p. Assume by induction hypothesis that there exists $0 < \tilde{r}_1 < \tilde{r}_2 < \cdots < \tilde{r}_{p-1}$ such that system (12) has a unique solution $\tilde{\lambda} = \lambda(\tilde{r}_1,\tilde{r}_2,\ldots,\tilde{r}_{p-1}) > 0$ and $\widetilde{M}_i = M_i(\tilde{r}_1,\tilde{r}_2,\ldots,\tilde{r}_{p-1}) > 0$ for $i = 1,2,\ldots,p-1$.

We need the next lemma.

Lemma 4. *There exists $\tilde{r}_p > \tilde{r}_{p-1}$ such that $\tilde{\lambda} = \lambda(\tilde{r}_1,\tilde{r}_2,\ldots,\tilde{r}_{p-1})$, $\widetilde{M}_i = M_i(\tilde{r}_1,\tilde{r}_2,\ldots,\tilde{r}_{p-1})$ for $i = 1,2,\ldots,p-1$ and $\widetilde{M}_p = 0$ is a solution of* (12).

Proof. Since $M_p = \widetilde{M}_p = 0$, we have that the first $p-1$ equation of (12) is satisfied when $\lambda = \tilde{\lambda}$, $M_i = \widetilde{M}_i$ for $i = 1,2,\ldots,p-1$ and $M_p = \widetilde{M}_p = 0$. Substituting this solution into the last equation of (12), we let

$$f\left(r_p\right) = \tilde{\lambda} + a_{p,1}\widetilde{M}_1 + \cdots + a_{p,p-1}\widetilde{M}_{p-1} - \frac{N}{\left|1+r_p^2\right|^{3/2}} - \frac{M_0}{r_p^3}. \tag{15}$$

We have that

$$\lim_{r_p\to+\infty} f\left(r_p\right) = \tilde{\lambda} > 0, \qquad \lim_{r_p\to\tilde{r}_{p-1}^+} f\left(r_p\right) = -\infty. \tag{16}$$

Therefore there exists at least a value $r_p = \tilde{r}_p > \tilde{r}_{p-1}$ satisfying equation $f(r_p) = 0$. This completes the proof of Lemma 4. □

By using the implicit function theorem, we will prove that the solution of (12) given in Lemma 3 can be continued to a solution with $M_p > 0$.

Let $s = (\lambda, r_1,\ldots,r_p, M_1,\ldots,M_p)$, we define

$$\begin{aligned} g_0(s) &= \lambda + a_{0,1}M_1 + \cdots + a_{0,p}M_p - b_0, \\ g_1(s) &= \lambda + a_{1,1}M_1 + \cdots + a_{1,p}M_p - b_1, \\ g_2(s) &= \lambda + a_{2,1}M_1 + \cdots + a_{2,p}M_p - b_2, \\ &\vdots \\ g_p(s) &= \lambda + a_{p,1}M_1 + \cdots + a_{p,p}M_p - b_p. \end{aligned} \tag{17}$$

It is not difficult to see that the system (12) is equivalent to $g_i(s) = 0$ for $i = 0,1,\ldots,p$.

Let $\tilde{s} = (\tilde{\lambda}, \widetilde{M}_1, \widetilde{M}_2,\ldots,\widetilde{M}_p, \tilde{r}_1,\ldots,\tilde{r}_p)$ be the solution of system (12) given in Lemma 4. The differential of (17) with

respect to the variables $(\lambda, M_1, M_2, \ldots, M_{p-1}, r_p)$ is

$$D_p(\tilde{s}) = \begin{vmatrix} 1 & a_{0,1} & a_{0,2} & \cdots & a_{0,p-1} & 0 \\ 1 & a_{1,1} & a_{1,2} & \cdots & a_{1,p-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_{p-1,1} & a_{p-1,2} & \cdots & a_{p-1,p-1} & 0 \\ 1 & a_{p,1} & a_{p,2} & \cdots & a_{p,p-1} & \dfrac{\partial g_p(\tilde{s})}{\partial r_p} \end{vmatrix}$$

$$= \left|A_{p-1}\left(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_{p-1}\right)\right| \frac{\partial g_p(\tilde{s})}{\partial r_p},$$

$$\frac{\partial g_p(\tilde{s})}{\partial r_p} = \sum_{i=1}^{p-1} \left[\frac{2}{\left|r_p - r_i\right|^3 r_p} + \frac{1}{\left|r_p - r_i\right|^2 r_p^2} + \frac{2}{\left|r_p + r_i\right|^3 r_p} + \frac{1}{\left|r_p + r_i\right|^3 r_p^2} \right] \widetilde{M}_i + \frac{\left(3Nr_p\right)}{\left|1 + r_p^2\right|^{5/2}} + \frac{3M_0}{r_p^4} > 0. \tag{18}$$

We have assumed that $0 < \tilde{r}_1 < \tilde{r}_2 < \cdots < \tilde{r}_{p-1}$ exists such that system (12) with $p-1$ instead of p has a unique solution, so $|A_{p-1}(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_{p-1})| \neq 0$; therefore $D(\tilde{s}) \neq 0$. Applying the Implicit Function Theorem, there exists a neighborhood U of $(\widetilde{M}_p, \tilde{r}_1, \tilde{r}_1, \ldots, \tilde{r}_{p-1})$, and unique analytic functions $\lambda = \lambda(\widetilde{M}_p, \tilde{r}_1, \tilde{r}_1, \ldots, \tilde{r}_{p-1})$, $M_i = M_i(\widetilde{M}_p, \tilde{r}_1, \tilde{r}_1, \ldots, \tilde{r}_{p-1})$ for $i = 1, \ldots, p-1$ and $r_p = r_p(M_p, r_1, r_2, \ldots, r_{p-1})$, such that $(\lambda, M_1, M_2, \ldots, M_p)$ is the solution of the system (12) for all $(M_p, r_1, r_2, \ldots, r_{p-1}) \in U$. The determinant is calculated as

$$\begin{aligned} &\left|A_p\left(r_1, r_2, \ldots, r_p\right)\right| \\ &\quad = a_{p,p}\left|A_{p-1}\left(r_1, r_2, \ldots, r_{p-1}\right)\right| + \sum_{i=0}^{p-1} a_{p,i} B_{p,i} \\ &\quad = \frac{-1}{\left|2r_p\right|^2}\left|A_{p-1}\left(r_1, r_2, \ldots, r_{p-1}\right)\right| + \sum_{i=0}^{p-1} a_{p,i} B_{p,i}, \end{aligned} \tag{19}$$

where $B_{p,i}$ is the algebraic cofactor of $a_{p,i}$.

We see that $|A_{p-1}(r_1, r_2, \ldots, r_{p-1})|$, $a_{p,i}$ and $B_{p,i}$ for $i = 0, 1, \ldots, p-1$ do not contain the factor $1/|2r_p|^2$. If we consider $|A_p(r_1, r_2, \ldots, r_p)|$ as a function of r_p, then $|A_p(r_1, r_2, \ldots, r_p)|$ is analytic and nonconstant. We can find $(\tilde{\tilde{r}}_1, \tilde{\tilde{r}}_2, \ldots, \tilde{\tilde{r}}_p)$ sufficiently close to $(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_p)$ such that $|A_p(\tilde{\tilde{r}}_1, \tilde{\tilde{r}}_2, \ldots, \tilde{\tilde{r}}_p)| \neq 0$ and therefore, a solution $(\lambda, M_1, \ldots, M_p)$ of system (12) is satisfying $\lambda > 0$, $M_i > 0$ for $i = 1, \ldots, p$.

The proof of Theorem 2 is completed.

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