Research Article

Results on Difference Analogues of Valiron-Mohon'ko Theorem

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The classical Valiron-Mohon'ko theorem has many applications in the study of complex equations. In this paper, we investigate rational functions in $f(z)$ and the shifts of $f(z)$. We get some results on their characteristic functions. These results may be viewed as difference analogues of Valiron-Mohon'ko theorem.

1. Introduction and Results

We use the basic notions of Nevanlinna's theory in this work (see [1, 2]). Let $f(z)$ be a meromorphic function. We say that a meromorphic function $\alpha(z)$ is a small function of $f(z)$ if $T(r, \alpha) = S(r, f)$, where $S(r, f) = o(T(r, f))$ outside a possible exceptional set of finite logarithmic measure.

The Valiron-Mohon'ko theorem has been proved to be an extremely useful tool in the study of meromorphic solutions of differential, difference, and functional equations. It is stated as follows.

Theorem A (see [3, page 29]). *Let f be a meromorphic function. Then for all irreducible rational functions in f*

$$R(z, f) = \frac{\sum_{i=0}^p a_i(z) f^i}{\sum_{j=0}^q b_j(z) f^j} \quad (1)$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$ such that

$$\begin{aligned} T(r, a_i) &= S(r, f), \quad i = 0, \dots, p, \\ T(r, b_j) &= S(r, f), \quad j = 0, \dots, q, \end{aligned} \quad (2)$$

the characteristic function of $R(r, f(z))$ satisfies

$$T(r, R(z, f)) = \max\{p, q\} T(r, f) + S(r, f). \quad (3)$$

Recently, a number of papers have focused on difference analogues of Nevanlinna's theory; see, for instance, [4–12].

Among these papers, difference polynomials are investigated extensively (see [5, 9–11]). But the difference analogues of Valiron-Mohon'ko theorem have not been established. In this paper, we are devoted to this work.

A difference polynomial of $f(z)$ is an expression of the form

$$H(z, f) = \sum_{\lambda \in J} a_{\lambda}(z) \prod_{j=1}^{\tau_{\lambda}} f(z + \delta_{\lambda,j})^{\mu_{\lambda,j}}, \quad (4)$$

where J is an index set, $\delta_{\lambda,j}$ are complex constants, and $\mu_{\lambda,j}$ are nonnegative integers. In what follows, we assume that the coefficients of difference polynomials are, unless otherwise stated, small functions. The maximal total degree of $H(z, f)$ in $f(z)$ and the shifts of $f(z)$ is defined by

$$\deg_f H = \max_{\lambda \in J} \sum_{j=1}^{\tau_{\lambda}} \mu_{\lambda,j}. \quad (5)$$

First, we investigate the rational function

$$R_1(z, f) = \frac{P(z, f)}{d_1(z) f(z + c) + d_0(z)}, \quad (6)$$

where c is an arbitrary complex number, and $d_0(z)$ and $d_1(z)$ are small functions of $f(z)$ with $d_0(z) \not\equiv 0$ or $d_1(z) \not\equiv 0$. Our result is stated as follows.

Theorem 1. *Let $f(z)$ be a meromorphic function of finite order such that $N(r, f) = S(r, f)$. Suppose that $P(z, f) \not\equiv 0$ is a*

difference polynomial in $f(z)$ and that $R_1(z, f)$ is of the form (6). Then

$$T(r, R_1) \leq (\deg_f P) T(r, f) + S(r, f). \quad (7)$$

In many papers (see, for instance, [7, 13, 14]), linear difference expressions often appear. Concerning their characteristic functions, we have the following corollary, which is obtained easily from Theorem 1.

Corollary 2. Let $f(z)$ be a meromorphic function of finite order such that $N(r, f) = S(r, f)$. Suppose that $L(z, f) \not\equiv 0$ is a linear combination in $f(z)$ and the shifts of $f(z)$. Then

$$T(r, L) \leq T(r, f) + S(r, f). \quad (8)$$

Next we consider the rational function

$$R_2(z, f) = \frac{P(z, f)}{f(z + c_1) \cdots f(z + c_n)}, \quad (9)$$

where $c_1, \dots, c_n$ are different complex constants. We get the following result.

Theorem 3. Let $f(z)$ be a meromorphic function of finite order such that $N(r, f) = S(r, f)$. Suppose that $P(z, f) \not\equiv 0$ is a difference polynomial in $f(z)$ and that $R_2(z, f)$ is of the form (9). Then

$$T(r, R_2) \leq \max\{\deg_f P, n\} T(r, f) + S(r, f). \quad (10)$$

As for the general rational function in $f(z)$ and the shifts of $f(z)$,

$$R_3(z, f) = \frac{P(z, f)}{Q(z, f)}, \quad (11)$$

we get the following two results.

Theorem 4. Let $f(z)$ be a meromorphic function of finite order such that $N(r, f) = S(r, f)$. Suppose that $P(z, f) \not\equiv 0$ and $Q(z, f) \not\equiv 0$ are difference polynomials in $f(z)$ and that $R_3(z, f)$ is of the form (11). Then

(i) If $\deg_f P \geq \deg_f Q$ and $P(z, f)$ contains just one term of maximal total degree, then

$$T(r, R_3) \geq (\deg_f P - \deg_f Q) T(r, f) + S(r, f). \quad (12)$$

(ii) If $\deg_f P \leq \deg_f Q$ and $Q(z, f)$ contains just one term of maximal total degree, then

$$T(r, R_3) \geq (\deg_f Q - \deg_f P) T(r, f) + S(r, f). \quad (13)$$

Theorem 5. Let $f(z)$ be a meromorphic function of finite order such that $N(r, f) + N(r, 1/f) = S(r, f)$. Suppose that $P(z, f) \not\equiv 0$ and $Q(z, f) \not\equiv 0$ are difference polynomials in $f(z)$ and that $R_3(z, f)$ is of the form (11). Then

$$T(r, R_3) \leq \max\{\deg_f P, \deg_f Q\} T(r, f) + S(r, f). \quad (14)$$

The following two examples show that the results in Theorems 1–5 are sharp; that is, “ $\leq$ ” and “ $\geq$ ” cannot be replaced by “ $<$ ”, “ $>$ ” or “ $=$ ”.

Example 6. Let $f(z) = e^z$ and

$$P(z, f) = f(z)^2 f(z + \pi i) + f(z)^2 + 2f(z + \pi i) f(z) + 2f(z) + f(z + \pi i) + 1. \quad (15)$$

Let

$$R_{11}(z, f) = \frac{P(z, f)}{f(z + \pi i) + 2}, \quad R_{12}(z, f) = \frac{P(z, f)}{f(z + \pi i) + 1}. \quad (16)$$

Then $R_{11}(z, f) = (1 + e^z)^2(1 - e^z)/(-e^z + 2)$ and $R_{12}(z, f) = (1 + e^z)^2$. Clearly,

$$\begin{aligned} T(r, R_{11}) &= 3T(r, f) + S(r, f), \\ T(r, R_{12}) &= 2T(r, f) + S(r, f). \end{aligned} \quad (17)$$

Therefore,

$$\begin{aligned} &(\deg_f P - 1) T(r, f) + S(r, f) \\ &< T(r, R_{11}) = (\deg_f P) T(r, f) + S(r, f), \\ &(\deg_f P - 1) T(r, f) + S(r, f) \\ &= T(r, R_{12}) < (\deg_f P) T(r, f) + S(r, f). \end{aligned} \quad (18)$$

Example 7. Let $f(z) = \sin z$ and

$$P(z, f) = f\left(z + \frac{\pi}{2}\right)^2 f(z) + f(z)^2 + f(z + \pi) f(z) - f(z). \quad (19)$$

Let

$$R_{21}(z, f) = \frac{P(z, f)}{f(z + \pi/2)^3}, \quad R_{22}(z, f) = \frac{P(z, f)}{f(z + \pi)^2}. \quad (20)$$

Then $R_{21}(z, f) = -\tan^3 z$ and $R_{22}(z, f) = -\sin z$. Clearly,

$$\begin{aligned} T(r, R_{21}) &= 3T(r, f) + S(r, f), \\ T(r, R_{22}) &= T(r, f) + S(r, f). \end{aligned} \quad (21)$$

Therefore,

$$\begin{aligned} &(\deg_f P - 3) T(r, f) + S(r, f) \\ &< T(r, R_{21}) = (\deg_f P) T(r, f) + S(r, f), \\ &(\deg_f P - 2) T(r, f) + S(r, f) \\ &= T(r, R_{22}) < (\deg_f P) T(r, f) + S(r, f). \end{aligned} \quad (22)$$

2. Proof of Theorem 1

We need the following lemmas for the proof of Theorem 1.

The difference analogue of the logarithmic derivative lemma was given by Halburd-Korhonen [8, Corollary 2.2] and Chiang-Feng [7, Corollary 2.6], independently. The following Lemma 8 is a variant of [8, Corollary 2.2].

Lemma 8. *Let $f(z)$ be a nonconstant meromorphic function of finite order, and let η_1, η_2 be two arbitrary complex numbers. Then,*

$$m\left(r, \frac{f(z + \eta_1)}{f(z + \eta_2)}\right) = S(r, f). \quad (23)$$

In the remark of [15, page 15], it is pointed out that the following lemma holds.

Lemma 9. *Let $f(z)$ be a nonconstant finite order meromorphic function and let $c \neq 0$ be an arbitrary complex number. Then,*

$$\begin{aligned} T(r + |c|, f) &= T(r, f) + S(r, f), \\ N(r + |c|, f) &= N(r, f) + S(r, f). \end{aligned} \quad (24)$$

Let $f(z)$ be a meromorphic function. It is shown in [16, page 66] that for an arbitrary $c \neq 0$, the following inequalities:

$$\begin{aligned} (1 + o(1)) T(r - |c|, f(z)) &\leq T(r, f(z + c)) \\ &\leq (1 + o(1)) T(r + |c|, f(z)) \end{aligned} \quad (25)$$

hold as $r \rightarrow \infty$. From its proof we see that the above relations are also true for counting functions. So by these relations and Lemma 9, we get the following lemma.

Lemma 10. *Let $f(z)$ be a nonconstant finite order meromorphic function and let $c \neq 0$ be an arbitrary complex number. Then,*

$$\begin{aligned} T(r, f(z + c)) &= T(r, f) + S(r, f), \\ N(r, f(z + c)) &= N(r, f) + S(r, f), \\ N\left(r, \frac{1}{f(z + c)}\right) &= N\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned} \quad (26)$$

Remark 11. In [7], Chiang and Feng proved a similar result. Let $f(z)$ be a meromorphic function with $\sigma(f) < \infty$, and let $\eta \neq 0$ be fixed; then for each $\varepsilon > 0$, we have

$$T(r, f(z + \eta)) = T(r, f) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r). \quad (27)$$

Proof of Theorem 1. Let

$$P(z, f) = \sum_{\lambda \in I} a_{\lambda}(z) \prod_{j=1}^{\sigma_{\lambda}} f(z + \alpha_{\lambda, j})^{l_{\lambda, j}}, \quad (28)$$

and $\deg_f P = p$.

Rearranging the expression of $P(z, f)$ by collecting together all terms having the same total degree, we get

$$P(z, f) = \sum_{i=0}^p h_i(z) f(z)^i, \quad (29)$$

where, for $i = 0, \dots, p$,

$$\begin{aligned} h_i(z) &= \sum_{\lambda \in I_i} a_{\lambda}(z) \prod_{j=1}^{\sigma_{\lambda}} \left(\frac{f(z + \alpha_{\lambda, j})}{f(z)} \right)^{l_{\lambda, j}}, \\ I_i &= \left\{ \lambda \in I \mid \sum_{j=1}^{\sigma_{\lambda}} l_{\lambda, j} = i \right\}. \end{aligned} \quad (30)$$

Since the coefficients $a_{\lambda}(z)$ of $P(z, f)$ are small functions of $f(z)$, we have

$$m(r, a_{\lambda}) \leq T(r, a_{\lambda}) = S(r, f). \quad (31)$$

So by Lemma 8, we have, for all $i = 0, 1, \dots, p$ the estimates

$$m(r, h_i) = S(r, f). \quad (32)$$

Without loss of generality, we may assume $c = 0$ in (6). Otherwise, substituting $z - c$ for z , we get

$$R_1(z - c, f) = \frac{P(z - c, f)}{d_1(z - c) f(z) + d_0(z - c)}. \quad (33)$$

By Lemma 10, we see that

$$T(r, R_1(z - c, f)) = T(r, R_1(z, f)) + S(r, f). \quad (34)$$

So, in the following discussion, we only discuss the form

$$R_1(z, f) = \frac{P(z, f)}{d_1(z) f(z) + d_0(z)}. \quad (35)$$

Assume first that $d_1(z) = 0$. Clearly, we may assume that $d_0(z) = 1$. By (29), we get

$$\begin{aligned} R_1(z, f) &= P(z, f) \\ &= h_p(z) f(z)^p + h_{p-1}(z) f(z)^{p-1} \\ &\quad + \dots + h_1(z) f(z) + h_0(z). \end{aligned} \quad (36)$$

If $p = 1$, then $R_1(z, f) = h_1(z) f(z) + h_0(z)$. So by (32), we get

$$m(r, R_1) \leq m(r, f) + S(r, f). \quad (37)$$

If $p > 1$, then rewrite $R_1(z, f)$ in the form

$$R_1(z, f) = f(z) (h_p(z) f(z)^{p-1} + \dots + h_1(z)) + h_0(z). \quad (38)$$

So we have

$$\begin{aligned} m(r, R_1) &\leq m(r, f) \\ &\quad + m(r, h_p(z) f(z)^{p-1} + \dots + h_1(z)) + S(r, f). \end{aligned} \quad (39)$$

By (39) and the inductive argument, we have

$$m(r, R_1) \leq pm(r, f) + S(r, f). \quad (40)$$

To estimate $N(r, R_1)$, we use the form

$$R_1(z, f) = P(z, f) = \sum_{\lambda \in I} a_\lambda(z) \prod_{j=1}^{\sigma_\lambda} f(z + \alpha_{\lambda,j})^{l_{\lambda,j}}. \quad (41)$$

Clearly,

$$\begin{aligned} N(r, R_1) &\leq \sum_{\lambda \in I} \left(N(r, a_\lambda) + \sum_{j=1}^{\sigma_\lambda} l_{\lambda,j} N(r, f(z + \alpha_{\lambda,j})) \right) + O(1). \end{aligned} \quad (42)$$

So by (31), $N(r, f) = S(r, f)$, and Lemma 10, we get

$$N(r, R_1) = S(r, f). \quad (43)$$

Combining this equality with (40), we get

$$T(r, R_1) \leq pT(r, f) + S(r, f), \quad (44)$$

and we have completed the case $d_1(z) = 0$.

We now proceed to the case $d_1(z) \neq 0$. Clearly, in this case we may assume that $d_1(z) = 1$. By (29), we see that (6) becomes

$$\begin{aligned} R_1(z, f) &= (h_p(z) f(z)^p + h_{p-1}(z) f(z)^{p-1} \\ &\quad + \cdots + h_1(z) f(z) + h_0(z)) \\ &\quad \times (f(z) + d_0(z))^{-1}. \end{aligned} \quad (45)$$

By (45), we get

$$\begin{aligned} R_1(z, f) &= h_p(z) f(z)^{p-1} \\ &\quad + (h_{p-1}^*(z) f(z)^{p-1} + h_{p-2}(z) f(z)^{p-2} \\ &\quad + \cdots + h_1(z) f(z) + h_0(z)) \\ &\quad \times (f(z) + d_0(z))^{-1} \\ &= h_p(z) f(z)^{p-1} + h_{p-1}^*(z) f(z)^{p-2} \\ &\quad + \frac{h_{p-2}^*(z) f(z)^{p-2} + \cdots + h_1(z) f(z) + h_0(z)}{f(z) + d_0(z)} \\ &= \cdots \\ &= h_p(z) f(z)^{p-1} + h_{p-1}^*(z) f(z)^{p-2} \\ &\quad + \cdots + h_2^*(z) f(z) + h_1^*(z) + \frac{h_0^*(z)}{f(z) + d_0(z)}, \end{aligned} \quad (46)$$

where

$$\begin{aligned} h_{p-1}^*(z) &= h_{p-1}(z) - h_p(z) d_0(z), \\ h_{p-2}^*(z) &= h_{p-2}(z) - h_{p-1}^*(z) d_0(z), \\ &\vdots \\ h_1^*(z) &= h_1(z) - h_2^*(z) d_0(z), \\ h_0^*(z) &= h_0(z) - h_1^*(z) d_0(z). \end{aligned} \quad (47)$$

By (32), we get, for $j = 0, 1, \dots, p-1$, the estimates

$$m(r, h_j^*) = S(r, f). \quad (48)$$

By (46), using the same method as in (36)–(40), we get

$$\begin{aligned} m(r, R_1) &\leq m(r, h_p(z) f(z)^{p-1} + h_{p-1}^*(z) f(z)^{p-2} + \cdots + h_1^*(z)) \\ &\quad + m\left(r, \frac{h_0^*(z)}{f(z) + d_0(z)}\right) \\ &\leq (p-1)m(r, f) \\ &\quad + m\left(r, \frac{1}{f(z) + d_0(z)}\right) + S(r, f). \end{aligned} \quad (49)$$

To estimate $N(r, R_1)$, we use the form

$$\begin{aligned} R_1(z, f) &= \frac{P(z, f)}{f(z) + d_0(z)} \\ &= \frac{\sum_{\lambda \in I} a_\lambda(z) \prod_{j=1}^{\sigma_\lambda} f(z + \alpha_{\lambda,j})^{l_{\lambda,j}}}{f(z) + d_0(z)}. \end{aligned} \quad (50)$$

By (31), $N(r, f) = S(r, f)$, and Lemma 10, we get

$$N(r, R_1) = N\left(r, \frac{1}{f(z) + d_0(z)}\right) + S(r, f). \quad (51)$$

Combining this equality with (49), we get

$$\begin{aligned} T(r, R_1) &\leq (p-1)m(r, f) \\ &\quad + T\left(r, \frac{1}{f(z) + d_0(z)}\right) + S(r, f) \\ &\leq pT(r, f) + S(r, f). \end{aligned} \quad (52)$$

Theorem 1 is proved. $\square$

3. Proof of Theorem 3

Proof. Let $P(z, f)$ be of the form (28) and $\deg_f P = p$. Rearranging the expression of $P(z, f)$, we get (29) and (30). We only discuss the case $p \geq n$ since the case $p < n$ is easier.

Rewrite $R_2(z, f)$ in the form

$$R_2(z, f) = \frac{P(z, f)}{s(z) f(z)^n}, \quad (53)$$

where

$$s(z) = \frac{f(z + c_1) \cdots f(z + c_n)}{f(z)^n}. \quad (54)$$

By Lemma 8, we get

$$m\left(r, \frac{1}{s}\right) = S(r, f). \quad (55)$$

By (29) and (53), we get

$$\begin{aligned} R_2(z, f) &= \frac{\sum_{i=0}^p h_i(z) f(z)^i}{s(z) f(z)^n} \\ &= \sum_{i=n}^p \frac{h_i(z)}{s(z)} f(z)^{i-n} \\ &\quad + \frac{h_{n-1}(z) f(z)^{n-1} + \cdots + h_0(z)}{s(z) f(z)^n} \\ &= \sum_{i=n}^p \frac{h_i(z)}{s(z)} f(z)^{i-n} \\ &\quad + \sum_{j=1}^n \frac{h_{n-j}(z)}{s(z)} \left(\frac{1}{f(z)}\right)^j. \end{aligned} \quad (56)$$

By (32) and (55), we have, for all $i = 0, \dots, p$, the estimates

$$m\left(r, \frac{h_i(z)}{s(z)}\right) = S(r, f). \quad (57)$$

By (57), using the same method as in (36)–(40), we get

$$\begin{aligned} m\left(r, \sum_{i=n}^p \frac{h_i(z)}{s(z)} f(z)^{i-n}\right) &\leq (p-n) m(r, f) + S(r, f), \\ m\left(r, \sum_{j=1}^n \frac{h_{n-j}(z)}{s(z)} \left(\frac{1}{f(z)}\right)^j\right) &\leq nm\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned} \quad (58)$$

Combining the above two inequalities with (56), we get

$$m(r, R_2) \leq (p-n) m(r, f) + nm\left(r, \frac{1}{f}\right) + S(r, f). \quad (59)$$

To estimate $N(r, R_2)$, we use the form

$$R_2(z, f) = \frac{\sum_{\lambda \in I} a_\lambda(z) \prod_{j=1}^{\sigma_\lambda} f(z + \alpha_{\lambda,j})^{l_{\lambda,j}}}{f(z + c_1) \cdots f(z + c_n)}. \quad (60)$$

By (31), $N(r, f) = S(r, f)$, and Lemma 10, we get

$$\begin{aligned} N(r, R_2) &= N\left(r, \frac{1}{f(z + c_1) \cdots f(z + c_n)}\right) \\ &\quad + S(r, f) \leq nN\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned} \quad (61)$$

Combining this inequality with (59), we get

$$\begin{aligned} T(r, R_2) &\leq (p-n) m(r, f) + nT\left(r, \frac{1}{f}\right) \\ &\quad + S(r, f) \leq pT(r, f) + S(r, f). \end{aligned} \quad (62)$$

Theorem 3 is proved. $\square$

4. Proof of Theorem 4

We need the following lemma for the proof of Theorem 4.

Lemma 12 (see [11]). *Let $f(z)$ be a meromorphic function of finite order such that $N(r, f) = S(r, f)$. Suppose that $H(z, f)$ is a difference polynomial in $f(z)$ and $H(z, f)$ contains just one term of maximal total degree. Then,*

$$T(r, H) = (\deg_f H) T(r, f) + S(r, f). \quad (63)$$

Proof of Theorem 4. We have the following.

Case 1. Suppose that $\deg_f P \geq \deg_f Q$ and $P(z, f)$ contains just one term of maximal total degree.

Let $\deg_f P = p$ and $\deg_f Q = q$. By Lemma 12, we get

$$T(r, P) = pT(r, f) + S(r, f). \quad (64)$$

By Theorem 1, we get

$$T(r, Q) \leq qT(r, f) + S(r, f). \quad (65)$$

By (11), we get

$$P(z, f) = R_3(z, f) Q(z, f). \quad (66)$$

By (64)–(66), we get

$$\begin{aligned} pT(r, f) + S(r, f) &= T(r, P(z, f)) \\ &= T(r, R_3(z, f) Q(z, f)) \\ &\leq T(r, R_3(z, f)) + T(r, Q(z, f)) \\ &\leq T(r, R_3(z, f)) + qT(r, f) + S(r, f). \end{aligned} \quad (67)$$

So we have,

$$T(r, R_3) \geq (p-q) T(r, f) + S(r, f). \quad (68)$$

Case 2. Suppose that $\deg_f P \leq \deg_f Q$ and $Q(z, f)$ contains just one term of maximal total degree.

In this case, we consider $1/R_3(z, f)$. Using the same method as in Case 1, we can easily get

$$T(r, R_3) = T\left(r, \frac{1}{R_3}\right) \geq (q-p) T(r, f) + S(r, f). \quad (69)$$

Theorem 4 is proved. $\square$

5. Proof of Theorem 5

Proof. Let $P(z, f)$ be of the form (28) and $\deg_f P = p$. Let

$$Q(z, f) = \sum_{\mu \in J} b_{\mu}(z) \prod_{j=1}^{\tau_{\mu}} f(z + \beta_{\mu,j})^{m_{\mu,j}}, \quad (70)$$

and $\deg_f Q = q$.

Rearranging the expression of $P(z, f)$, we get (29) and (30).

Similarly, rearranging the expression of $Q(z, f)$, we get

$$Q(z, f) = \sum_{k=0}^q t_k(z) f(z)^k, \quad (71)$$

where, for $k = 0, \dots, q$,

$$t_k(z) = \sum_{\mu \in J_k} b_{\mu}(z) \prod_{j=1}^{\tau_{\mu}} \left(\frac{f(z + \beta_{\mu,j})}{f(z)} \right)^{m_{\mu,j}}, \quad (72)$$

$$J_k = \left\{ \lambda \in J \mid \sum_{j=1}^{\tau_{\lambda}} m_{\lambda,j} = k \right\}.$$

By (29) and (71), we get

$$R_3(z, f) = \frac{\sum_{i=0}^p h_i(z) f(z)^i}{\sum_{k=0}^q t_k(z) f(z)^k}. \quad (73)$$

Since $N(r, f) + N(r, 1/f) = S(r, f)$, by Lemma 10, we have, for an arbitrary η ,

$$\begin{aligned} N\left(r, \frac{f(z+\eta)}{f(z)}\right) &\leq N\left(r, \frac{1}{f}\right) + N(r, f(z+\eta)) \\ &= N\left(r, \frac{1}{f}\right) + N(r, f) + S(r, f) \\ &= S(r, f). \end{aligned} \quad (74)$$

By (74) and Lemma 8, we have, for an arbitrary η ,

$$T\left(r, \frac{f(z+\eta)}{f(z)}\right) = S(r, f). \quad (75)$$

Since the coefficients $a_{\lambda}(z)$ and $b_{\mu}(z)$ of $P(z, f)$ and $Q(z, f)$ are small functions of $f(z)$, by (30), (72), and (75), we get

$$\begin{aligned} T(r, h_i) &= S(r, f), \quad i = 0, \dots, p \\ T(r, t_k) &= S(r, f), \quad k = 0, \dots, q. \end{aligned} \quad (76)$$

By (73), we are not clear whether $R_3(z, f)$ is an irreducible rational function in $f(z)$. So by Theorem A, we get

$$T(r, R_3) \leq \max\{p, q\} T(r, f) + S(r, f). \quad (77)$$

Theorem 5 is proved. $\square$

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