Research Article

A Note on the Tail Behavior of Randomly Weighted Sums with Convolution-Equivalently Distributed Random Variables

Yang Yang,^{1,2} Jun-feng Liu,¹ and Yu-lin Zhang²

¹ School of Mathematics and Statistics, Nanjing Audit University, Nanjing 210029, China

² School of Economics and Management, Southeast University, Nanjing 210096, China

Correspondence should be addressed to Yang Yang; yyangmath@gmail.com

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We investigate the tailed asymptotic behavior of the randomly weighted sums with increments with convolution-equivalent distributions. Our obtained result can be directly applied to a discrete-time insurance risk model with insurance and financial risks and derive the asymptotics for the finite-time probability of the above risk model.

1. Introduction and Main Result

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) real-valued random variables with common distribution F , and let $\{Y_n, n \geq 1\}$ be another sequence of i.i.d. nonnegative r.v.s with common distribution G and right endpoint $\hat{x}_G = \sup\{x : \mathbb{P}(Y_1 \leq x) < 1\}$. Assume that $\{X_n, n \geq 1\}$ are independent of $\{Y_n, n \geq 1\}$. In this paper, we are interested in the randomly weighted sum

$$S_n^Y = \sum_{i=1}^n X_i \prod_{j=1}^i Y_j, \quad n \geq 1. \quad (1)$$

This is because the study for the tail probability $\mathbb{P}(S_n^Y > x)$ can be directly applied to risk theory. Consider a discrete-time insurance risk model. Within period i , $i \geq 1$, the net insurance loss is denoted by a real-valued (r.v.) X_i . The insurer makes both risk-free and risky investments, leading to an overall stochastic discounted factor Y_i from time i to time $i-1$. In the terminology of Norberg [1], the sequences $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are called the insurance and financial risks, respectively. Then, the randomly weighted sum S_n^Y in (1) represents the stochastic discounted value of aggregate net

losses up to time n , $n \geq 1$. As usual, the probability of ruin by time n can be defined by

$$\Psi(x, n) = \mathbb{P}\left(\max_{1 \leq m \leq n} \sum_{i=1}^m X_i \prod_{j=1}^i Y_j > x\right), \quad n \geq 1, \quad (2)$$

where $x \geq 0$ is interpreted as the initial capital reserve of an insurance company. Clearly, for each $n \geq 1$,

$$\mathbb{P}(S_n^Y > x) \leq \Psi(x, n) \leq \mathbb{P}\left(\sum_{i=1}^n X_i^+ \prod_{j=1}^i Y_j > x\right), \quad (3)$$

where $X_i^+ = X_i \mathbf{1}_{\{X_i \geq 0\}}$ denotes the positive part of X_i , $i \geq 1$. If we can establish an asymptotic formula for $\mathbb{P}(S_n^Y > x)$ while doing so does not require $F(0-) > 0$, then the same asymptotic formula should hold for the right-hand side of (3) as well. In this way the ruin probability $\Psi(x, n)$ has the same asymptotic behavior as that of the tail probability $\mathbb{P}(S_n^Y > x)$ as x tends to infinity.

There has been a vast amount of literature studying the asymptotic behavior of the tail probability of the randomly weighted sum S_n^Y . Many works have considered the heavy-tailed case; that is, the distribution F of X belongs to some classes of heavy-tailed distributions, even under some

dependence structures. For example, one can refer to Tang and Tsitsiashvili [2, 3], Wang and Tang [4], Zhang et al. [5], Shen et al. [6], Chen and Yuen [7], Gao and Wang [8], and Yi et al. [9] among others for some details in this direction, where the distribution F is heavily heavy tailed; as for some lightly heavy-tailed distribution F , some related results were obtained by Tang and Tsitsiashvili [3, 10], Chen and Su [11], Hashorva et al. [12], Yang et al. [13], Yang and Hashorva [14], and Yang and Wang [15] among others. We pointed out that Tang and Tsitsiashvili [3] achieved some interesting results on the asymptotics for the tail probability $\mathbb{P}(S_n^Y > x)$ in some cases where F belongs to the intersection between the subexponential distribution class and the rapidly varying distribution class.

In this paper, we aim to consider the light-tailed case, more exactly, to investigate the asymptotic behavior of the tail probability of the randomly weighted sums with increments with convolution-equivalent distributions.

Hereafter, all the limit relationships hold for x tending to infinity. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$; write $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$; and write $a(x) = O(b(x))$ if $\limsup a(x)/b(x) < \infty$.

Firstly we introduce some definitions on some classes of convolution-equivalent distributions. A distribution V on $[0, \infty)$ belongs to the class of convolution-equivalent distributions, denoted by $\mathcal{S}(\gamma)$, $\gamma \geq 0$, if for any $y \in \mathbb{R}$,

$$\lim_{x \rightarrow \infty} \frac{\bar{V}(x-y)}{\bar{V}(x)} = e^{\gamma y}, \quad (4)$$

$$\lim_{x \rightarrow \infty} \frac{\bar{V}^{*2}(x)}{\bar{V}(x)} = 2 \int_0^\infty e^{\gamma u} V(du) < \infty, \quad (5)$$

where V^{*2} denotes the convolution of V with itself. More generally, a distribution V on $\mathbb{R}$ belongs to the class $\mathcal{S}(\gamma)$, $\gamma \geq 0$, if and only if its right-hand distribution $V^+(x) = V(x)\mathbf{1}_{\{x \geq 0\}}$ belongs to this class; see Corollary 2.1 of Pakes [16]. The class $\mathcal{S} := \mathcal{S}(0)$ is called the class of subexponential distributions. A distribution V on $\mathbb{R}$ belongs to the class $\mathcal{L}(\gamma)$, $\gamma \geq 0$ if only relation (4) holds. In the case $\gamma = 0$, we say that $\mathcal{L} := \mathcal{L}(0)$ is the class of long-tailed distributions. Similarly, a positive function $f(\cdot)$ is said to be long tailed if $\lim_{x \rightarrow \infty} f(x-y)/f(x) = 1$ for any $y \in \mathbb{R}$. Clearly, if a distribution $V \in \mathcal{L}$, then its tail probability $\bar{V}(x)$ is long tailed. Closely related is the class $\mathcal{A}$, which was introduced by Konstantinides et al. [17]. A distribution V on $\mathbb{R}$ belongs to the class $\mathcal{A}$ if V is subexponential, and, for some $\gamma > 1$,

$$\limsup_{x \rightarrow \infty} \frac{\bar{V}(xy)}{\bar{V}(x)} < 1. \quad (6)$$

Clearly, all distributions in the classes $\mathcal{A}$, $\mathcal{S}$, and $\mathcal{L}$ are heavy tailed. A distribution V of r.v. ξ is said to be heavy tailed if $\mathbb{E}e^{s\xi} = \infty$ for any $s > 0$; otherwise it is said to be light tailed.

For each $n \geq 1$, denote the distribution of $X_n \prod_{j=1}^n Y_j$ by H_n , by convention, $H = H_1$. Now we state our main result as follows.

Theorem 1. *If $F \in \mathcal{S}(\gamma)$ for some $\gamma > 0$, $\hat{x}_G = \infty$, and, for all $u > 0$,*

$$\bar{G}(ux) = o(\bar{H}(x)), \quad (7)$$

then, for each $n \geq 1$,

$$\mathbb{P}(S_n^Y > x) \sim \bar{H}_n(x). \quad (8)$$

Remark 2. We remark that Tang [18] considered a similar result for $0 < \hat{x}_G < \infty$, whereas Theorem 1 deals with the case with $\hat{x}_G = \infty$ for a complement.

Remark 3. In Theorem 1, relation (7) is a mild condition. According to Corollary 1.1 of Tang [19], relation (7) can be further implied by either

- (a) $\bar{G}(\nu x) = o(\bar{G}(x))$ for some $\nu > 1$ or
- (b) $\bar{G}(\nu x) = o(\bar{F}(x))$ for some $\nu > 0$.

2. Proof of the Main Result

We start this section by a series of lemmas. The first two lemmas are due to Lemma 3.2 and Theorem 2.1 of Tang [20].

Lemma 4. *For two distributions G and H with $\bar{G}(x) > 0$ and $\bar{H}(x) > 0$ for all $x \geq 0$, relation (7) holds for each $u > 0$, if and only if there is a nonnegative function $a(\cdot)$ such that*

$$a(x) \nearrow \infty, \quad \frac{a(x)}{x} \searrow 0, \quad \bar{G}(a(x)) = o(\bar{H}(x)). \quad (9)$$

Lemma 5. *Consider the product XY . The distribution H of the product belongs to the class $\mathcal{A}$ if and only if $F \in \mathcal{A}$ and relation (7) holds for all $u > 0$.*

Tang [19] obtained an interesting result to show that a light-tailed random variable can be transferred into a heavy-tailed one through multiplier.

Lemma 6. *Consider the product XY with $F \in \mathcal{S}(\gamma)$ for some $\gamma > 0$ and $\hat{x}_G = \infty$. If relation (7) holds for all $u > 0$, then $H \in \mathcal{S}$.*

The last lemma can be found in, for example, Theorem 3.14 of Foss et al. [21].

Lemma 7. *Let a reference distribution V on $\mathbb{R}$ belong to the class $\mathcal{S}$. Assume that distributions $V_1, \dots, V_n$ on $\mathbb{R}$ satisfy that, for each $i = 1, \dots, n$, the function $\bar{V} + \bar{V}_i$ is long tailed and $\bar{V}_i(x) = O(\bar{V}(x))$. Then, it holds that*

$$\overline{V_1 * \dots * V_n}(x) = \bar{V}_1(x) + \dots + \bar{V}_n(x) + o(\bar{V}(x)). \quad (10)$$

Proof of Theorem 1. Now we begin to prove the main result of Theorem 1.

For each $n \geq 1$, write

$$T_n := \sum_{i=1}^n X_i \prod_{j=i}^n Y_j \stackrel{d}{=} S_n^Y, \quad (11)$$

where $\stackrel{d}{=}$ stands for equality in distribution. Since $F \in \mathcal{S}(\gamma) \subset \mathcal{L}(\gamma)$, $\gamma > 0$, its tail distribution $\bar{F}$ is rapidly varying in the sense that

$$\lim_{x \rightarrow \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} = 0, \quad \forall y > 1. \quad (12)$$

By Lemma 6, we get that $H \in \mathcal{S}$. Further, by Lemma 4, there exists a nonnegative function $a(\cdot)$ such that (9) holds. Thus, by (9) and (12), for any $y > 1$,

$$\begin{aligned} & \limsup \frac{\bar{H}^+(xy)}{\bar{H}^+(x)} \\ &= \limsup \frac{\left(\int_0^{a(x)} + \int_{a(x)}^\infty \right) \bar{F}(xy/u) G(du)}{\bar{H}^+(x)} \\ &\leq \limsup \frac{\int_0^{a(x)} \bar{F}(xy/u) G(du)}{\int_0^{a(x)} \bar{F}(x/u) G(du)} + \limsup \frac{\bar{G}(a(x))}{\bar{H}^+(x)} \\ &\leq \limsup \sup_{0 \leq u \leq a(x)} \frac{\bar{F}(xy/u)}{\bar{F}(x/u)} \\ &= \limsup \frac{\bar{F}(xy)}{\bar{F}(x)} = 0 < 1, \end{aligned} \quad (13)$$

which, together with $H \in \mathcal{S}$, implies that $H \in \mathcal{A}$.

We proceed to prove relation (8) by induction on n . Trivially, the distribution $H_1 = H$ of T_1 or S_1 belongs to the class $\mathcal{A}$, and relation (8) holds for $n = 1$. Assume that $H_n \in \mathcal{A}$ and (8) holds for n . We aim to prove that $H_{n+1} \in \mathcal{A}$ and (8) holds for $n + 1$, which, by (11), is equivalent to

$$\mathbb{P}(T_{n+1} > x) \sim \bar{H}_{n+1}(x). \quad (14)$$

First of all, according to Lemma 2.17 of Foss et al. [21] and $H_n \in \mathcal{A} \subset \mathcal{L}$, we have, that for any $s > 0$,

$$\lim_{x \rightarrow \infty} \bar{H}_n(x) e^{sx} = \infty, \quad (15)$$

which, together with $F \in \mathcal{S}(\gamma)$, $\gamma > 0$, implies that

$$\bar{F}(x) = o(\bar{H}_n(x)). \quad (16)$$

By (16) and $H_n \in \mathcal{A} \subset \mathcal{L}$, we have that for any $y \in \mathbb{R}$,

$$\begin{aligned} & \lim \frac{\bar{H}_n(x-y) + \bar{F}(x-y)}{\bar{H}_n(x) + \bar{F}(x)} \\ &= \lim \left(\left(\frac{\bar{H}_n(x-y)}{\bar{H}_n(x)} \right) + \left(\frac{\bar{F}(x-y)}{\bar{H}_n(x)} \right) \right. \\ & \quad \cdot \left. \left(\frac{\bar{H}_n(x)}{\bar{H}_n(x)} \right) \right) \times \left(1 + \left(\frac{\bar{F}(x)}{\bar{H}_n(x)} \right) \right)^{-1} \\ &= 1, \end{aligned} \quad (17)$$

which shows that the function $\bar{H}_n(x) + \bar{F}(x)$ is long tailed. Since T_n and X_{n+1} are independent of each other, thus, by $H_n \in \mathcal{A} \subset \mathcal{S}$, we can apply Lemma 7 to derive from the induction assumption and (16) that

$$\begin{aligned} \mathbb{P}(T_n + X_{n+1} > x) &= \bar{H}_n(x) + \bar{F}(x) + o(\bar{H}_n(x)) \\ &= (1 + o(1)) \bar{H}_n(x). \end{aligned} \quad (18)$$

For the above-mentioned nonnegative function $a(\cdot)$, from (9) and (18), we obtain that

$$\begin{aligned} & \mathbb{P}(T_{n+1} > x) \\ &= \mathbb{P}((T_n + X_{n+1}) Y_{n+1} > x) \\ &= \left(\int_0^{a(x)} + \int_{a(x)}^\infty \right) \mathbb{P}\left(T_n + X_{n+1} > \frac{x}{u}\right) G(du) \\ &= (1 + o(1)) \int_0^{a(x)} \bar{H}_n\left(\frac{x}{u}\right) G(du) + o(\bar{H}(x)) \\ &= (1 + o(1)) \bar{H}_{n+1}(x), \end{aligned} \quad (19)$$

where the last step used the fact that $\bar{H}(x) = O(\bar{H}_{n+1}(x))$, because, for any $x \geq 0$,

$$\begin{aligned} \bar{H}_{n+1}(x) &\geq \mathbb{P}\left(X_{n+1} \prod_{j=1}^{n+1} Y_j > x, Y_1 > 1, \dots, Y_n > 1\right) \\ &\geq \bar{H}(x) (\bar{G}(1))^n, \end{aligned} \quad (20)$$

and $\bar{G}(1) > 0$ by $\hat{x}_G = \infty$. Relation (19) means that (14) holds. Finally, by (9) and (20), we have that

$$\bar{G}(a(x)) = o(\bar{H}_{n+1}(x)), \quad (21)$$

from which and $H_n \in \mathcal{A}$, Lemma 5 gives that $H_{n+1} \in \mathcal{A}$.

This completes the proof of Theorem 1. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] R. Norberg, "Ruin problems with assets and liabilities of diffusion type," *Stochastic Processes and Their Applications*, vol. 81, no. 2, pp. 255–269, 1999.
- [2] Q. Tang and G. Tsitsiashvili, "Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks," *Stochastic Processes and Their Applications*, vol. 108, no. 2, pp. 299–325, 2003.
- [3] Q. Tang and G. Tsitsiashvili, "Finite- and infinite-time ruin probabilities in the presence of stochastic returns on investments," *Advances in Applied Probability*, vol. 36, no. 4, pp. 1278–1299, 2004.
- [4] D. Wang and Q. Tang, "Tail probabilities of randomly weighted sums of random variables with dominated variation," *Stochastic Models*, vol. 22, no. 2, pp. 253–272, 2006.
- [5] Y. Zhang, X. Shen, and C. Weng, "Approximation of the tail probability of randomly weighted sums and applications," *Stochastic Processes and Their Applications*, vol. 119, no. 2, pp. 655–675, 2009.
- [6] X.-m. Shen, Z.-y. Lin, and Y. Zhang, "Uniform estimate for maximum of randomly weighted sums with applications to ruin theory," *Methodology and Computing in Applied Probability*, vol. 11, no. 4, pp. 669–685, 2009.
- [7] Y. Chen and K. C. Yuen, "Sums of pairwise quasi-asymptotically independent random variables with consistent variation," *Stochastic Models*, vol. 25, no. 1, pp. 76–89, 2009.
- [8] Q. Gao and Y. Wang, "Randomly weighted sums with dominated varying-tailed increments and application to risk theory," *Journal of the Korean Statistical Society*, vol. 39, no. 3, pp. 305–314, 2010.
- [9] L. Yi, Y. Chen, and C. Su, "Approximation of the tail probability of randomly weighted sums of dependent random variables with dominated variation," *Journal of Mathematical Analysis and Applications*, vol. 376, no. 1, pp. 365–372, 2011.
- [10] Q. Tang and G. Tsitsiashvili, "Randomly weighted sums of subexponential random variables with application to ruin theory," *Extremes*, vol. 6, no. 3, pp. 171–188, 2003.
- [11] Y. Chen and C. Su, "Finite time ruin probability with heavy-tailed insurance and financial risks," *Statistics & Probability Letters*, vol. 76, no. 16, pp. 1812–1820, 2006.
- [12] E. Hashorva, A. G. Pakes, and Q. Tang, "Asymptotics of random contractions," *Insurance*, vol. 47, no. 3, pp. 405–414, 2010.
- [13] Y. Yang, R. Leipus, and J. Šiaulyš, "Tail probability of randomly weighted sums of subexponential random variables under a dependence structure," *Statistics & Probability Letters*, vol. 82, no. 9, pp. 1727–1736, 2012.
- [14] Y. Yang and E. Hashorva, "Extremes and products of multivariate AC-product risks," *Insurance*, vol. 52, no. 2, pp. 312–319, 2013.
- [15] Y. Yang and Y. Wang, "Tail behavior of the product of two dependent random variables with applications to risk theory," *Extremes*, vol. 16, no. 1, pp. 55–74, 2013.
- [16] A. G. Pakes, "Convolution equivalence and infinite divisibility," *Journal of Applied Probability*, vol. 41, no. 2, pp. 407–424, 2004.
- [17] D. Konstantinides, Q. Tang, and G. Tsitsiashvili, "Estimates for the ruin probability in the classical risk model with constant interest force in the presence of heavy tails," *Insurance*, vol. 31, no. 3, pp. 447–460, 2002.
- [18] Q. Tang, "On convolution equivalence with applications," *Bernoulli*, vol. 12, no. 3, pp. 535–549, 2006.
- [19] Q. Tang, "From light tails to heavy tails through multiplier," *Extremes*, vol. 11, no. 4, pp. 379–391, 2008.
- [20] Q. Tang, "The subexponentiality of products revisited," *Extremes*, vol. 9, no. 3–4, pp. 231–241, 2006.
- [21] S. Foss, D. Korshunov, and S. Zachary, *An Introduction to Heavy-Tailed and Subexponential Distributions*, Springer, New York, NY, USA, 2011.