On the Stability of Trigonometric Functional Equations in Distributions and Hyperfunctions

Jaeyoung Chung$^1$ and Jeongwook Chang$^2$

$^1$ Department of Mathematics, Kunsan National University, Kunsan 573-701, Republic of Korea
$^2$ Department of Mathematics Education, Dankook University, Yongin 448-701, Republic of Korea

Correspondence should be addressed to Jeongwook Chang; jchang@dankook.ac.kr

Received 6 February 2013; Accepted 10 April 2013

Academic Editor: Adem Kilicman

Copyright © 2013 J. Chung and J. Chang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider the Hyers-Ulam stability for a class of trigonometric functional equations in the spaces of generalized functions such as Schwartz distributions and Gelfand hyperfunctions.

1. Introduction

Hyers-Ulam stability problems of functional equations go back to 1940 when Ulam proposed the following question [1].

Let $f$ be a mapping from a group $G_1$ to a metric group $G_2$ with metric $d(\cdot,\cdot)$ such that

$$d(f(xy),f(x)f(y)) \leq \epsilon. \quad (1)$$

Then does there exist a group homomorphism $h$ and $\delta_\epsilon > 0$ such that

$$d(f(x),h(x)) \leq \delta_\epsilon \quad (2)$$

for all $x \in G_1$?

This problem was solved affirmatively by Hyers [2] under the assumption that $G_2$ is a Banach space. After the result of Hyers, Aoki [3] and Bourgin [4, 5] treated with this problem; however, there were no other results on this problem until 1978 when Rassias [6] treated again with the inequality of Aoki [3]. Following Rassias’ result, a great number of papers on the subject have been published concerning numerous functional equations in various directions [6–10, 10–25]. In 1990 Székelyhidi [24] has developed his idea of using invariant subspaces of functions defined on a group or semigroup in connection with stability questions for the sine and cosine functional inequalities. We refer the reader to [9, 10, 18, 19, 25] for Hyers-Ulam stability of functional equations of trigonometric type. In this paper, following the method of Székelyhidi [24] we consider a distributional analogue of the Hyers-Ulam stability problem of the trigonometric functional inequalities

$$\|f(x+y)-f(x)-f(y)\| \leq \Phi(x,y), \text{ for } x, y \in X \quad (3)$$

with $\Phi(x,y) = \epsilon(||x||^p + ||y||^p)$ ($\epsilon \geq 0$, $0 \leq p < 1$), then there exists a unique additive function $A : X \to Y$ such that $\|f(x) - A(x)\| \leq 2\epsilon|x|^p/(2 - 2^p)$ for all $x \in X$. In 1951 Bourgin [4, 5] stated that if $\Phi$ is symmetric in $\|x\|$ and $\|y\|$ with $\sum_{j=1}^{\infty} \Phi(2^j x, 2^j y)/2^j < \infty$ for each $x \in X$, then there exists a unique additive function $A : X \to Y$ such that $\|f(x) - A(x)\| \leq \sum_{j=1}^{\infty} \Phi(2^j x, 2^j y)/2^j$ for all $x \in X$. Unfortunately, there was no use of these results until 1978 when Rassias [7] treated with the inequality of Aoki [3]. Following Rassias’ result, a great number of papers on the subject have been published concerning numerous functional equations in various directions [6–10, 10–25]. In 1990 Székelyhidi [24] has developed his idea of using invariant subspaces of functions defined on a group or semigroup in connection with stability questions for the sine and cosine functional equations. We refer the reader to [9, 10, 18, 19, 25] for Hyers-Ulam stability of functional equations of trigonometric type. In this paper, following the method of Székelyhidi [24] we consider a distributional analogue of the Hyers-Ulam stability problem of the trigonometric functional inequalities

$$|f(x+y) - f(x)g(y) + g(x)f(y)| \leq \psi(y), \quad (4)$$

$$|g(x+y) - g(x)g(y) - f(x)f(y)| \leq \psi(y),$$

where $f, g : \mathbb{R}^n \to \mathbb{C}$ and $\psi : \mathbb{R}^n \to [0, \infty)$ is a continuous function. As a distributional version of the inequalities (4), we
consider the inequalities for the generalized functions \( u, v \in G'(\mathbb{R}^n) \) (resp., \( \delta'(\mathbb{R}^n) \)),
\[
\left\| u \circ (x - y) - u_x \otimes v_y + v_x \otimes u_y \right\| \leq \psi(y), \\
\left\| v \circ (x - y) - v_x \otimes v_y - u_x \otimes u_y \right\| \leq \psi(y),
\]
where \( \circ \) and \( \otimes \) denote the pullback and the tensor product of generalized functions, respectively, and \( \psi : \mathbb{R}^n \to [0, \infty) \) denotes a continuous infraexponential function of order 2 (resp., a function of polynomial growth). For the proof we employ the tensor product \( E_t(x)E_s(y) \) of \( n \)-dimensional heat kernel
\[
E_t(x) = (4\pi t)^{-n/2} \exp \left( -\frac{|x|^2}{4t} \right), \quad x \in \mathbb{R}^n, \quad t > 0.
\]
For the first step, convolving \( E_t(x)E_s(y) \) in both sides of (5) we convert (5) to the Hyers-Ulam stability problems of trigonometric-hyperbolic type functional inequalities, respectively,
\[
\begin{align*}
|u(x, t + s) - u(x, t) V(y, s) + V(x, t) U(y, s)| & \leq \psi(y), \\
|v(x, t + s) - v(x, t) V(y, s) - U(x, t) U(y, s)| & \leq \psi(y),
\end{align*}
\]
for all \( x, y \in \mathbb{R}^n, t, s > 0 \), where \( U, V \) are the Gauss transforms of \( u, v \), respectively, given by
\[
\begin{align*}
U(x, t) &= u \ast E_t(x) = \left\langle u, E_t(x - y) \right\rangle, \\
V(x, t) &= v \ast E_t(x),
\end{align*}
\]
which are solutions of the heat equation, and
\[
\psi(y) = \int \psi(\eta) E_s(\eta - y) d\eta = (\psi \ast E_s)(y).
\]
For the second step, using similar idea of Székelyhidi [24] we prove the Hyers-Ulam stabilities of inequalities (7). For the final step, taking initial values as \( t \to 0^+ \) for the results we arrive at our results.

2. Generalized Functions

We first introduce the spaces \( \delta' \) of Schwartz tempered distributions and \( \delta' \) of Gelfand hyperfunctions (see [26–29] for more details of these spaces). We use the notations: \( |\alpha| = \alpha_1 + \cdots + \alpha_n, \alpha! = \alpha_1! \cdots \alpha_n! \), \( |x| = x_1^2 + \cdots + x_n^2 \), \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \), and \( \delta^\alpha = \delta_1^{\alpha_1} \cdots \delta_n^{\alpha_n} \), for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \), where \( \mathbb{N}_0 \) is the set of nonnegative integers and \( \partial_j = \partial/\partial x_j \).

Definition 1 (see [29]). One denotes by \( \delta' \) or \( \delta'(\mathbb{R}^n) \) the Schwartz space of all infinitely differentiable functions \( \varphi \) in \( \mathbb{R}^n \) such that
\[
\| \varphi \|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^n} \left| x^\alpha \delta^\beta \varphi(x) \right| < \infty
\]
for all \( \alpha, \beta \in \mathbb{N}_0^n \), equipped with the topology defined by the seminorms \( \| \cdot \|_{\alpha, \beta} \). The elements of \( \delta' \) are called rapidly decreasing functions, and the elements of the dual space \( \delta' \) are called tempered distributions.

Definition 2 (see [26]). One denotes by \( \mathcal{S} \) or \( \mathcal{S}(\mathbb{R}^n) \) the Gelfand space of all infinitely differentiable functions \( \varphi \) in \( \mathbb{R}^n \) such that
\[
\| \varphi \|_{h,k} = \sup_{x \in \mathbb{R}^n, a \in \mathbb{R}^n} \left| x^a \delta^b \varphi(x) \right| < \infty
\]
for some \( h, k > 0 \). One says that \( \varphi_j \to 0 \) as \( j \to \infty \) if \( \| \varphi_j \|_{h,k} \to 0 \) as \( j \to \infty \) for some \( h, k \), and one denotes by \( \mathcal{S}' \) the dual space of \( \mathcal{S} \) and calls its elements Gelfand hyperfunctions.

It is well known that the following topological inclusions hold:
\[
\mathcal{S} \hookrightarrow \delta', \quad \delta' \hookrightarrow \mathcal{S}'.
\]

It is known that the space \( \mathcal{S}(\mathbb{R}^n) \) consists of all infinitely differentiable functions \( \varphi(x) \) on \( \mathbb{R}^n \) which can be extended to an entire function on \( \mathbb{C}^n \) satisfying
\[
|\varphi(x + iy)| \leq C \exp \left( -a|x|^2 + b|y|^2 \right), \quad x, y \in \mathbb{R}^n
\]
for some \( a, b, \) and \( C > 0 \) (see [26]).

By virtue of Theorem 6.12 of [27, p. 134] we have the following.

Definition 3. Let \( u_j \in \mathcal{S}'(\mathbb{R}^n) \) for \( j = 1, 2 \), with \( n_1 \geq n_2 \), and let \( \lambda : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2} \) be a smooth function such that, for each \( x \in \mathbb{R}^{n_1} \), the Jacobian matrix \( \nabla \lambda(x) \) of \( \lambda \) at \( x \) has rank \( n_2 \). Then there exists a unique continuous linear map \( \lambda^* : \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^{n_2}) \to \mathcal{S}'(\mathbb{R}^{n_2}) \) such that \( \lambda^* u = u \circ \lambda \) when \( u \) is a continuous function. One calls \( \lambda^* \) the pullback of \( u \) by \( \lambda \) which is often denoted by \( u \circ \lambda \).

In particular, let \( \lambda : \mathbb{R}^{2n} \to \mathbb{R}^{n} \) be defined by \( \lambda(x, y) = x - y, x, y \in \mathbb{R}^n \). Then in view of the proof of Theorem 6.12 of [27, p. 134] we have
\[
\langle u \circ \lambda, \varphi(x, y) \rangle = \left\langle u, \int \varphi(x - y, y) dy \right\rangle.
\]

Definition 4. Let \( u_x \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n), u_y \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n) \). Then the tensor product \( u_x \otimes u_y \) and \( u_x \), defined by
\[
\langle u_x \otimes u_y, \varphi(x, y) \rangle = \left\langle u_x, \langle u_y, \varphi(x, y) \rangle \right\rangle
\]
for \( \varphi(x, y) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \), belongs to \( \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n) \).

For more details of pullback and tensor product distributions of distributions we refer the reader to Chapter V-VI of [27].

3. Main Theorems

Let \( f \) be a Lebesgue measurable function on \( \mathbb{R}^n \). Then \( f \) is said to be an infraexponential function of order 2 (resp,
a function of polynomial growth) if for every ε > 0 there exists 
\( C_\varepsilon > 0 \) (resp., there exist positive constants \( C, N, \) and \( d \)) such that

\[
|f(x)| \leq C_\varepsilon e^{\varepsilon |x|^d} \quad \text{[resp. \( \leq C|x|^N + d \)]} \tag{17}
\]

for all \( x \in \mathbb{R}^n \). It is easy to see that every infraexponential function \( f \) of order 2 (resp., every function of polynomial growth) defines an element of \( \mathcal{S}(\mathbb{R}^n) \) (resp., \( \delta'(\mathbb{R}^n) \)) via the correspondence

\[
(f, \varphi) = \int f(x) \varphi(x) \, dx \tag{18}
\]

for \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) (resp., \( \delta'(\mathbb{R}^n) \)).

Let \( u, v \in \mathcal{S}(\mathbb{R}^n) \) (resp., \( \delta'(\mathbb{R}^n) \)). We prove the stability of the following functional inequalities:

\[
\|u \circ (x - y) - u_x \otimes v_y + v_x \otimes u_y \| \leq \psi(y), \tag{19}
\]

\[
\|v \circ (x - y) - v_x \otimes v_y - u_x \otimes u_y \| \leq \psi(y), \tag{20}
\]

where \( \circ \) and \( \otimes \) denote the pullback and the tensor product of generalized functions, respectively, \( \psi : \mathbb{R}^n \to [0, \infty) \) denotes a continuous infraexponential functional of order 2 (resp. a continuous function of polynomial growth) with \( \psi(0) = 0 \), and \( \| \cdot \| \leq \psi \) means that \( |\langle \cdot, \varphi \rangle| \leq \|\varphi\|_1 \) for all \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) (resp., \( \delta'(\mathbb{R}^n) \)).

In view of (14) it is easy to see that the \( n \)-dimensional heat kernel

\[
E_t(x) = (4\pi t)^{-n/2} \exp \left( -\frac{|x|^2}{4t} \right), \quad t > 0, \tag{21}
\]

belongs to the Gelfand space \( \mathcal{S}(\mathbb{R}^n) \) for each \( t > 0 \). Thus the convolution \( (u \ast E_t)(x) := \langle u, E_t(x - y) \rangle \) is well defined for all \( u \in \mathcal{S}(\mathbb{R}^n) \). It is well known that \( U(x, t) = (u \ast E_t)(x) \) is a smooth solution of the heat equation \( \partial_t U - \Delta U = 0 \) in \( \{(x, t) : x \in \mathbb{R}^n, t > 0 \} \) and \( (u \ast E_t)(x) \to u \) as \( t \to 0^+ \) in the sense of generalized functions that is, for every \( \varphi \in \mathcal{S}(\mathbb{R}^n) \),

\[
\langle u, \varphi \rangle = \lim_{t \to 0} \int (u \ast E_t)(x) \varphi(x) \, dx. \tag{22}
\]

We call \( (u \ast E_t)(x) \) the Gauss transform of \( u \).

A function \( A \) from a semigroup \( (S, +) \) to the field \( \mathbb{C} \) of complex numbers is said to be an additive function provided that \( A(x + y) = A(x) + A(y) \), and \( m : S \to \mathbb{C} \) is said to be an exponential function provided that \( m(x + y) = m(x)m(y) \).

For the proof of stabilities of (19) and (20) we need the following.

**Lemma 5** (see [15]). Let \( S \) be a semigroup and \( \mathbb{C} \) the field of complex numbers. Assume that \( f, g : S \to \mathbb{C} \) satisfy the inequality; for each \( y \in S \) there exists a positive constant \( M_y \) such that

\[
|f(x + y) - f(x)g(y)| \leq M_y \tag{23}
\]

for all \( x \in S \). Then either \( f \) is a bounded function or \( g \) is an exponential function.

**Proof.** Suppose that \( g \) is not exponential. Then there are \( y, z \in S \) such that \( g(y + z) \neq g(y)g(z) \). Now we have

\[
f(x + y + z) - f(x + y)g(z) = (f(x + y + z) - f(x)g(y + z))
\]

\[
- g(z)(f(x + y) - f(x)g(y))
\]

\[
+ f(x)(g(y + z) - g(y)g(z)),
\]

and hence

\[
f(x) = (g(y + z) - g(y)g(z))^{-1} \times ((f(x + y + z) - f(x + y)g(z))
\]

\[
- (f(x + y + z) - f(x)g(y + z))
\]

\[
+ g(z)(f(x + y) - f(x)g(y))).
\]

In view of (23) the right side of (25) is bounded as a function of \( x \). Consequently, \( f \) is bounded. \( \square \)

**Lemma 6** (see [30, p. 122]). Let \( f(x, t) \) be a solution of the heat equation. Then \( f(x, t) \) satisfies

\[
|f(x, t)| \leq M, \quad x \in \mathbb{R}^n, \ t \in (0, 1) \tag{26}
\]

for some \( M > 0 \), if and only if

\[
f(x, t) = (f_0 \ast E_t)(x) = \int f_0(y)E_t(x - y) \, dy \tag{27}
\]

for some bounded measurable function \( f_0 \) defined in \( \mathbb{R}^n \). In particular, \( f(x, t) \to f_0(x) \) in \( \mathcal{S}(\mathbb{R}^n) \) as \( t \to 0^+ \).

We discuss the solutions of the corresponding trigonometric functional equations

\[
u \circ (x - y) - u_x \otimes v_y + v_x \otimes u_y = 0, \tag{28}
\]

\[
u \circ (x - y) - v_x \otimes v_y - u_x \otimes u_y = 0, \tag{29}
\]

in the space \( \mathcal{S}' \) of Gelfand hyperfunctions. As a consequence of the results [8, 31, 32] we have the following.

**Lemma 7.** The solutions \( u, v \in \mathcal{S}'(\mathbb{R}^n) \) of (28) and (29) are equal, respectively, to the continuous solutions \( f, g : \mathbb{R}^n \to \mathbb{C} \) of corresponding classical functional equations

\[
f(x + y) - f(x)g(y) + g(x)f(y) = 0, \tag{30}
\]

\[
f(x + y) - g(x)g(y) - f(x)f(y) = 0. \tag{31}
\]

The continuous solutions \( f, g \) of the functional equation (30) are given by one of the following:

(i) \( f = 0 \) and \( g \) is arbitrary,

(ii) \( f(x) = c_1 \cdot x, g(x) = 1 + c_2 \cdot x \) for some \( c_1, c_2 \in \mathbb{C}^n \),

(iii) \( f(x) = \lambda_1 \sin(c \cdot x) \) and \( g(x) = \cos(c \cdot x) + \lambda_2 \sin(c \cdot x) \) for some \( \lambda_1, \lambda_2 \in \mathbb{C}, c \in \mathbb{C}^n \).
Also, the continuous solutions \((f, g)\) of the functional equation (31) are given by one of the following:

(i) \(g(x) = \lambda \) and \(f(x) = \pm \sqrt{\lambda - \lambda^2} \) for some \(\lambda \in \mathbb{C}\),

(ii) \(g(x) = \cos(c \cdot x)\) and \(f(x) = \sin(c \cdot x)\) for some \(c \in \mathbb{C}^n\).

For the proof of the stability of (19) we need the following:

**Lemma 8.** Let \(G\) be an Abelian group and let \(U, V : G \times (0, \infty) \rightarrow \mathbb{C}\) satisfy the inequality; there exists a nonnegative function \(\Psi : G \times (0, \infty) \rightarrow \mathbb{R}\) such that

\[
|U(x - y, t + s) - U(x, t)V(y, s) + V(x, t)U(y, s)| \leq \Psi(y, s)
\]

for all \(x, y \in G, t, s > 0\). Then either there exist \(\lambda_1, \lambda_2 \in \mathbb{C}\), not both are zero, and \(M > 0\) such that

\[
|\lambda_1 U(x, t) - \lambda_2 V(x, t)| \leq M,
\]

or else

\[
U(x - y, t + s) - U(x, t)V(y, s) + V(x, t)U(y, s) = 0
\]

for all \(x, y \in G, t, s > 0\).

**Proof.** Suppose that inequality (33) holds only when \(\lambda_1 = \lambda_2 = 0\). Let

\[
K(x, y, t, s) = U(x + y, t + s) - U(x, t)V(-y, s) + V(x, t)U(-y, s),
\]

and choose \(y_1\) and \(s_1\) satisfying \(U(-y_1, s_1) \neq 0\). Now it can be easily calculated that

\[
V(x, t) = \lambda_0 U(x, t) + \lambda_1 U(x + y_1, t + s_1) - \lambda_1 K(x, y_1, t, s_1),
\]

where \(\lambda_0 = V(-y_1, s_1)/U(-y_1, s_1)\) and \(\lambda_1 = -1/U(-y_1, s_1)\).

By (35) we have

\[
U(x + (y + z), t + (s + r)) = U(x, t)V(-y - z, s + r) - V(x, t)U(-y - z, s + r) + K(x, y + z, t, s + r).
\]

Also by (35) and (36) we have

\[
U((x + y) + z, (t + s) + r) = U(x + y, t + s)V(-z, r) - V(x + y, t + s)U(-z, r) + K(x + y, z, t + s, r)
\]

\[
= (U(x, t)V(-y, s) - V(x, t)U(-y, s)) + K(x, y, t, s) + \lambda_0 V(-y - y_1, s + s_1) + \lambda_1 U(-y - y_1, t + s + s_1)
\]

\[
= -K(x, y, t, s)V(-z, r) + \lambda_0 K(x, y + y_1, t + s + s_1) + \lambda_1 K(x + y + y_1, t + s + s_1) + K(x + y, z, t + s, r).
\]

From (37) and (38) we have

\[
(V(-y, s)V(-z, r) - \lambda_0 V(-y, s)U(-z, r) + \lambda_1 U(-y, s)U(-z, r) + K(x, y, t, s))V(x, t)
\]

\[
= -K(x, y, t, s)V(-z, r) + \lambda_0 K(x, y + y_1, t + s + s_1) + \lambda_1 K(x + y + y_1, t + s + s_1) + K(x + y, z, t + s, r).
\]
Since $K(x, y, t, s)$ is bounded by $\Psi(-y, s)$, if we fix $y, z, r,$ and $s$, the right hand side of (39) is bounded by a constant $M$, where

$$M = \Psi(-y, s) |V(-z, r)| + \Psi(-y, s) |\lambda_1 U(-z, r)|$$

$$+ \Psi(-y - y_1, s + s_1) |\lambda_1 U(-z, r)|$$

$$+ \Psi(-y, s_1) |\lambda_1 U(-z, r)| + \Psi(-z, r)$$

$$+ \Psi(-y - z, r + s).$$

So by our assumption, the left hand side of (39) vanishes, so is the right hand side. Thus we have

$$-\lambda_0 K(x, y, t, s) - \lambda_1 K(x, y + y_1, t, s + s_1)$$

$$+ \lambda_1 K(x + y, y_1, t + s, s_1) U(-z, r)$$

$$+ K(x, y, t, s) V(-z, r) = K(x, y + z, t, s + r)$$

Now by the definition of $K$ we have

$$K(x + y, z, t + s, r) - K(x, y + z, t, s + r)$$

$$= U(x + y, z, t + s + r) - U(x + y, t + s) V(-z, r)$$

$$+ V(x + y, t + s) U(-z, r) - U(x + y + z, t + s + r)$$

$$+ U(x, t) V(-y - z, s + r) - V(x, t) U(-y - z, s + r)$$

$$= U(-y - z - x, s + r + t) - U(-y - z, s + r) V(x, t)$$

$$+ V(-y - z, s + r) U(x, t) - U(-z - x - y, r + t + s)$$

$$+ U(-z, r) V(x + y, t + s) - V(-z, r) U(x + y, t + s)$$

$$= K(-y - z - x, s + r, t) - K(-z, -x - y, r, t + s).$$

Hence the left hand side of (41) is bounded by $\Psi(x, t) + \Psi(x + y, y, t + s)$. So if we fix $x, y, t, s$ and $s$ in (41), the left hand side of (41) is a bounded function of $y$ and $r$. Thus $K(x, y, t, s) \equiv 0$ by our assumption. This completes the proof. \( \Box \)

In the following lemma we assume that $\Psi: \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{C}$ is a continuous function such that

$$\Psi(x) := \lim_{t \rightarrow 0^+} \Psi(x, t)$$

exists and satisfies the conditions $\Psi(0) = 0$ and

$$\Phi_1(x) := \sum_{k=0}^{\infty} 2^{-k} \Psi(-2^k x) < \infty \quad (44)$$

or

$$\Phi_2(x) := \sum_{k=1}^{\infty} 2^{k} \Psi(-2^{-k} x) < \infty. \quad (45)$$

**Lemma 9.** Let $U, V : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{C}$ be continuous functions satisfying

$$|U(x - y, t + s) - U(x, t) V(y, s) + V(x, t) U(y, s)| \leq \Psi(y, s)$$

(46)

for all $x, y \in \mathbb{R}^n, t, s > 0$, and there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both are zero, and $M > 0$ such that

$$|\lambda_1 U(x, t) - \lambda_2 V(x, t)| \leq M. \quad (47)$$

Then $(U, V)$ satisfies one of the following:

(i) $U = 0, V$ is arbitrary.

(ii) $U$ and $V$ are bounded functions,

(iii) $V(x, t) = \lambda U(x, t) + e^{ic-bt}$ for some $\lambda \in \mathbb{C}^n, c(\neq 0) \in \mathbb{R}^n$, and $b \in \mathbb{C}$, and $f(x) := \lim_{t \rightarrow 0^-} U(x, t)$ is a continuous function; in particular, there exists $\delta : (0, \infty) \rightarrow [0, \infty)$ with $\delta(t) \rightarrow 0$ as $t \rightarrow 0^-$ such that

$$|U(x, t) - f(x) e^{-bt}| \leq \delta(t) \quad (48)$$

for all $x \in \mathbb{R}^n, t > 0$, and satisfies the condition; there exists $d \geq 0$ satisfying

$$|f(x)| \leq \psi(-x) + d \quad (49)$$

for all $x \in \mathbb{R}^n$,

(iv) $V(x, t) = \lambda U(x, t) + e^{-bt}$ for some $\lambda \in \mathbb{C}^n, b \in \mathbb{C}$, and $f(x) := \lim_{t \rightarrow 0^-} U(x, t)$ is a continuous function; in particular, there exists $\delta : (0, \infty) \rightarrow [0, \infty)$ with $\delta(t) \rightarrow 0$ as $t \rightarrow 0^+$ such that

$$|U(x, t) - f(x) e^{-bt}| \leq \delta(t) \quad (50)$$

for all $x \in \mathbb{R}^n, t > 0$, and satisfies one of the following conditions; there exists $a_1 \in \mathbb{C}^n$ such that

$$|f(x) - a_1 \cdot x| \leq \Phi_1(x) \quad (51)$$

for all $x \in \mathbb{R}^n$, or there exists $a_2 \in \mathbb{C}^n$ such that

$$|f(x) - a_2 \cdot x| \leq \Phi_2(x) \quad (52)$$

for all $x \in \mathbb{R}^n$.

**Proof.** If $U = 0, V$ is arbitrary which is case (i). If $U$ is a nontrivial bounded function, in view of (46) $V$ is also bounded which gives case (ii). If $U$ is unbounded, it follows from (47) that $\lambda_2 \neq 0$ and

$$V(x, t) = \lambda U(x, t) + R(x, t) \quad (53)$$

for some $\lambda \in \mathbb{C}$ and a bounded function $R$. Putting (53) in (46) we have

$$|U(x - y, t + s) - U(x, t) R(y, s) + R(x, t) U(y, s)| \leq \Psi(y, s) \quad (54)$$
for all \( x, y \in \mathbb{R}^n, t, s > 0 \). Replacing \( y \) by \(-y\) and using the triangle inequality, we have, for some \( C > 0 \),
\[
|U(x + y, t + s) - U(x, t) R(y, s)| \\
\leq C |U(y, s)| + \Psi(-y, s)
\]
for all \( x, y \in \mathbb{R}^n, t, s > 0 \). By Lemma 5, \( R(-y, s) \) is an exponential function. If \( R = 0 \), putting \( y = 0, s \to 0^+ \) in (54) we have
\[
|U(x, t)| \leq \psi(0) = 0.
\]
Thus we have \( R \neq 0 \) since \( U \) is unbounded. Given the continuity of \( U \) and \( V \) we have
\[
R(x, t) = e^{ic \cdot x - bt}
\]
for some \( c \in \mathbb{R}^n, b \in \mathbb{C} \) with \( |Rb| \geq 0 \). Putting \( y = 0 \) and \( s = 1 \) in (54), dividing \( R(0, 1) \), and using the triangle inequality we have
\[
|U(x, t)| \leq |R(0, 1)|^{-1} |U(x, t + 1)| + C |U(0, 1)| + \Psi(0, 1)
\]
for all \( x \in \mathbb{R}^n, t > 0 \).

From (58) and the continuity of \( U \) it is easy to see that
\[
\lim_{t \to 0^+} U(x, t) := f(x)
\]
exists. Putting \( x = y = 0 \) and replacing \( s \) and \( t \) by \( t/2 \) in (54) we have
\[
|U(0, t)| \leq \Psi(0, \frac{t}{2})
\]
for all \( t > 0 \).

Fixing \( x \), putting \( y = 0 \) letting \( t \to 0^+ \) so that \( U(x, t) \to f(x) \) in (54), and using the triangle inequality and (60) we have
\[
|U(x, s) - f(x) e^{-bt}| \leq \Psi \left(0, \frac{s}{2}\right) + \Psi(0, s) := \delta(s)
\]
for all \( x \in \mathbb{R}^n, s > 0 \). Letting \( s \to 0^+ \) in (61) we have
\[
\lim_{s \to 0^+} U(x, s) = f(x)
\]
for all \( x \in \mathbb{R}^n \). From (61) the continuity of \( f \) can be checked by a usual calculus. Letting \( t \to 0^+ \) in (60) we see that \( f(0) = 0 \). Letting \( t, s \to 0^+ \) in (54) we have
\[
|f(x - y) - f(x)e^{iy} + e^{kx} f(y)| \leq \psi(y)
\]
for all \( x, y \in \mathbb{R}^n \). Putting \( x = 0 \) in (63) and replacing \( y \) by \(-y\) we have
\[
|f(-y) + f(y)| \leq \psi(-y)
\]
for all \( y \in \mathbb{R}^n \).

Replacing \( y \) by \(-y\) and using (64) and the triangle inequality we have
\[
|f(x + y) - f(x)e^{iy} - e^{kx} f(y)| \leq 2\psi(-y)
\]
for all \( x, y \in \mathbb{R}^n \). Now we divide (65) into two cases: \( c = 0 \) and \( c \neq 0 \). First we consider the case \( c \neq 0 \). Replacing \( x \) by \( y \) and \( y \) by \( x \) in (65) we have
\[
|f(x + y) - f(y)e^{ix} - e^{-ky} f(x)| \leq 2\psi(-x)
\]
for all \( x, y \in \mathbb{R}^n \). From (65) and (66), using the triangle inequality and dividing \(|e^{ic\cdot y} - e^{-ic\cdot y}| \) we have
\[
|f(x)| \leq \frac{2(\psi(-x) + \psi(y) + |f(y)|)}{|e^{ic\cdot y} - e^{-ic\cdot y}|}
\]
for all \( x, y \in \mathbb{R}^n \) such that \( c \cdot y \neq 0 \). Choosing \( y_0 \in \mathbb{R}^n \) so that \( c \cdot y_0 = \pi/2 \) and putting \( y = y_0 \) in (67) we have
\[
|f(x)| \leq \psi(x) + d,
\]
where \( d = \psi(\pi/2) + |f(\pi/2)| \), which gives (iii). Now we consider the case \( c = 0 \). It follows from (65) that
\[
|f(x + y) - f(x) - f(y)| \leq 2\psi(-y)
\]
for all \( x, y \in \mathbb{R}^n \). By the well-known results in [3], there exists a unique additive function \( A_1(x) \) given by
\[
A_1(x) = \lim_{n \to \infty} 2^{-n} f(2^{-n} x)
\]
such that
\[
|f(x) - A_1(x)| \leq \Phi_1(x)
\]
if \( \Phi_1(x) := \sum_{k=0}^{\infty} 2^{-k} \psi(-2^{-k} x) < \infty \), and there exists a unique additive function \( A_2(x) \) given by
\[
A_2(x) = \lim_{n \to \infty} 2^n f(2^n x)
\]
such that
\[
|f(x) - A_2(x)| \leq \Phi_2(x)
\]
if \( \Phi_2(x) := \sum_{k=0}^{\infty} 2^k \psi(-2^k x) < \infty \). Now by the continuity of \( U \) and inequality (61), it is easy to see that \( f \) is continuous. In view of (70) and (72), \( A_j(x), j = 1, 2 \), are Lebesgue measurable functions. Thus there exist \( a_1, a_2 \in C^n \) such that \( A_1(x) = a_1 \cdot x \) and \( A_2(x) = a_2 \cdot x \) for all \( x \in \mathbb{R}^n \), which gives (iv). This completes the proof.

In the following we assume that \( \psi \) satisfies (44) or (45).

**Theorem 10.** Let \( u, v \in \mathcal{G} \) satisfy (19). Then \( (u, v) \) satisfies one of the followings:

(i) \( u = 0 \), and \( v \) is arbitrary,

(ii) \( u \) and \( v \) are bounded measurable functions,
(iii) \( v(x) = \lambda u(x) + e^{i c \cdot x} \) for some \( \lambda \in \mathbb{C}, c( \neq 0) \in \mathbb{R}^n \), where \( u \) is a continuous function satisfying the condition; there exists \( d \geq 0 \)
\[ |u(x)| \leq \psi(-x) + d \] (74)
for all \( x \in \mathbb{R}^n \),
(iv) \( v(x) = \lambda u(x) + 1 \) for some \( \lambda \in \mathbb{C} \), where \( u \) is a continuous function satisfying one of the following conditions; there exists \( a_i \in \mathbb{C} \) such that
\[ |u(x) - a_i \cdot x| \leq \Phi_i(x) \] (75)
for all \( x \in \mathbb{R}^n \), or there exists \( a_2 \in \mathbb{C}^n \) such that
\[ |u(x) - a_2 \cdot x| \leq \Phi_2(x) \] (76)
for all \( x \in \mathbb{R}^n \),
(v) \( u = \lambda \sin(c \cdot x) \), \( v = \cos(c \cdot x) + \lambda \sin(c \cdot x) \), for some \( c \in \mathbb{C}^n \), \( \lambda \in \mathbb{C} \).

Proof. Convolving in (19) the tensor product \( E_i(x)E_j(y) \) of \( n \)-dimensional heat kernels in both sides of inequality (19) we have
\begin{align*}
[u \circ (x-y) \ast (E_i(\xi)E_j(\eta))](x, y) &= \left\langle u_2, \int E_i(x - \xi) \ast E_j(y - \eta) \, d\eta \right\rangle \\
&= \left\langle u_2, (E_i \ast E_j)(x-y, -\xi) \right\rangle \\
&= \left\langle u_2, (E_{i+1})(x-y, -\xi) \right\rangle \\
&= U(x-y,t+s).
\end{align*}

Similarly we have
\begin{align*}
[(u \otimes v) \ast (E_i(\xi)E_j(\eta))](x, y) &= U(x,t)V(y,s), \\
[(v \otimes u) \ast (E_i(\xi)E_j(\eta))](x, y) &= V(x,t)U(y,s),
\end{align*}
(78)
where \( U, V \) are the Gauss transforms of \( u, v \), respectively. Thus we have the following inequality:
\[ |U(x-y,t+s) - U(x,t)V(y,s) + V(x,t)U(y,s)| \leq \Psi(y,s) \] (79)
for all \( x, y \in \mathbb{R}^n \), \( t, s > 0 \), where
\[ \Psi(y,s) = \int \Psi(\eta)E_i(x-\xi)E_j(y-\eta) \, d\xi \, d\eta \] (80)

By Lemma 8 there exist \( \lambda_1, \lambda_2 \in \mathbb{C} \), not both are zero, and \( M > 0 \) such that
\[ |
\lambda_1 U(x,t) - \lambda_2 V(x,t)| \leq M, \] (81)
or else \( U, V \) satisfy
\[ U(x-y, t+s) - U(x,t)V(y,s) + V(x,t)U(y,s) = 0 \] (82)
for all \( x, y \in \mathbb{R}^n \), \( t, s > 0 \). Assume that (81) holds. Applying Lemma 9, case (i) follows from (i) of Lemma 9. Using (ii) of Lemma 9, it follows from Lemma 7 the initial values \( u, v \) of \( U(x,t), V(x,t) \) as \( t \to 0^+ \) are bounded measurable functions, respectively, which gives (ii). For case (iii), it follows from (50) that, for all \( \varphi \in \mathcal{S}(\mathbb{R}^n) \),
\[ |\langle u, \varphi \rangle - \langle f, \varphi \rangle| \leq \lim_{t \to 0^+} \int U(x,t) \varphi(x) \, dx - \int f(x) \varphi(x) \, dx \]
\[ = \lim_{t \to 0^+} \int \left( U(x,t) - f(x) e^{-bt} \right) \varphi(x) \, dx \]
\[ \leq \lim_{t \to 0^+} \int |U(x,t) - f(x) e^{-bt}| |\varphi(x)| \, dx \]
\[ \leq \lim_{t \to 0^+} \delta(t) \int |\varphi(x)| \, dx = 0. \]
Thus we have \( u = f \) in \( \mathcal{S}^0(\mathbb{R}^n) \). Letting \( t \to 0^+ \) in (iii) of Lemma 9 we get case (iii). Finally we assume that (82) holds. Letting \( t, s \to 0^+ \) in (82) we have
\[ u \circ (x-y) - u \otimes v \oslash v + v \otimes u \oslash u = 0. \] (84)
By Lemma 6 the solutions of (84) satisfy (i), (iv), or (v). This completes the proof. \( \square \)

Let \( \psi(x) = e|x|^p > 0 \). Then \( \psi \) satisfies the conditions assumed in Theorem 10. In view of (44) and (45) we have
\[ \Phi_1(x) = \frac{2e|x|^p}{2 - 2^p} \] (85)
if \( 0 < p < 1 \), and
\[ \Phi_2(x) = \frac{2e|x|^p}{2^p - 2} \] (86)
if \( p > 1 \). Thus as a direct consequence of Theorem 10 we have the following.

**Corollary 11.** Let \( 0 < p < 1 \) or \( p > 1 \). Suppose that \( u, v \in \mathcal{S}^0 \) satisfy
\[ \|u \circ (x-y) - u \otimes v \oslash v + v \otimes u \oslash u\| \leq e|y|^p. \] (87)
Then \( u, v \) satisfies one of the followings:
\begin{enumerate}
\item \( u = 0 \), and \( v \) is arbitrary,
\item \( u \) and \( v \) are bounded measurable functions,
\end{enumerate}
(iii) \( v(x) = \lambda u(x) + e^{ic \cdot x} \) for some \( \lambda \in \mathbb{C}, c (\neq 0) \in \mathbb{R}^n \), where \( u \) is a continuous function satisfying the condition; there exists \( d \geq 0 \)
\[ |u(x)| \leq |x|^p + d \] (88)
for all \( x \in \mathbb{R}^n \),
(iv) \( v(x) = \lambda u(x) + 1 \) for some \( \lambda \in \mathbb{C} \), where \( u \) is a continuous function satisfying the conditions; there exists \( a \in \mathbb{C}^n \) such that
\[ |u(x) - a \cdot x| \leq 2|y|^p \] (89)
for all \( x, y \in \mathbb{R}^n \),
(v) \( u = \lambda \sin(c \cdot x), V = \cos(c \cdot x) + \lambda \sin(c \cdot x) \), for some \( c \in \mathbb{C}^n, \lambda \in \mathbb{C} \).

Now we prove the stability of (20). For the proof we need the following.

Lemma 12. Let \( U, V : G \times (0, \infty) \to \mathbb{C} \) satisfy the inequality; there exists a \( \Psi : G \times (0, \infty) \to [0, \infty) \) such that
\[ |V(x - y, t + s) - V(x, t)V(y, s) - U(x, t)U(y, s)| \leq \Psi(y, s) \] (90)
for all \( x, y \in \mathbb{R}^n, t, s > 0 \). Then either there exist \( \lambda_1, \lambda_2 \in \mathbb{C} \), not both are zero, and \( M > 0 \) such that
\[ |\lambda_1 U(x, t) - \lambda_2 V(x, t)| \leq M, \] (91)
or else
\[ V(x - y, t + s) - V(x, t)V(y, s) - U(x, t)U(y, s) = 0 \] (92)
for all \( x, y \in G, t, s > 0 \).

Proof. As in Lemma 9, suppose that \( \lambda_1 U(x, t) - \lambda_2 V(x, t) \) is bounded only when \( \lambda_1 = \lambda_2 = 0 \), and let
\[ L(x, y, t, s) = V(x + y, t + s) - V(x, t)V(y, s) - U(x, t)U(y, s). \] (93)
Since we may assume that \( U \) is nonconstant, we can choose \( y_1 \) and \( s_1 \) satisfying \( U(-y_1, s_1) \neq 0 \). Now it can be easily got that
\[ U(x, t) = \lambda_0 V(x, t) + \lambda_1 V(x + y_1, t + s_1) \]
\[ - \lambda_1 L(x, y_1, t, s_1), \] (94)
where \( \lambda_0 = -V(-y_1, s_1)/U(-y_1, s_1) \) and \( \lambda_1 = 1/U(-y_1, s_1) \).
From the definition of \( L \) and the use of (94), we have the following two equations:
\[ V((x + y) + z, (t + s) + r) \]
\[ = V(x + y, t + s) V(-z, r) + U(x + y, t + s) U(-z, r) \]
\[ + L(x + y, z, t + s, r) \]
\[ = (V(x, t) V(-y, s) + U(x, t) U(-y, s)) \]
\[ + L(x, y, t, s)) V(-z, r) \]
\[ + \lambda_0 V(x + y, t + s) + \lambda_1 V(x + y + y_1, t + s + s_1) \]
\[ - \lambda_1 L(x + y, y_1, t + s, s_1)) U(-z, r) \]
\[ + L(x + y, z, t + s, r) \]
\[ = (V(x, t) V(-y, s) + U(x, t) U(-y, s)) \]
\[ + L(x, y, t, s)) V(-z, r) \]
\[ + \lambda_0 V(x, t) V(-y, s) + U(x, t) U(-y, s) \]
\[ + L(x, y, t, s)) U(-z, r) \]
\[ + \lambda_1 V(x, t) V(-y - y_1, s + s_1) + U(x, t) U(-y - y_1, s + s_1) \]
\[ + L(x, y + y_1, t, s + s_1)) U(-z, r) \]
\[ - \lambda_1 L(x + y, y_1, t + s, s_1)) U(-z, r) \]
\[ + L(x + y, z, t + s, r), \] (95)
\[ V((x + (y + z), t + (s + r)) \]
\[ = V(x, t) V(-y - z, s + r) + U(x, t) U(-y - z, s + r) \]
\[ + L(x, y + z, t, s + r). \] (96)

By equating (95) and (96), we have
\[ V(x, t) (V(-y, s)) V(-z, r) + \lambda_0 V(-y, s) U(-z, r) \]
\[ + \lambda_1 V(-y - y_1, s + s_1) U(-z, r) \]
\[ - V(-y - z, s + r)) \]
\[ + U(x, t) U(-y, s)) V(-z, r) + \lambda_0 U(-y, s) U(-z, r) \]
\[ + \lambda_1 U(-y - y_1, s + s_1)) U(-z, r) \]
\[ - U(-y - z, s + r)) \]
Abstract and Applied Analysis

\[ -L(x, y, t, s) = -L(x, y, t, s) - v(-z, r) - \lambda_0 L(x, y, t, s) U(-z, r) \\
- \lambda_1 L(x + y, t + s) U(-z, r) \\
+ \lambda_1 L(x + y + y_1, t + s + s_1) U(-z, r) \\
- L(x + y, z, t + s, r) + L(x, y + z, t + s, r). \tag{97} \]

In (97), when \( y, z, s, r \) are fixed, the right hand side is bounded; so by our assumption we have

\[ L(x, y, t, s) V(-z, r) \\
+ (\lambda_0 L(x, y, t, s) + \lambda_1 L(x + y, t + s, s_1)) U(-z, r) \\
- \lambda_1 L(x + y, y_1, t, s_1) U(-z, r) \\
= L(x, y + z, t + s, r) - L(x + y, z, t + s, r). \tag{99} \]

Here, we have

\[ L(x, y + z, t + s, r) - L(x + y, z + t + s, r) \]
\[ = \Psi(x, t) + \Psi(x + y, t + s). \tag{100} \]

Considering (98) as a function of \( z \) and \( r \) for all fixed \( x, y, t, \) and \( s \) again, we have \( L(x, y, t, s) = 0 \). This completes the proof. \( \square \)

In the following lemma we assume that \( \Psi : \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty) \) is a continuous function such that

\[ \psi(x) := \lim_{t \to 0} \Psi(x, t) \tag{101} \]

exists and satisfies the condition \( \psi(0) = 0 \).

Lemma 13. Let \( U, V : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{C} \) be continuous functions satisfying

\[ |V(x - y, t + s) - V(x, t) V(y, s) - U(x, t) U(y, s)| \]
\[ \leq \Psi(y, s) \tag{102} \]

for all \( x, y \in \mathbb{R}^n, t, s > 0 \) and there exist \( \lambda_1, \lambda_2 \in \mathbb{C} \), not both zero, and \( M > 0 \) such that

\[ |\lambda_1 U(x, t) - \lambda_2 V(x, t)| \leq M. \tag{103} \]

Then \( (U, V) \) satisfies one of the followings:

(i) \( U \) and \( V \) are bounded functions in \( \mathbb{R}^n \times (0, 1) \),

(ii) \( \pm i U(x, t) = V(x, t) - e^{i\lambda x} e^{-t} \) for some \( a \in \mathbb{R}^n \), \( b \in \mathbb{C} \), and \( g(x) := \lim_{t \to 0} V(x, t) \) is a continuous function; in particular, there exists \( \delta : (0, \infty) \rightarrow (0, \infty) \) with \( \delta(t) \to 0 \) as \( t \to 0^+ \) such that

\[ |V(x, t) - g(x) e^{-t}| \leq \delta(t) \tag{104} \]

for all \( x \in \mathbb{R}^n, t > 0 \), and \( g \) satisfies

\[ |g(x) - \cos(a \cdot x)| \leq \frac{1}{2} \psi(x) \tag{105} \]

for all \( x \in \mathbb{R}^n \).

Proof. If \( U \) is bounded, then in view of inequality (100), for each \( y, s \), \( V(x + y, t + s) - V(x, t) V(-y, s) \) is also bounded. It follows from Lemma 5 that \( V \) is (101). If \( V \) is bounded, case (i) follows. If \( V \) is a nonzero exponential function, then by the continuity of \( V \) we have

\[ V(x, t) = e^{\lambda x} e^{-t} \tag{107} \]

for some \( c \in \mathbb{C}^n, b \in \mathbb{C} \). Putting (105) in (107) and using the triangle inequality we have for some \( d > 0 \)

\[ |e^{\lambda x} e^{b(t + d)} (e^{c y} - e^{-y})| \leq \Psi(y, s) + d \tag{108} \]

for all \( x, y \in \mathbb{R}^n, t, s > 0 \). In view of (106) it is easy to see that \( c = ia, a \in \mathbb{R}^n \). Thus \( V(x, t) \) is bounded on \( \mathbb{R}^n \times (0, 1) \). If \( U \) is unbounded; then in view of (101) \( V \) is also unbounded, hence \( \lambda_1 \lambda_2 \neq 0 \) and

\[ U(x, t) = \lambda V(x, t) + R(x, t) \tag{109} \]

for some \( \lambda \neq 0 \) and a bounded function \( R \). Putting (107) in (101), replacing \( y \) by \(-y \), and using the triangle inequality we have

\[ |V(x + y, t + s) - V(x, t) (\lambda^2 + 1) V(-y, s) + \lambda R(-y, s)| \]
\[ \leq |(\lambda V(-y, s) + R(-y, s)) R(x, t)| + \Psi(-y, s). \tag{110} \]

From Lemma 5 we have

\[ (\lambda^2 + 1) V(y, s) + \lambda R(y, s) = m(y, s) \tag{111} \]

for some exponential function \( m \). From (107) and (109), \( m \) is continuous, and we have

\[ m(x, t) = e^{\lambda x} e^{-t} \tag{112} \]

for some \( c \in \mathbb{C}^n, b \in \mathbb{C} \). If \( \lambda^2 \neq -1 \), we have

\[ U = \frac{\lambda m + R}{\lambda^2 + 1} \quad V = \frac{m - \lambda R}{\lambda^2 + 1}. \tag{113} \]

Putting (113) in (101), multiplying \(|\lambda^2 + 1|\) in the result, and using the triangle inequality we have, for some \( d \geq 0 \),

\[ |m(x, t) (m(-y, s) - m(y, s))| \leq |\lambda^2 + 1| \Psi(y, s) + d \tag{114} \]
for all \( x, y \in \mathbb{R}^n, t, s > 0. \) Since \( m \) is unbounded, we have
\[
m(y, s) = m(-y, s) \tag{113}
\]
for all \( y \in \mathbb{R} \) and \( s > 0. \) Thus it follows that \( m(x, t) = e^{bt} \) and that \( U, V \) are bounded in \( \mathbb{R}^n \times (0, 1). \) If \( \lambda^2 = -1, \) we have
\[
U = \pm i (V - m), \tag{114}
\]
where \( m \) is a bounded exponential function. Putting (114) in (101) we have
\[
|V (x - y, t + s) - V (x, t) m(y, s) - V(y, s) m(x, t)| \leq \Psi(y, s) \tag{115}
\]
for all \( x, y \in \mathbb{R}^n, t, s > 0. \) Since \( m \) is a bounded continuous function, we have
\[
m(x, t) = e^{\lambda x - bt} \tag{116}
\]
for some \( a \in \mathbb{R}^n, b \in \mathbb{C} \) with \( \Re b \geq 0. \)

Similarly as in the proof of Lemma 9, by (101) and the continuity of \( V, \) it is easy to see that
\[
\limsup_{t \to 0^+} |V(x, t)| = g(x) \tag{117}
\]
exists. Putting \( x = y = 0 \) in (115), multiplying \( [e^{bt}] \) in both sides of the result, and using the triangle inequality we have
\[
|V(0, s) - e^{-bt}| \leq \left| e^{bt} \right| \left( |V(0, t + s) - V(0, t) e^{-bt}| + \Psi(0, s) \right) \tag{118}
\]
for all \( t, s > 0. \) Letting \( s \to 0^+ \) in (118) we have
\[
\lim_{t \to 0^+} V(0, t) = 1. \tag{119}
\]
Putting \( y = 0, \) fixing \( x, \) letting \( t \to 0^+ \) in (115) so that \( V(x, t) \to g(x), \) and using the triangle inequality we have
\[
|V(x, s) - g(x) e^{-bt}| \leq |V(0, s) - e^{-bt}| + \Psi(0, s) \tag{120}
\]
for all \( x \in \mathbb{R}^n, s > 0. \) Letting \( s \to 0^+ \) in (120) we have
\[
\lim_{s \to 0^+} V(x, s) = g(x) \tag{121}
\]
for all \( x \in \mathbb{R}^n. \) The continuity of \( g \) follows from (120). Letting \( t, s \to 0^+ \) in (115) we have
\[
|g(x - y) - g(x) e^{i\lambda y} - g(y) e^{i\lambda x} + e^{i\lambda (x+y)}| \leq |\Psi(y)| \tag{122}
\]
for all \( x, y \in \mathbb{R}^n. \) Replacing \( y \) by \( x \) in (122) and dividing the result by \( 2e^{i\lambda x} \) we have
\[
|g(x) - \cos(a \cdot x)| \leq \frac{1}{2} |\Psi(x)|. \tag{123}
\]
From (114), (116), (120) and (123) we get (ii). This completes the proof.

**Theorem 14.** Let \( u, v \in \mathcal{S}' \) satisfy (20). Then \( (u, v) \) satisfies one of the followings:

(i) \( u \) and \( v \) are bounded measurable functions,

(ii) \( v(x) = \cos(a \cdot x) + r(x) \) for some \( a \in \mathbb{R}^n, \) where \( r(x) \) is a continuous function satisfying
\[
|r(x)| \leq \frac{1}{2} |\Psi(x)| \tag{124}
\]
for all \( x \in \mathbb{R}^n. \)

(iii) \( v(x) = \cos(c \cdot x) \) and \( u(x) = \sin(c \cdot x) \) for some \( c \in \mathbb{C}^n. \)

**Proof.** Similarly as in the proof of Theorem 10 convolving in (20) the tensor product \( E_i(x) E_j(y) \) we obtain the inequality
\[
|V(x - y, t + s) - V(x, t) V(y, s) - U(x, t) U(y, s)| \leq \Psi(y, s) \tag{125}
\]
for all \( x, y \in \mathbb{R}^n, t, s > 0, \) where \( U, V \) are the Gauss transforms of \( u, v, \) respectively, and
\[
\Psi(y, s) = \left| \int \psi(\xi) E_i(x - \xi) E_j(y - \xi) d\xi d\eta \right|
\]
\[
= \left| \int \psi(\eta) E_i(\eta - y) d\eta \right| = (\psi \ast E_i)(y). \tag{126}
\]
By Lemma 12 there exist \( \lambda_1, \lambda_2 \in \mathbb{C}, \) not both zero, and \( M > 0 \) such that
\[
|\lambda_1 U(x, t) - \lambda_2 V(x, t)| \leq M, \tag{127}
\]
or else \( U, V \) satisfy
\[
V(x - y, t + s) - V(x, t) V(y, s) - U(x, t) U(y, s) = 0 \tag{128}
\]
for all \( x, y \in \mathbb{R}^n, t, s > 0. \)

Firstly we assume that (127) holds. Letting \( t \to 0^+ \) in (i) of Lemma 13, by Lemma 6, the initial values \( u, v \) of \( U(x, t), V(x, t) \) as \( t \to 0^+ \) are bounded measurable functions, respectively, which gives case (i). Using the same approach of the proof of case (iii) of Theorem 10, we have \( v = g \in \mathcal{S}'. \) It follows from (104) that
\[
v(x) = \cos(a \cdot x) + r(x), \tag{129}
\]
where \( r(x) \) is a continuous function satisfying
\[
|r(x)| \leq \frac{1}{2} |\Psi(x)| \tag{130}
\]
for all \( x \in \mathbb{R}^n. \) Letting \( t \to 0^+ \) in (ii) of Lemma 13 we have
\[
\pm iu(x) = v(x) - e^{i\lambda x}. \tag{131}
\]
Putting (129) in (131) we have
\[
\pm u(x) = \sin(a \cdot x) + ir(x). \tag{132}
\]
Secondly we assume that (128) holds. Letting \( t, s \to 0^+ \) in (127) we have
\[
v \ast (x - y) - v_x \otimes v_y - u_x \otimes u_y = 0.
\] (133)
By Lemma 7 the solution of (133) satisfies (i) or (iii). This completes the proof. 

Every infraexponential function \( f \) of order 2 defines an element of \( R_c(\mathbb{R}^n) \) via the correspondence
\[
(f, \varphi) = \int f(x) \varphi(x) \, dx
\] (134)
for \( \varphi \in R_c \). Thus as a direct consequence of Corollary 11 and Theorem 14 we have the followings:

Corollary 15. Let \( 0 < p < 1 \) or \( p > 1 \). Suppose that \( f, g \) are infraexponential functions of order 2 satisfying the inequality
\[
|f(x - y) - f(x)g(y) + g(x)f(y)| \leq \varepsilon |x|^p
\] for almost every \((x, y) \in \mathbb{R}^{2n}\). Then \((f, g)\) satisfies one of the following:

(i) \( f(x) = 0 \), almost everywhere \( x \in \mathbb{R}^n \), and \( g \) is arbitrary.

(ii) \( f \) and \( g \) are bounded in almost everywhere.

(iii) \( f(x) = f_0(x), g(x) = \lambda f_0(x) + e^{\alpha x} \) for almost every \( x \in \mathbb{R}^n \), where \( \lambda \in \mathbb{C}, \alpha \neq 0 \in \mathbb{R}^n \), and \( f_0 \) is a continuous function satisfying the condition; there exists \( d > 0 \)
\[
|f_0(x)| \leq \varepsilon |x|^p + d
\] (136)
for all \( x \in \mathbb{R}^n \).

(iv) \( f(x) = f_0(x), g(x) = \lambda f_0(x) + 1 \) for a.e. \( x \in \mathbb{R}^n \), where \( \lambda \in \mathbb{C} \) and \( f_0 \) is a continuous function satisfying the condition; there exists \( a \in C^n \) such that
\[
|f_0(x) - a \cdot x| \leq \frac{2\varepsilon|x|^p}{|\alpha|^2 - 2}  
\] (137)
for all \( x \in \mathbb{R}^n \).

(v) \( f(x) = \lambda \sin(c \cdot x), g(x) = \cos(c \cdot x) + \lambda \sin(c \cdot x) \) for a.e. \( x \in \mathbb{R}^n \), where \( c \in \mathbb{C}, \lambda \in \mathbb{C} \).

Corollary 16. Suppose that \( f, g \) are infraexponential functions of order 2 satisfying the inequality
\[
|g(x - y) - g(x)g(y) - f(x)f(y)| \leq \varepsilon |y|^p
\] (138)
for almost every \((x, y) \in \mathbb{R}^{2n}\). Then \((f, g)\) satisfies one of the followings:

(i) \( f \) and \( g \) are bounded in almost everywhere.

(ii) there exists \( a \in \mathbb{R}^n \) such that
\[
|g(x) - \cos(a \cdot x)| \leq \frac{1}{2} \varepsilon |x|^p,  
\] (139)
\[
|f(x) - \sin(a \cdot x)| \leq \frac{1}{2} \varepsilon |x|^p  
\] (140)
for almost every \( x \in \mathbb{R}^n \),

(iii) \( g(x) = \cos(c \cdot x) \) and \( f(x) = \sin(c \cdot x) \) for a.e. \( x \in \mathbb{R}^n \), where \( c \in \mathbb{C} \).

Remark 17. Taking the growth of \( u = e^{cx} \) as \( |x| \to \infty \) into account, \( u \in \mathcal{D}^\prime(\mathbb{R}^n) \) only when \( c = ia \) for some \( a \in \mathbb{R}^n \). Thus Theorems 10 and 14 are reduced to the following:

Corollary 18. Let \( u, v \in \mathcal{D}^\prime \) satisfy (19). Then \((u, v)\) satisfies one of the followings:

(i) \( u = 0 \), and \( v \) is arbitrary,

(ii) \( u \) and \( v \) are bounded measurable functions,

(iii) \( v(x) = \lambda u(x) + e^{icx} \) for some \( \lambda \in \mathbb{C}, c \neq 0 \in \mathbb{R}^n \), where \( u \) is a continuous function satisfying the condition; there exists \( d > 0 \)
\[
|u(x)| \leq \psi(-x) + d  
\] (141)
for all \( x \in \mathbb{R}^n \),

(iv) \( v(x) = \lambda u(x) + 1 \) for some \( \lambda \in \mathbb{C} \), where \( u \) is a continuous function satisfying one of the following conditions; there exists \( a_1 \in C^n \) such that
\[
|u(x) - a_1 \cdot x| \leq \Phi_1(x)  
\] (142)
for all \( x \in \mathbb{R}^n \), or there exists \( a_2 \in C^n \) such that
\[
|u(x) - a_2 \cdot x| \leq \Phi_2(x)  
\] (143)
for all \( x \in \mathbb{R}^n \).

Corollary 19. Let \( u, v \in \mathcal{D}^\prime \) satisfy (20). Then \((u, v)\) satisfies one of the followings:

(i) \( u \) and \( v \) are bounded measurable functions,

(ii) \( v(x) = \cos(a \cdot x) + r(x), u(x) = \sin(a \cdot x) + ir(x) \) for some \( a \in \mathbb{R}^n \), where \( r(x) \) is a continuous function satisfying
\[
|r(x)| \leq \frac{1}{2} \psi(x)  
\] (144)
for all \( x \in \mathbb{R}^n \).

Acknowledgments

The first author was supported by Basic Science Research Program through the National Foundation of Korea (NRF) funded by the Ministry of Education Science and Technology (MEST) (no. 2012008507), and the second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science, and Technology (2011-0005235).
References


