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Research Article

Multilinear Singular Integrals and their Commutators with Nonsmooth Kernels on Weighted Morrey Spaces

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Some multilinear maximal functions and the generalized Calderón-Zygmund operators and their commutators with nonsmooth kernels are studied. The purpose of this paper is to establish that these operators are bounded on certain product Morrey spaces. Some multilinear maximal functions and the generalized Calderón-Zygmund operators and their commutators with nonsmooth kernels are studied. The purpose of this paper is to establish that these operators are bounded on certain product Morrey spaces.

1. Introduction

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing functions and tempered distributions, respectively. Let $T$ be a multilinear operator initially defined on the $m$-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n).$$

Following [1], the $m$-multilinear Calderón-Zygmund operator $T$ satisfies the following conditions:

(S1) there exist $q_i < \infty$ ($i = 1, \ldots, m$), it extends to such that a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to $L^q$, where

$$\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m};$$

(S2) there exists a function $K$, defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(\vec{f})(x) = T(f_1, \ldots, f_m)(x)$$

$$= \int_{(\mathbb{R}^n)^n} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) \, dy_1 \cdots dy_m,$$

for all $x \notin \bigcap_{j=1}^m \text{supp } f_j$ and $f_1, \ldots, f_m \in \mathcal{S}(\mathbb{R}^n)$, where

$$|K(y_0, y_1, \ldots, y_m)| \leq \frac{A}{(\sum_{j=0}^m |y_j - y_k|)^{\delta}};$$

for some $\delta > 0$ and all $0 \leq j \leq m$, whenever $|y_j - y_k| \leq (1/2) \max_{0 \leq k \leq m} |y_j - y_k|$.

We also take some notation following [2]. Given a locally integrable vector function $b = (b_1, \ldots, b_m) \in (\text{BMO})^m$. The commutator of $b$ and the $m$-linear Calderón-Zygmund operator $T$, denoted here by $T_{b_1}^\mathcal{S}(\vec{f})$, was introduced by Pérez and Torres in [3] and is defined via

$$T_{b_1}^\mathcal{S}(\vec{f}) = \sum_{j=1}^m T_{b_j}^\mathcal{S}(\vec{f}),$$

where

$$T_{b_j}^\mathcal{S}(\vec{f}) = b_j T(\vec{f}) - T(f_1, \ldots, b_j f_j, \ldots, f_m).$$
And the iterated commutators $T_{\Pi^m}$ are defined by
\[
T_{\Pi^m}(\vec{f}) = [b_1, \ldots, [b_{m-1}, [b_m T], \ldots, \cdot\cdot\cdot, 1]_{j=1}^m(\vec{f}).
\]
(8)

To clarify the notations, if $T$ is associated in the usual way with a Calderón-Zygmund kernel $K$, then at a formal level
\[
T_{\Pi^m}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \prod_{j=1}^m (b_j(x) - b_j(y_j))
\times K(x, y_1, \ldots, y_m f_1(y_1) \cdots
f_m(y_m) dy_1 \cdots dy_m,
\]
(9)

In this paper, we will consider $T$ to be associated with the kernel satisfying a weaker regularity conditions introduced by [4,5]. A special example is the $m$th Calderón commutator.

Let $\{A_i\}_{i=0}$ be a class of integral operators, which play the role of the approximation to the identity. We always assume that the operators $A_i$ are given by kernels $a_i(x, y)$ in the sense that
\[
A_i f(x) = \int_{\mathbb{R}^n} a_i(x, y) f(y) dy,
\]
(10)
for all $f \in \mathcal{C}_c[1,\infty] L^p$ and $x \in \mathbb{R}^n$, and the kernels $a_i(x, y)$ satisfy the following conditions:
\[
|a_i(x, y)| \leq h_i(x, y) := t^{-n/h_i} \left(\frac{|x-y|}{t^{1/j_i}}\right),
\]
(11)
where $s$ is a positive fixed constant and $h$ is a positive, bounded, decreasing function satisfying that for some $\eta > 0$
\[
\lim_{r \to \infty} r^{m/h_i}(r^j) = 0.
\]
(12)

Recall that the $j$th transpose $T^{* j}$ of the $m$-linear operator $T$ is defined via
\[
\left< T^{* j}(f_1, \ldots, f_m), g \right> = \left< T^{* j}(f_1, \ldots, f_{j-1}, g, f_{j+1}, \ldots, f_m), f_j \right>,
\]
(13)
for all $f_1, \ldots, f_m, g$ in $\mathcal{C}(\mathbb{R}^n)$. It is seen that the kernel $K^{* j}$ of $T^{* j}$ is related to the kernel $K$ of $T$ via the identity
\[
K^{* j}(x, y_1, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_m) = K(y_j, y_1, \ldots, y_{j-1}, x, y_{j+1}, \ldots, y_m).
\]
(14)

If an $m$-linear operator $T$ maps a product of Banach spaces $X_1 \times \cdots \times X_m$ into another Banach space $X$, then the transpose $T^{* j}$ maps $X_1 \times \cdots \times X_{j-1} \times X \times X_{j+1} \times \cdots \times X_m$ to $X_j$. Moreover, the norms of $T$ and $T^{* j}$ are equal. To maintain uniform notation, we may occasionally denote $T$ by $T^{* 0}$ and $K$ by $K^{* 0}$.

**Assumption 1.** Assume that for each $i = 1, \ldots, m$ there exist operators $\{A_i^{(i)}\}_{i=0}$ with kernels $a_i^{(i)}(x, y)$ that satisfy conditions (11) and (12) with constants $s$ and $\eta$ and that, for every $j = 0, 1, 2, \ldots, m$, there exist kernel $K^{*, j(i)}(x, y_1, \ldots, y_m)$ such that
\[
\left< T^{* j}(f_1, \ldots, f_m), g \right> = \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)^m} K^{*, j(i)}(x, y_1, \ldots, y_m) f_1(y_1) \cdots
f_m(y_m) g(x) dy_1 \cdots dy_m dx,
\]
(15)
for all $f_1, \ldots, f_m, g$ in $\mathcal{S}(\mathbb{R}^n)$ with $\supp f_i \cap \supp g = 0$. Also assume that there exist a function $\phi \in C(\mathbb{R})$ with $\supp \phi \subset [-1, 1]$ and constants $c > 0$ and $A$ so that for every $j = 0, 1, 2, \ldots, m$ and every $i = 1, 2, \ldots, m$, we have
\[
\left| K^{*, j}(x, y_1, \ldots, y_m) - K^{*, j(i)}(x, y_1, \ldots, y_m) \right| 
\leq \frac{A}{(\sum_{k=1}^m |x - y_k|)^m} \sum_{k=1}^m \phi \left( \frac{|y_k - y_k|}{t^{1/j_k}} \right)
\]
(16)
whenever $t^{1/j_k} \leq |x - y_k|/2$.

If $T$ satisfies Assumption 1 we will say that $T$ is an $m$-linear operator with generalized Calderón-Zygmund kernel $K$. The collection of function $K$ satisfying (15) and (16) with parameters $m, A, s, \eta$ and $\epsilon$ will be denoted by $m$-linear GCZK$(A, s, \eta, \epsilon)$. We say that $T$ is of class $m$-GCZO$(A, s, \eta, \epsilon)$ if $T$ has an associated kernel $K$ in $m$-GCZK$(A, s, \eta, \epsilon)$. Throughout this paper, we always assume that the $m$-linear operator $T$ satisfies the following assumption.

**Assumption 2.** Assume that there exist some $1 \leq q_1, \ldots, q_m < \infty$ and some $0 < q < \infty$ with $1/q = 1/q_1 + \cdots + 1/q_m$, such that $T$ maps $L^q(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{p^{\infty}}(\mathbb{R}^n)$.

**Theorem 3** (see [4]). Assume that $T$ is a multilinear operator in $m$-GCZO$(A, s, \eta, \epsilon)$. Let $1/m \leq p < \infty$, $1 \leq p_i \leq \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, all the following statement are valid:

(i) when all $p_i > 1$, then $T$ can be extended to be a bounded operator from the $m$-fold product $L^p(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^{p^{\infty}}(\mathbb{R}^n)$;

(ii) when some $p_i = 1$, then $T$ can be extended to be a bounded operator from the $m$-fold product $L^p(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^{p^{\infty}}(\mathbb{R}^n)$.

Moreover, there exists a constant $C(n, m, p_1, q_i)$ such that
\[
\|T\|_{L^{p^{\infty}}(\mathbb{R}^n) \to L^{p^{\infty}}(\mathbb{R}^n)} \leq C(n, m, p_1, q_i) (A + \|T\|_{L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)}).
\]
(17)
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Assumption 4. Assume that there exist operators \( \{B_t\}_{t>0} \) with kernels \( b_t(x,y) \) that satisfy condition (11) and (12) with constants \( s \) and \( \eta \). Let

\[
K_i^{(0)}(x,y_1,\ldots,y_m) = \int_{\mathbb{R}^s} K(z,y_1,\ldots,y_m)b_i(x,z) \, dz.
\]
(18)

We assume that the kernels \( K_i^{(0)}(x,y_1,\ldots,y_m) \) satisfy the following estimates; there exist a function \( \phi \in \mathcal{C}_0^\infty(\mathbb{R}) \) with \( \text{supp} \phi \subseteq [-1,1] \) and constants \( \epsilon > 0 \) and \( A \) such that

\[
|K_i^{(0)}(x,y_1,\ldots,y_m)| \leq \frac{A}{(\sum_{k=1}^m |x-y_k|)^{\epsilon\eta n}},
\]
whenever \( 2t^{1/s} \leq \min_{1 \leq j \leq m} |x-y_j| \), and

\[
|K(x,y_1,\ldots,y_m) - K_i^{(0)}(x',y_1,\ldots,y_m)|
\leq \frac{A}{(\sum_{k=1}^m |x-y_k|)^{\epsilon\eta n}} \sum_{k=1}^m \phi \left( \frac{|y_k-y_j|}{t^{1/s}} \right)
\]

whenever \( 2|x-x'| \leq t^{1/s} \leq \max_{1 \leq j \leq m} |x-y_j|/2 \).

It is known that condition (16) is weaker than, and indeed a consequence of, the Calderón-Zygmund kernel condition (5) from the proof of Proposition 2.1 in [4]. And also it is pointed out that Assumption 4 is weaker than the condition (5) for \( K(x,y_1,\ldots,y_m) \) in [6].

For \( T \) be an \( m \)-linear Calderón-Zygmund operator, \( \tilde{\omega} \in A_\beta \) and \( \nu = \prod_{j=1}^m \omega_j^{1/p} \) with \( 1/p = 1/p_1 + \cdots + 1/p_m \) and \( \tilde{b} \in \text{BMO}^m \), Lerner et al. [7] proved that \( T \) and \( T_{\tilde{b}_\omega} \) bounded from \( L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m) \) to \( L^p(\nu) \) and Pérez et al. [2] extended the result to \( T_{\tilde{b}_\omega} \) when all \( 1 < p_i < \infty \), in the case of the endpoint, that is, some \( p_i = 1 \), weak type estimates have been established; for some details refer [2, 7]. To obtain the same results for the multilinear singular integral operators \( T \) in \( m \)-GCZO(A, \( s, \eta, e \)) with kernel satisfying Assumption 4, some authors have done so much work. Duong et al. [5] obtained that \( T \) maps \( L^{p_1}(\omega) \times \cdots \times L^{p_m}(\omega) \) to \( L^p(\omega) \), where \( \omega \in A_\beta \) with \( \beta = \text{min}(p_1,\ldots,p_m) > 1 \). Grafakos et al. [8] proved that \( T \) maps \( L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m) \) to \( L^p(\nu) \) where all \( p_i > 1 \) and \( \tilde{\omega} \in A_\beta \), and maps \( L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m) \) to \( L^{p_\infty}(\nu) \) with some \( p_i = 1 \). For \( \tilde{\omega} \in \prod_{j=1}^m A_\beta \), with \( p_j > 1, j = 1,\ldots,m \), and \( \tilde{\omega} \in A_\beta \), established that \( T_{\tilde{b}_\omega} \) are of boundedness from \( L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m) \) to \( L^p(\nu) \); after that, Chen and Wu [9] extended the results of Lerner et al. [7] and Pérez et al. [2] to the multilinear singular integral operators \( T \) in \( m \)-GCZO(A, \( s, \eta, e \)) without the endpoint case.

Definition 5. Some multilinear maximal function used in Theorem 6 will be listed in the following, which are introduced by Lerner et al. [7] and Grafakos et al. [8]:

\[
\mathcal{M}(\tilde{f}(x)) = \sup_{Q \ni x} \prod_{j=1}^m \frac{1}{|Q|} \int_Q |f_j(y_j)| \, dy_j,
\]
\[
\mathcal{M}_{r}(\tilde{f}(x)) = \sup_{Q \ni x} \prod_{j=1}^m \frac{1}{|Q|} \left( \int_Q |f_j(y_j)|^r \, dy_j \right)^{1/r},
\]
\[
\mathcal{M}_{L,L_Q \log L} \log L(\tilde{f}(x)) = \sup_{Q \ni x} \prod_{j=1}^m \left\| f_j \right\|_{L,L_Q \log L}. \log L \right\|_{Q}
\]

The following relationship with the above three maximal functions is easy to check:

\[
\mathcal{M}(\tilde{f}(x)) \leq \mathcal{M}_{L,L_Q \log L} \log L(\tilde{f}(x)) \leq \mathcal{M}_{r}(\tilde{f}(x)).
\]
(22)

Let \( r > 1, 1 \leq l < m, \sigma = \{j_1,\ldots,j_l\} \subset \{1,\ldots,m\} \), and \( \sigma^c = \{1,\ldots,m\} \setminus \sigma \). We define the following multilinear maximal functions:

\[
\mathcal{M}(\tilde{f}(x))
\]

\[
= \sup_{Q \ni x} \prod_{j=1}^m \frac{1}{|Q|} \left( \int_Q |f_j(y_j)| \, dy_j \right)^{1/r}
\]

\[
\times \prod_{j \in \sigma^c} \left( \frac{1}{|Q|} \int_Q |f_j(y_j)| \, dy_j \right)^{1/r},
\]
\[
\mathcal{M}_{L,L_Q \log L, \log L}(\tilde{f}(x))
\]

\[
= \sup_{Q \ni x} \prod_{j \in \sigma^c} \left( \frac{1}{|Q|} \int_Q \left\| f_j \right\|_{L,L_Q \log L} \prod_{j \in \sigma^c} \left\| f_j \right\|_{L,L_Q \log L} \right). \log L \right\|_{Q}
\]

We have that

\[
\mathcal{M}(\tilde{f}(x)) \leq \mathcal{M}_{L,L_Q \log L, \log L}(\tilde{f}(x)) \leq \mathcal{M}_{r}(\tilde{f}(x)).
\]
(24)

The following statements are our main results.

Theorem 6. Let \( 0 < k < 1, 1 \leq p_1,\ldots,p_m < \infty, 1/p = 1/p_1 + \cdots + 1/p_m \), and \( \tilde{\omega} \in \prod_{j=1}^m A_\beta \). Let \( 1 \leq j < m, \sigma = \{i_1,\ldots,i_j\} \subset \{1,\ldots,m\} \), and for some \( r > 1 \) (depending only on \( \tilde{\omega} \)), if all \( p_j > 1 \), then \( \mathcal{M}(\tilde{f}(x)) \) are bounded from \( L^{p_j,k}(\omega_j) \times \cdots \times L^{p_m,k}(\omega_m) \) to \( L^{p,k}(\nu) \), and or else, bounded from \( L^{p_j,k}(\omega_j) \times \cdots \times L^{p_m,k}(\omega_m) \) to \( WL^{p,k}(\nu) \).
Corollary 7. Under the same assumptions as in Theorem 6, $M, ML\log L, M\sigma, M\sigma, L\log L$ are bounded from $L^{p,k}(\omega) \times \cdots \times L^{p,k}(\omega_m)$ to $L^{p,k}(v_0)$ or $WL^{p,k}(v_0)$.

Theorem 8. Assume that $T$ is a multilinear operator in $m$-GCZOA($A, s, \eta, e$) with kernel $K$ satisfying Assumption 4. Let $0 < k < 1$, $1/m \leq p < \infty$, $1 \leq p_j \leq \infty$ with $1/p_j = 1/p_1 + \cdots + 1/p_m$ and $\omega_j \in A_{p_j}$. Then we have the following:

(i) when all $p_j > 1$, there exists a constant $C$ such that

$$\|T(\tilde{f})\|_{L^{p,k}(\omega)} \leq C \|f\|_{L^{p,k}(\omega)},$$

(ii) when some $p_j = 1$, there exists a constant $C$ such that

$$\|T(\tilde{f})\|_{L^{p,k}(\omega)} \leq C \|f\|_{L^{p,k}(\omega)},$$

where $v_\omega = \prod_{j=1}^m \omega_j^{p/p_j}$.

Theorem 9. Assume that $T$ is a multilinear operator in $m$-GCZOA($A, s, \eta, e$) with kernel $K$ satisfying Assumption 4. Let $0 < k < 1$, $\tilde{\omega} = (\omega_1, \ldots, \omega_m) = \prod_{j=1}^m A_{p_j}$, and $v_\omega = \prod_{j=1}^m \omega_j^{p/p_j}$ with $1/p = 1/p_1 + \cdots + 1/p_m$ and $1 < p_j < \infty$, $j = 1, \ldots, m$ and $\tilde{b} = (b_1, \ldots, b_m) \in BMO^m$. Then, there exists a constant $C$ such that

$$\|T_{\tilde{b}}(\tilde{f})\|_{L^{p,k}(\omega)} \leq C \|f\|_{L^{p,k}(\omega)},$$

$$\|T_{\tilde{b}}(\tilde{f})\|_{L^{p,k}(\omega)} \leq C \|f\|_{L^{p,k}(\omega)}.$$
As remarked in [7], \( \prod_{j=1}^{m} A_{p_j} \) is strictly contained in \( A \vec{P} \); moreover, in general \( \vec{\omega} \in A_{\vec{P}} \) does not imply \( \omega_j \in L^1_{\text{loc}} \) for any \( j \), but instead

\[
\vec{\omega} \in A_{\vec{p}} \iff \begin{cases} 
\omega_j \in A_{mmp'},
\omega_j^{1/p_j'} \in A_{mp_j'} \ ,
\end{cases} \quad j = 1, \ldots, m, \tag{35}
\]

where the condition \( \omega_j^{1/p_j'} \in A_{mp_j'} \) in the case \( p_j = 1 \) is understood as \( \omega_j^{1/m} \in A_1 \).

Definition 13 (see [10]). Let \( 1 \leq p < \infty \), \( 0 < k < 1 \), and \( \omega \) be a weight function on \( \mathbb{R}^n \). The weighted Morrey space is define by

\[
L^{p,k}(\omega) = \left\{ f \in L^p_{\text{loc}} : \| f \|_{L^{p,k}(\omega)} < \infty \right\}, \tag{36}
\]

where

\[
\| f \|_{L^{p,k}(\omega)} = \sup_Q \left( \frac{1}{\omega(Q)^{1/p}} \int_Q |f(x)|^p \omega(x) \right)^{1/p}. \tag{37}
\]

The weighted weak Morrey space is defined by

\[
WL^{p,k}(\omega) = \left\{ f \text{ measurable} : \| f \|_{WL^{p,k}(\omega)} < \infty \right\}, \tag{38}
\]

where

\[
\| f \|_{WL^{p,k}(\omega)} = \sup_Q \sup_{\lambda > 0} \frac{\lambda}{\omega(Q)^{1/p}} \omega\left( \{ x \in Q : |f(x)| > \lambda \} \right)^{1/p}. \tag{39}
\]

We say that a weight \( \omega \) satisfies the doubling condition, denoting \( \omega \in \Delta_2 \), if there is a constant \( C > 0 \) such that \( \omega(2Q) \leq C \omega(Q) \) holds for any cube \( Q \). If \( \omega \in A_p \) with \( 1 \leq p < \infty \), we know that \( \omega(\lambda Q) \leq \lambda^{mp}[\omega]_{A_p} \omega(Q) \) for all \( \lambda > 1 \), then \( \omega \in \Delta_2 \).

Lemma 14 (see [10]). Suppose \( \omega \in \Delta_2 \), then there exists a constant \( D > 1 \) such that

\[
\omega(2Q) \geq D \omega(Q), \tag{40}
\]

for any cube.

Lemma 15 (see [11]). If \( \omega_j \in A_{\infty} \), then for any cube \( Q \), we have

\[
\int_Q \prod_{j=1}^{m} \omega_j^\theta_j(x) \ dx \geq \prod_{j=1}^{m} \left( \int_Q \omega_j(x) \ dx \right)^{\theta_j}, \tag{41}
\]

where \( \sum_{j=1}^{m} \theta_j = 1, 0 \leq \theta_j \leq 1 \).

Lemma 16 (see [12]). Suppose \( \omega \in A_{\infty} \), then \( \| b \|_{\text{BMO}(\omega)} \approx \| b \|_{\text{BMO}} \). Here

\[
\text{BMO}(\omega) = \left\{ b : \| b \|_{\text{BMO}(\omega)} \right\} \tag{42}
\]

where

\[
\| b \|_{\text{BMO}(\omega)} = \sup_Q \frac{1}{\omega(Q)} \int_Q |b(x) - b_{Q,\omega}| \omega(x) \ dx < \infty, \tag{43}
\]

\[
b_{Q,\omega} = \frac{1}{\omega(Q)} \int_Q b(x) \omega(x) \ dx. \tag{44}
\]

From the fact \( |b_{2,\omega} - b_{2,Q,\omega}| \leq C \| b \|_{\text{BMO}} \) and Lemma 16, we can deduce that \( |b_{2,\omega} - b_{2,Q,\omega}| \leq C \| b \|_{\text{BMO}} \).

Lemma 17 (see [8]). Assume that \( T \) is a multilinear operator in \( m\text{-GCZO}(A,s,\eta,\epsilon) \) with kernel \( K \) satisfying Assumption 4. Let \( 1/m \leq p < \infty \), \( 1 \leq p_j \leq \infty \) with \( 1/p = 1/p_1 + \cdots + 1/p_m \) and \( \omega_j \in A_{p_j}, j = 1, \ldots, m \). Then we have the following:

(i) \( T \) extends to a bounded operators from \( L^p(\omega_1) \times \cdots \times L^p(\omega_m) \) to \( L^p(\omega) \) if all the exponents \( p_j \) are strictly greater than \( 1 \);

(ii) \( T \) extends to a bounded operators from \( L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m) \) to \( L^{p,\infty}(\omega) \) if some exponents \( p_j \) are equal to \( 1 \).

In either case, the norm of \( T \) is bounded by \( C(A + \| T \|_{L^{m\times \cdots \times m} \rightarrow L^p}) \), where \( C \) is a positive constant depending on \( A, s, \eta, \epsilon, \) and \( [\omega]_{A_p} \).

Lemma 18 (see [6]). Assume that \( T \) is a multilinear operator in \( m\text{-GCZO}(A,s,\eta,\epsilon) \) with kernel \( K \) satisfying Assumption 4. Let \( \bar{b} \in \text{BMO}^m \) with \( \| \bar{b} \|_{\text{BMO}} = 1 \) and \( 1/p = 1/p_1 + \cdots + 1/p_m \) with \( 1 < p_j < \infty, j = 1, \ldots, m \). Then we have the following:

(i) there exists a constant \( C \) such that

\[
\| T_{\Sigma\bar{b}}(\bar{f}) \|_{L^p(\omega_{\bar{b}})} \leq C \prod_{j=1}^{m} \| f_j \|_{L^{p_j}(\omega_j)}, \tag{43}
\]

(ii) if \( \omega_j \in A_{p_j} \), then there exists a constant \( C \) such that

\[
\| T_{\Sigma\bar{b}}(\bar{f}) \|_{L^p(\omega_{\bar{b}})} \leq C \prod_{j=1}^{m} \| f_j \|_{L^{p_j}(\omega_j)}, \tag{44}
\]

where \( \omega_{\bar{b}} = \prod_{j=1}^{m} \omega_{j}^{p_j/p} \).

Lemma 19 (see [9]). Assume that \( T \) is a multilinear operator in \( m\text{-GCZO}(A,s,\eta,\epsilon) \) with kernel \( K \) satisfying Assumption 4. Let \( \bar{b} \in \text{BMO}^m \) with \( \| \bar{b} \|_{\text{BMO}} = 1 \) and \( 1/p = 1/p_1 + \cdots + 1/p_m \) with \( 1 < p_j < \infty, j = 1, \ldots, m \). If \( \omega_j \in A_{p_j} \) with \( \bar{P} = (p_1, \ldots, p_m) \), then there exists a constant \( C \) such that

\[
\| T_{\Sigma\bar{b}}(\bar{f}) \|_{L^p(\omega_{\bar{b}})} \leq C \prod_{j=1}^{m} \| f_j \|_{L^{p_j}(\omega_j)}, \tag{45}
\]

where \( \omega_{\bar{b}} = \prod_{j=1}^{m} \omega_{j}^{p_j/p} \).
3. Proof of Theorems

Proof of Theorem 6. Here, we only prove the boundedness of $\mathcal{M}_{\sigma_j}$. From [9], there exists some $t \in (0, 1)$ only depend on $\tilde{\omega}$ such that

$$
\mathcal{M}\left(\tilde{f}(x)\right) \leq C\prod_{j=1}^{m}\left\{ M_{\tilde{\omega}}^c\left(\left\{ |f_j|^p \tilde{\omega}_j / \gamma_{ao}\right\}^t\right)\right\}^{1/p},
$$

(46)

where $M_{\tilde{\omega}}^c$ is the weighted centered maximal operator. Then by the H"older inequality,

$$
\|\mathcal{M}_{\sigma_j}(\tilde{f})(x)\|_{L^{p_k}(\gamma_{ao})} \\
\leq C\prod_{j=1}^{m}\left\{ M_{\tilde{\omega}}\left(\left\{ |f_j|^p \tilde{\omega}_j / \gamma_{ao}\right\}^t\right)\right\}^{1/p} \\
\leq C\prod_{j=1}^{m}\left\{ M_{\tilde{\omega}}\left(\left\{ |f_j|^p \tilde{\omega}_j / \gamma_{ao}\right\}^t\right)\right\}^{1/p} \\
\leq C\prod_{j=1}^{m}\left\{ M_{\tilde{\omega}}\left(\left\{ |f_j|^p \tilde{\omega}_j / \gamma_{ao}\right\}^t\right)\right\}^{1/p} \\
\leq C\prod_{j=1}^{m}\|f_j\|_{L^{k}(\gamma_{ao})}.
$$

(47)

The weak version is a very similar process by the H"older inequality for the weak spaces. We omit the details. □

Proof of Theorem 8. For any $B = B(x_B, r_B) \subset \mathbb{R}^n$, we split $f_i = f_i^0 + f_i^\infty$ where $f_i^0 = f_i(x_B, i = 1, 2, \ldots, m, B^* = 8B$; then

$$
m\|T(f_1, \ldots, f_m)(x)\|_{L^{p_k}(\gamma_{ao})} \\
= \sum_{\alpha_1, \ldots, \alpha_m \in (0, \infty)} m\|T(f^\alpha_1(\gamma_{ao}), \ldots, f^\alpha_m(\gamma_{ao}))\|_{L^{p_k}(\gamma_{ao})} \\
= \sum_{\alpha_1, \ldots, \alpha_m \in (0, \infty)} \sum_{i=1}^{m} f_{i}^\alpha(y_i) + \sum_{i=1}^{m} f_{i}^\alpha(y_i) \ldots f_{i}^\alpha(y_m),
$$

(48)

where each term of $\sum'$ contains at least one $\alpha_i \neq 0$. Write then

$$
\frac{1}{\gamma_{ao}(B)^{k/p}} \left(\int_B |T(f_1, \ldots, f_m)(x)|^p \gamma_{ao}(x) \, dx\right)^{1/p} \\
\leq \frac{1}{\gamma_{ao}(B)^{k/p}} \left(\int_B |T(f_1^0, \ldots, f_m^0)(x)|^p \gamma_{ao}(x) \, dx\right)^{1/p} \\
+ \sum_{i=1}^{m} \frac{1}{\gamma_{ao}(B)^{k/p}} \left(\int_B |T(f_1^\alpha, \ldots, f_m^\alpha)(x)|^p \gamma_{ao}(x) \, dx\right)^{1/p} \\
= I^{0,0} + \sum_{i=1}^{m} I^{\alpha_i, \ldots, \alpha_m}.
$$

(49)

From Definition 12, Lemma 17, we can get

$$
I^{0,0} \leq \frac{C}{\gamma_{ao}(B)^{k/p}} \int_B \left(\int_B |f_1^0(\gamma_{ao})(x)|^p \gamma_{ao}(x) \, dx\right)^{1/p} \\
\leq C \int_B \left(\int_B |f_1^0(\gamma_{ao})(x)|^p \gamma_{ao}(x) \, dx\right)^{1/p} \\
\leq C \int_B \left(\int_B |f_1^0(\gamma_{ao})(x)|^p \gamma_{ao}(x) \, dx\right)^{1/p} \\
\leq C \int_B \left(\int_B |f_1^0(\gamma_{ao})(x)|^p \gamma_{ao}(x) \, dx\right)^{1/p}.
$$

(50)

The last inequality holds by Lemma 15. For $\sum' I^{\alpha_1, \ldots, \alpha_m}$, we first consider the case when $\alpha_1 = \ldots = \alpha_m = \infty$. Taking $t = (2r_B)^{1/p}$, since $x \in B$ and $y_i \in \mathbb{R}^n \setminus 8B$, we get

$$
|y_i - x| > 7r_B > 2t^{1/p}, \quad \text{for all } j = 1, \ldots, m;
$$

(51)

hence, $h(|y_i - x|^{1/p}) = 0$. By Assumption 4, we have that

$$
\left| \mathcal{K}(x, y_1, \ldots, y_m) - \mathcal{K}_{(0)}(x, y_1, \ldots, y_m) \right| \\
\leq \left(\sum_{i=1}^{m} |x - y_i|^{\alpha_i} \right)^{1/m} \leq A \left(\sum_{i=1}^{m} |x - y_i|^{\alpha_i} \right)^{1/m}.
$$

(52)

For any $x \in B$, then by Assumption 4,

$$
\left| T(f_1^\infty, \ldots, f_m^\infty)(x) \right| \\
\leq \left| \mathcal{K}(x, y_1, \ldots, y_m) - \mathcal{K}_{(0)}(x, y_1, \ldots, y_m) \right| \\
\times \prod_{i=1}^{m} |f_i^\infty(y_i)| \, dy \\
+ \left| \mathcal{K}_{(0)}(x, y_1, \ldots, y_m) \right| \prod_{i=1}^{m} |f_i^\infty(y_i)| \, dy \\
\leq C \left(\sum_{i=1}^{m} |x - y_i|^{\alpha_i} \right)^{1/m} \prod_{i=1}^{m} |f_i^\infty(y_i)| \, dy \\
\leq C \left(\sum_{i=1}^{m} \left| \mathcal{K}_{(0)}(x, y_1, \ldots, y_m) \right| \prod_{i=1}^{m} |f_i^\infty(y_i)| \, dy \right)^{1/m}.
$$

(53)
Since \( \nu_0 \in A_{mp} \), then there is a positive \( \delta \) such that
\[
\frac{\nu_0(B)}{\nu_0(B^+)} \leq C \left( \frac{|B|}{|B^+|} \right)^{\delta}.
\] (54)

Hence
\[
\nu_0(B) \leq C \sum_{i=1}^{\infty} \left( \frac{|B|}{|B^+|} \right)^{\delta^{(i-k)/p}} \prod_{i=1}^{m} \| f_i \|_{L^p(B)}
\] (55)

\[
\leq C \prod_{i=1}^{m} \| f_i \|_{L^p(B)},
\]

It remains to estimate the terms with \( \alpha_{i_1} = \cdots = \alpha_{i_j} = 0 \) for some \( \{i_1, \ldots, i_j\} \subset \{1, \ldots, m\} \) and \( 1 \leq j < m \). We have
\[
|T(1_{i_1} \cdots , f_{m}^\alpha)(x)|
\]
\[
\leq \int_{(\mathbb{R}^m)^m} |K(x, y_1, \ldots, y_m) - K^{(0)}(x, y_1, \ldots, y_m)| \prod_{i=1}^{m} |f_i^{\alpha_i}(y_i)| \, dy
\]
\[
+ \int_{(\mathbb{R}^m)^m} |K^{(0)}(x, y_1, \ldots, y_m)| \prod_{i=1}^{m} |f_i^{\alpha_i}(y_i)| \, dy
\]
\[
\leq C \prod_{i \in \{1, \ldots, j\}} \int_{B^0} |f_i(y_i)| \, dy \times \left( \int_{(\mathbb{R}^m)^{m-j}} \left( \sum_{i \not\in \{1, \ldots, j\}} |x - y_i| \right)^{m-j-1} \prod_{i \not\in \{1, \ldots, j\}} |f_i(y_i)| \, dy \right)
\]
\[
\leq C \prod_{i \in \{1, \ldots, j\}} \left( \int_{B^0} |f_i(y_i)| \, dy \right) \times \left( \int_{(\mathbb{R}^m)^{m-j}} \left( \sum_{i \not\in \{1, \ldots, j\}} |x - y_i| \right)^{m-j-1} \prod_{i \not\in \{1, \ldots, j\}} |f_i(y_i)| \, dy \right)
\]
\[
\leq C \frac{1}{|B^0|} \prod_{i \in \{1, \ldots, j\}} \left( \int_{B^0} |f_i(y_i)| \, dy \right) \times \left( \int_{(\mathbb{R}^m)^{m-j}} \left( \sum_{i \not\in \{1, \ldots, j\}} |x - y_i| \right)^{m-j-1} \prod_{i \not\in \{1, \ldots, j\}} |f_i(y_i)| \, dy \right)
\]
\[
\leq C \sum_{i=1}^{\infty} \frac{\nu_0(8B)}{|B|} \prod_{i=1}^{m} \| f_i \|_{L^p(B)}.
\] (56)

Therefore, we also have
\[
T^\alpha \leq C \prod_{i=1}^{m} \| f_i \|_{L^p(B)}.
\] (57)

Combining the above estimates and then taking the supremum over all balls \( B \) in \( \mathbb{R}^n \), we have proved the previous part of Theorem 8.

Next, we turn to complete the proof of the weak inequality. For any \( \lambda > 0 \), we can write
\[
\nu_0(\{|x \in B : |T(f_1^\alpha, \ldots, f_m^\alpha)(x)| > \lambda\})^{1/p}
\]
\[
\leq \nu_0(\{|x \in B : |T(f_1^\phi, \ldots, f_m^\phi)(x)| > \lambda\})^{1/p}
\]
\[
+ \sum_{i=1}^{m} \nu_0(\{|x \in B : |T(f_i^\phi, \ldots, f_m^\phi)(x)| > \lambda\})^{1/p}
\]
\[
= \mathcal{L}^{0,\ldots,0} + \sum_{i=1}^{m} \mathcal{L}^{\alpha_i,\ldots,0}.
\] (58)

By Lemmas 17 and 15, we can easily check that
\[
\mathcal{L}^{\alpha_1,\ldots,0} \leq C \frac{m}{\lambda} \prod_{i=1}^{m} \left( \int_{B^0} f_i^0(y_i)^p \omega_i(y_i) \, dy \right)^{1/p},
\]
\[
\leq C \frac{\nu_0(B)^{k/p}}{\lambda} \prod_{i=1}^{m} \| f_i \|_{L^p(B)}.
\] (59)

From the proof of (53) and (56), we have the following pointwise estimate:
\[
|T(f_1^\alpha, \ldots, f_m^\alpha)(x)| \leq C \sum_{i=1}^{\infty} \frac{1}{|B|} \int_{B} |f_i(y_i)| \, dy,
\]
\[
|T(f_1^\alpha, \ldots, f_m^\alpha)(x)| \leq C \sum_{i=1}^{\infty} \frac{1}{|B|} \int_{B} |f_i(y_i)| \, dy.
\] (60)

Since at least one \( p_j = 1 \), we can assume that \( \{i_1, \ldots, i_j\} \subset \{1, \ldots, m\} \) such that \( p_{i_1} = \cdots = p_{i_j} = 1 \) and others greater than 1. Then,
\[
|T(f_1^\alpha, \ldots, f_m^\alpha)(x)|
\]
\[
\leq C \sum_{i=1}^{\infty} \frac{1}{|B|} \int_{B} \left( \inf_{y \in B} \omega_i(y_i) \right)^{-1} \times \left( \int_{B} |f_i(y_i)| \, dy \right)
\]
\[
\times \prod_{i \not\in \{1, \ldots, j\}} \frac{1}{|B|} \int_{B} \left( \int_{B} |f_i(y_i)|^p \omega_i(y_i) \, dy \right)^{1/p_i}
\]
\[
\times \left( \int_{B} \omega_i(y_i)^{1-p_i} \, dy \right)^{1/p_i}
\]
\[
\leq \frac{C \nu_0(B)^{1-k/p}}{\lambda} \prod_{i=1}^{m} \| f_i \|_{L^p(B)},
\]

Suppose that \( \{x \in B : |T(f_1^\alpha, \ldots, f_m^\alpha)(x)| > \lambda\} \neq \emptyset \); then we have that
\[
\nu_0(B)^{1/p} \leq \frac{C \nu_0(B)^{k/p}}{\lambda} \prod_{i=1}^{m} \| f_i \|_{L^p(B)}.
\] (62)
therefore,
\[
II^{α_1,\ldots,α_m} ≤ \nu_ω(B)^{1/p} ≤ \frac{C_{ν_ω(B)}}{\lambda} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(ω_i)}.
\] (63)

Taking the supremum over all balls \(B \subset \mathbb{R}^n\) and all \(\lambda > 0\), we complete the proof of Theorem 6.

Proof of Theorem 9. We will show the proof for \(T_{\Pi B}\) because the proof for \(T_{\Sigma B}\) is very similar but easier. Moreover, for simplicity of the expansion, we only present the case \(m = 2\).

For any cube \(B\), we also split \(f_i\) as \(f_i = f_i^0 + f_i^{co}\) with \(f_i^0 = f_i \chi_{B}\) and \(f_i^{co} = f_i - f_i^0\). Then it remains only to verify the following inequalities:

\[
I = \left( \frac{1}{\nu_ω(Q)^k} \int_Q |T_{Π B}(f_1^0, f_2^0)(x)|^p \nu_ω(x) \, dx \right)^{1/p}
\]
\[
\leq C \prod_{j=1}^2 \|b_j\|_{\text{BMO}} \prod_{j=1}^2 \|f_j\|_{L^{p_j}(ω_j)} \cdot \norm{f_i}_{BMO}(x)
\]
\[
II = \left( \frac{1}{\nu_ω(Q)^k} \int_Q |T_{Π B}(f_1^{co}, f_2^{co})(x)|^p \nu_ω(x) \, dx \right)^{1/p}
\]
\[
\leq C \prod_{j=1}^2 \|b_j\|_{\text{BMO}} \prod_{j=1}^2 \|f_j\|_{L^{p_j}(ω_j)} \cdot \norm{f_i}_{BMO}(x)
\]
\[
III = \left( \frac{1}{\nu_ω(Q)^k} \int_Q |T_{Π B}(f_1^{co}, f_2^0)(x)|^p \nu_ω(x) \, dx \right)^{1/p}
\]
\[
\leq C \prod_{j=1}^2 \|b_j\|_{\text{BMO}} \prod_{j=1}^2 \|f_j\|_{L^{p_j}(ω_j)} \cdot \norm{f_i}_{BMO}(x)
\]
\[
IV = \left( \frac{1}{\nu_ω(Q)^k} \int_Q |T_{Π B}(f_1^{co}, f_2^{co})(x)|^p \nu_ω(x) \, dx \right)^{1/p}
\]
\[
\leq C \prod_{j=1}^2 \|b_j\|_{\text{BMO}} \prod_{j=1}^2 \|f_j\|_{L^{p_j}(ω_j)} \cdot \norm{f_i}_{BMO}(x)
\]

From Lemma 19, Lemma 15, and Hölder’s inequality, we can get

\[
I ≤ \frac{1}{\nu_ω(Q)^{k/p}} \prod_{j=1}^2 \|b_j\|_{\text{BMO}} \left( \int_{\mathbb{R}^n} |f_j^0(x)|^p \omega_j(x) \, dx \right)^{1/p}
\]
\[
≤ \frac{1}{\nu_ω(Q)^{k/p}} \prod_{j=1}^2 \|b_j\|_{\text{BMO}}^{\omega_j(2Q)^{k/p}} \|f_j\|_{L^{p_j}(ω_j)}
\]
\[
≤ C \prod_{j=1}^2 \|b_j\|_{\text{BMO}} \|f_j\|_{L^{p_j}(ω_j)}.
\] (64)

Since \(II\) and \(III\) are symmetric, we only estimate \(II\). Taking \(\lambda_j = (b_j)_{B,ω_j}, T_{Π B}\) can be divided into four part:

\[
T_{Π B}(f_1^0, f_2^{co})(x)
\]
\[
= (b_1(x) - λ_1)(b_2(x) - λ_2) T(f_1^0, f_2^{co})(x)
\]
\[
- (b_1(x) - λ_1) T(f_1^0, (b_2 - λ_2) f_2^{co})(x)
\]
\[
- (b_2(x) - λ_2) T((b_1 - λ_1) f_1^0, f_2^{co})(x)
\]
\[
+ T((b_1 - λ_1) f_1^0, (b_2 - λ_2) f_2^{co})(x)
\]
\[
= II_1 + II_2 + II_3 + II_4.
\]

From the proof of Theorem 8 we know that, for any \(x \in B\),

\[
|T(f_1^0, f_2^{co})(x)| ≤ \frac{C_0}{\nu_ω(Q)} \sum_{j=1}^{\infty} \nu_ω(2^{j+1}Q)^{k-1/p} \|f_j\|_{L^{p_j}(ω_j)}.
\] (67)

Applying (67), Hölder’s inequality and Lemma 16, we have

\[
\left( \frac{1}{\nu_ω(Q)^k} \int_Q |II_1|^p \nu_ω(x) \, dx \right)^{1/p}
\]
\[
≤ \frac{1}{\nu_ω(Q)^{k/p}} \left( \int_Q |(b_1(x) - λ_1)(b_2(x) - λ_2)|^p \nu_ω(x) \, dx \right)^{1/p}
\]
\[
× \nu_ω(x) dx
\]
\[
× \prod_{j=1}^2 \|f_j\|_{L^{p_j}(ω_j)} \sum_{j=1}^{\infty} \nu_ω(2^{j+1}Q)^{k-1/p}
\]
\[
≤ \frac{1}{\nu_ω(Q)^{k/p}} \prod_{j=1}^2 \left( \frac{1}{\nu_ω(Q)} \int_Q |(b_j(x) - λ_1)|^2 \nu_ω(x) \, dx \right)^{1/p}
\]
\[
× \prod_{j=1}^2 \|f_j\|_{L^{p_j}(ω_j)} \sum_{j=1}^{\infty} \nu_ω(2^{j+1}Q)^{k-1/p}
\]
\[
≤ \prod_{j=1}^2 \|b_j\|_{\text{BMO}} \|f_j\|_{L^{p_j}(ω_j)}.
\] (68)

The last inequality is obtained by the property of \(A_{∞}\); there is a constant \(δ > 0\) such that

\[
\frac{\nu_ω(Q)}{\nu_ω(2^{j+1}Q)} ≤ C \left( \frac{|Q|}{|2^{j+1}Q|} \right)^δ.
\] (69)
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For $II_2$, by the Assumption 4, Lemma 15, and Lemma 16, it follows that

$$\left| T\left( f^0, (b_2 - \lambda_2) f_2^{\infty}\right)(x) \right|$$

$$\leq \int_{\mathbb{R}^p^2} \left| K(x, y_1, y_2) - K_1^{(0)}(x, y_1, y_2) \right| \times \left| f^0(x_1)(b_2(y_2) - \lambda_2) f_2^{\infty}(y_2) \right| dy_1 dy_2$$

$$+ \int_{\mathbb{R}^n^2} \left| K_1^{(0)}(x, y_1, y_2) \right| \left| f^0(x_1)(b_2(y_2) - \lambda_2) \right| f_2^{\infty}(y_2) dy_1 dy_2$$

$$\leq C \sum_{j=1}^\infty \int_{\mathbb{R}^n} |f^0(x_1)| \sum_{l=1}^\infty \frac{1}{8^l B_l} \times \int_{2^{l+1}Q_2^Q} \left| (b_2(y_2) - \lambda_2) f_2^{\infty}(y_2) \right| dy_2$$

$$\leq C \sum_{j=1}^\infty \| b_2 \|_{BMO} \left( \int_{2^{l+1}B} |f_l^0(y_1)\omega_j(y_1) dy_1 \right)^{1/p_1} \times \left( \int_{2^{l+1}Q} |f_2(y_2)\omega_j(y_2) dy_2 \right)^{1/p_2}$$

$$\times \left( \int_{2^{l+1}Q} |b_2(y_2) - \lambda_2|^{p/(p_2)} \omega_j(y_2) dy_2 \right)^{1/p_2}$$

$$\leq C \sum_{l=1}^\infty \left( \prod_{j=1}^{\infty} \| b_j \|_{BMO} \left( \int_{2^{l+1}B} |f_l^0(y_1)\omega_j(y_1) dy_1 \right)^{1/p_l} \times \left( \int_{2^{l+1}Q} |f_2(y_2)\omega_j(y_2) dy_2 \right)^{1/p_2} \right.$$}

\begin{equation}
\leq C \sum_{l=1}^\infty \left( \prod_{j=1}^{\infty} \| b_j \|_{BMO} \right) \sum_{l=1}^\infty \| f_l^0 \|_{L^{p/(p_2)}(\omega_j)} \| f_2 \|_{L^{p_2}} \| b_2 \|_{BMO} \||b_2(y_2) - \lambda_2|^{p/(p_2)} \omega_j(y_2) dy_2 \right)^{1/p_l} \times \left( \int_{2^{l+1}Q} |f_2(y_2)\omega_j(y_2) dy_2 \right)^{1/p_2}$$

\end{equation}

Hölder's inequality and Lemma 16 tell us that

\begin{equation}
\left( \frac{1}{\varphi_\infty(Q)^p} \int_Q |II_3|^p \varphi_\infty(x) dx \right)^{1/p} \leq C \frac{1}{\varphi_\infty(Q)^{1/p}} \left( \int_Q |b_1(x) - \lambda_1|^p \varphi_\infty(x) dx \right)^{1/p} \times \prod_{j=1}^\infty \| f_j \|_{L^{p/(p_2)}(\omega_j)} \sum_{l=1}^\infty \| f_l^0 \|_{L^{p/(p_2)}(\omega_j)} \| f_2 \|_{L^{p_2}} \| b_2 \|_{BMO} \||b_2(y_2) - \lambda_2|^{p/(p_2)} \omega_j(y_2) dy_2 \right)^{1/p_l} \times \left( \int_{2^{l+1}Q} |f_2(y_2)\omega_j(y_2) dy_2 \right)^{1/p_2}$$

Similarly, we also have that

\begin{equation}
\left( \frac{1}{\varphi_\infty(Q)^p} \int_Q |II_3|^p \varphi_\infty(x) dx \right)^{1/p} \leq C \prod_{j=1}^\infty \| b_j \|_{BMO} \| f_j \|_{L^{p/(p_2)}(\omega_j)} \times \left( \int_{2^{l+1}Q} |f_2(y_2)\omega_j(y_2) dy_2 \right)^{1/p_2}$$

By Assumption 4, Lemma 15, and Lemma 16, a similar way deduces that

\begin{equation}
\left| T\left( f^0, (b_2 - \lambda_2) f_2^{\infty}\right)(x) \right| \leq C \sum_{l=1}^\infty \| b_2 \|_{BMO} \left( \int_{2^{l+1}B} |f_l^0(y_1)\omega_j(y_1) dy_1 \right)^{1/p_l} \times \left( \int_{2^{l+1}Q} |f_2(y_2)\omega_j(y_2) dy_2 \right)^{1/p_2} \times \left( \int_{2^{l+1}Q} |b_2(y_2) - \lambda_2|^{p/(p_2)} \omega_j(y_2) dy_2 \right)^{1/p_2}$$

and so,

\begin{equation}
\left( \frac{1}{\varphi_\infty(Q)^p} \int_Q |II_4|^p \varphi_\infty(x) dx \right)^{1/p} \leq C \prod_{j=1}^\infty \| b_j \|_{BMO} \| f_j \|_{L^{p/(p_2)}(\omega_j)} \times \left( \int_{2^{l+1}Q} |f_2(y_2)\omega_j(y_2) dy_2 \right)^{1/p_2}$$

Finally, we still decompose $T_{\Pi_2}(f_1^{\infty}, f_2^{\infty})(x)$ into four terms:

\begin{equation}
T_{\Pi_2}(f_1^{\infty}, f_2^{\infty})(x) = (b_1(x) - \lambda_1)(b_2(x) - \lambda_2) T\left( f_1^{\infty}, f_2^{\infty}\right)(x) + (b_1(x) - \lambda_1) T\left( f_1^{\infty}, (b_2 - \lambda_2) f_2^{\infty}\right)(x) + (b_2(x) - \lambda_2) T\left( (b_1 - \lambda_1) f_1^{\infty}, f_2^{\infty}\right)(x) + T\left( (b_1 - \lambda_1) f_1^{\infty}, (b_2 - \lambda_2) f_2^{\infty}\right)(x)$$

Because each term of $IV_j$ is completely analogous to $II_j$, $j = 1, 2, 3, 4$ with a bit difference, so we get the following estimate without details:

\begin{equation}
\left( \frac{1}{\varphi_\infty(Q)^p} \int_Q |IV|^p \varphi_\infty(x) dx \right)^{1/p} \leq C \prod_{j=1}^\infty \| b_j \|_{BMO} \| f_j \|_{L^{p/(p_2)}(\omega_j)}$$

To this, we end the proof of Theorem 9. \hfill \Box

**Conflict of Interests**

The authors declare that they have no conflict of interests.

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