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Research Article

Multilinear Singular Integrals and their Commutators with Nonsmooth Kernels on Weighted Morrey Spaces

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Some multilinear maximal functions and the generalized Calderón-Zygmund operators and their commutators with nonsmooth kernels are studied. The purpose of this paper is to establish that these operators are bounded on certain product Morrey spaces $L^{p,k}(\mathbb{R}^n)$. Based on the boundedness of these operators from $L^{p_1}(\omega_i)\times\cdots\times L^{p_m}(\omega_m)$ to $L^p(\prod_{j=1}^m\omega_j^{p/p_j})$, we obtained that they are also bounded from $L^{p_1,k}(\omega_i)\times\cdots\times L^{p_m,k}(\omega_m)$ to $L^{p,k}(\prod_{j=1}^m\omega_j^{p/p_j})$ with $0<k<1$, $1<p_j<\infty$, $1/p=1/p_1,\ldots,p_m$ and $\omega_j\in A_{p_j}$, $j=1,\ldots,m$.

1. Introduction

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing functions and tempered distributions, respectively. Let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T:\mathcal{S}(\mathbb{R}^n)\times\cdots\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}'(\mathbb{R}^n). \tag{1}$$

Following [1], the m-multilinear Calderón-Zygmund operator T satisfies the following conditions:

(S1) there exist $q_i<\infty$ $(i=1,\ldots,m)$, it extends to such that a bounded multilinear operator from $L^{q_1}\times\cdots\times L^{q_m}$ to L^q, where

$$\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}; \tag{2}$$

(S2) there exists a function K, defined off the diagonal $x=y_1=\cdots=y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$\begin{aligned}T\left(\vec{f}\right)(x)&=T\left(f_1,\ldots,f_m\right)(x)\\&=\int_{(\mathbb{R}^m)^n}K\left(x,y_1,\ldots,y_m\right)f_1\left(y_1\right)\cdots\\&\quad f_m\left(y_m\right)dy_1\cdots dy_m,\end{aligned} \tag{3}$$

for all $x\notin\cap_{j=1}^m\operatorname{supp}f_j$ and $f_1,\ldots,f_m\in\mathcal{S}(\mathbb{R}^n)$, where

$$\left|K\left(y_0,y_1,\ldots,y_m\right)\right|\le\frac{A}{\left(\sum_{l,k=0}^m\left|y_l-y_k\right|\right)^{mn}}; \tag{4}$$

$$\begin{aligned}&\left|K\left(y_0,\ldots,y_j,\ldots,y_m\right)-K\left(y_0,\ldots,y_j,\ldots,y_m\right)\right|\\&\quad\le\frac{A\left|y_j-y_j'\right|^{\epsilon}}{\left(\sum_{l,k=0}^m\left|y_l-y_k\right|\right)^{mn+\epsilon}},\end{aligned} \tag{5}$$

for some $\epsilon>0$ and all $0\le j\le m$, whenever $|y_j-y_j|\le(1/2)\max_{0\le k\le m}|y_j-y_k|$.

We also take some notation following [2]. Given a locally integrable vector function $\mathbf{b}=(b_1,\ldots,b_m)\in(\mathrm{BMO})^m$. The commutator of $\mathbf{b}$ and the m-linear Calderón-Zygmund operator T, denoted here by $T_{\Sigma\mathbf{b}}$, was introduced by Pérez and Torres in [3] and is defined via

$$T_{\Sigma\vec{b}}\left(\vec{f}\right)=\sum_{j=1}^m T_{b_j}^j\left(\vec{f}\right), \tag{6}$$

where

$$T_{b_j}^j\left(\vec{f}\right)=b_jT\left(\vec{f}\right)-T\left(f_1,\ldots,b_jf_j,\ldots,f_m\right). \tag{7}$$

And the iterated commutators $T_{\Pi \mathbf{b}}$ are defined by

$$T_{\Pi \vec{b}}\left(\vec{f}\right) = \left[b_1, \ldots, \left[b_{m-1}, \left[b_m, T\right]_m\right]_{m-1} \ldots\right]_1 \left(\vec{f}\right). \quad (8)$$

To clarify the notations, if T is associated in the usual way with a Calderón-Zygmund kernel K, then at a formal level

$$\begin{aligned} T_{\Sigma \vec{b}}\left(\vec{f}\right)(x) &= \int_{(\mathbb{R}^n)^m} \sum_{j=1}^{m} \left(b_j(x) - b_j\left(y_j\right)\right) \\ &\quad \times K\left(x, y_1, \ldots, y_m\right) f_1\left(y_1\right) \cdots \\ &\quad f_m\left(y_m\right) dy_1 \cdots dy_m, \\ T_{\Pi \vec{b}}\left(\vec{f}\right)(x) &= \int_{(\mathbb{R}^n)^m} \prod_{j=1}^{m} \left(b_j(x) - b_j\left(y_j\right)\right) \\ &\quad \times K\left(x, y_1, \ldots, y_m\right) f_1\left(y_1\right) \cdots \\ &\quad f_m\left(y_m\right) dy_1 \cdots dy_m. \end{aligned} \quad (9)$$

In this paper, we will consider T to be associated with the kernel satisfying a weaker regularity conditions introduced by [4, 5]. A special example is the mth Calderón commutator.

Let $\{A_t\}_{t>0}$ be a class of integral operators, which play the role of the approximation to the identity. We always assume that the operators A_t are given by kernels $a_t(x, y)$ in the sense that

$$A_t f(x) = \int_{\mathbb{R}^n} a_t(x, y) f(y)\, dy, \quad (10)$$

for all $f \in \cup_{p\in[1,\infty]} L^p$ and $x \in \mathbb{R}^n$, and the kernels $a_t(x, y)$ satisfy the following conditions:

$$\left|a_t(x, y)\right| \le h_t(x, y) := t^{-n/s} h\left(\frac{|x-y|}{t^{1/s}}\right), \quad (11)$$

where s is a positive fixed constant and h is a positive, bounded, decreasing function satisfying that for some $\eta > 0$

$$\lim_{r\to\infty} r^{n+\eta} h\left(r^s\right) = 0. \quad (12)$$

Recall that the jth transpose $T^{*,j}$ of the m-linear operator T is defined via

$$\begin{aligned} &\left\langle T^{*,j}\left(f_1, \ldots, f_m\right), g\right\rangle \\ &\quad = \left\langle T^{*,j}\left(f_1, \ldots, f_{j-1}, g, f_{j+1}, \ldots, f_m\right), f_j\right\rangle, \end{aligned} \quad (13)$$

for all $f_1, \ldots, f_m, g$ in $\mathcal{S}(\mathbb{R}^n)$. It is seen that the kernel $K^{*,j}$ of $T^{*,j}$ is related to the kernel K of T via the identity

$$\begin{aligned} &K^{*,j}\left(x, y_1, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, f_m\right) \\ &\quad = K\left(y_j, y_1, \ldots, y_{j-1}, x, y_{j+1}, \ldots, f_m\right). \end{aligned} \quad (14)$$

If an m-linear operator T maps a product of Banach spaces $X_1 \times \cdots \times X_m$ into another Banach space X, then the transpose $T^{*,j}$ maps $X_1 \times \cdots \times X_{j-1} \times X \times X_{j+1} \times \cdots \times X_m$ to X_j. Moreover, the norms of T and $T^{*,j}$ are equal. To maintain uniform notation, we may occasionally denote T by $T^{*,0}$ and K by $K^{*,0}$.

Assumption 1. Assume that for each $i = 1, \ldots, m$ there exist operators $\{A_t^{(i)}\}_{t>0}$ with kernels $a_t^{(i)}(x, y)$ that satisfy conditions (11) and (12) with constants s and η and that, for every $j = 0, 1, 2, \ldots, m$, there exist kernel $K^{*,j,(i)}(x, y_1, \ldots, y_m)$ such that

$$\begin{aligned} &\left\langle T^{*,j}\left(f_1, \ldots, A_t^{(i)} f_i, \ldots, f_m\right), g\right\rangle \\ &\quad = \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)^m} K_t^{*,j,(i)}\left(x, y_1, \ldots, f_m\right) f_1\left(y_1\right) \cdots \\ &\qquad f_m\left(y_m\right) g(x)\, dy_1 \cdots dy_m dx, \end{aligned} \quad (15)$$

for all $f_1, \ldots, f_m, g$ in $\mathcal{S}(\mathbb{R}^n)$ with $\cap_{k=1}^m \operatorname{supp} f_k \cap \operatorname{supp} g = \emptyset$. Also assume that there exist a function $\phi \in \mathscr{C}(\mathbb{R})$ with $\operatorname{supp}\phi \subset [-1, 1]$ and constants $\epsilon > 0$ and A so that for every $j = 0, 1, 2, \ldots, m$ and every $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} &\left|K^{*,j}\left(x, y_1, \ldots, y_m\right) - K_t^{*,j,(i)}\left(x, y_1, \ldots, y_m\right)\right| \\ &\quad \le \frac{A}{\left(\sum_{k=1}^m |x - y_k|\right)^{mn}} \sum_{k=1, k\ne i}^{m} \phi\left(\frac{|y_i - y_k|}{t^{1/s}}\right) \\ &\qquad + \frac{A t^{\epsilon/t}}{\left(\sum_{k=1}^m |x - y_k|\right)^{mn+\epsilon}} \end{aligned} \quad (16)$$

whenever $t^{1/s} \le |x - y_i|/2$.

If T satisfies Assumption 1 we will say that T is an m-linear operator with generalized Calderón-Zygmund kernel K. The collection of function K satisfying (15) and (16) with parameters m, A, s, η, and ϵ will be denoted by m-linear $GCZK(A, s, \eta, \epsilon)$. We say that T is of class m-$GCZO(A, s, \eta, \epsilon)$ if T has an associated kernel K in m-$GCZK(A, s, \eta, \epsilon)$. Throughout this paper, we always assume that the m-linear operator T satisfies the following assumption.

Assumption 2. Assume that there exist some $1 \le q_1, \ldots, q_m < \infty$ and some $0 < q < \infty$ with $1/q = 1/q_1 + \cdots + 1/q_m$, such that T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$.

Theorem 3 (see [4]). *Assume that T is a multilinear operator in m-$GCZO(A, s, \eta, \epsilon)$. Let $1/m \le p < \infty$, $1 \le p_j \le \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, all the following statement are valid:*

(i) *when all $p_j > 1$, then T can be extended to be a bounded operator from the m-fold product $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$;*

(ii) *when some $p_j = 1$, then T can be extended to be a bounded operator from the m-fold product $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$.*

Moreover, there exists a constant $C(n, m, p_j, q_j)$ such that

$$\begin{aligned} &\|T\|_{L^1\times\cdots\times L^1 \to L^{1/m,\infty}} \\ &\quad \le C\left(n, m, p_j, q_j\right)\left(A + \|T\|_{L^{q_1}\times\cdots\times L^{q_m} \to L^{q,\infty}}\right). \end{aligned} \quad (17)$$

Assumption 4. Assume that there exist operators $\{B_t\}_{t>0}$ with kernels $b_t(x,y)$ that satisfy condition (11) and (12) with constants s and η. Let

$$K_t^{(0)}(x,y_1,\dots,y_m)=\int_{\mathbb{R}^n}K(z,y_1,\dots,y_m)\,b_t(x,z)\,dz. \tag{18}$$

We assume that the kernels $K_t^{(0)}(x,y_1,\dots,y_m)$ satisfy the following estimates; there exist a function $\phi\in\mathscr{C}(\mathbb{R})$ with $\operatorname{supp}\phi\subset[-1,1]$ and constants $\epsilon>0$ and A such that

$$\left|K_t^{(0)}(x,y_1,\dots,y_m)\right|\le\frac{A}{\left(\sum_{k=1}^m|x-y_k|\right)^{mn}}, \tag{19}$$

whenever $2t^{1/s}\le\min_{1\le j\le m}|x-y_j|$, and

$$\begin{aligned}&\left|K(x,y_1,\dots,y_m)-K_t^{(0)}(x',y_1,\dots,y_m)\right|\\&\quad\le\frac{A}{\left(\sum_{k=1}^m|x-y_k|\right)^{mn}}\sum_{k=1}^m\phi\left(\frac{|y_i-y_k|}{t^{1/s}}\right)\\&\qquad+\frac{At^{\epsilon/s}}{\left(\sum_{k=1}^m|x-y_k|\right)^{mn+\epsilon}},\end{aligned} \tag{20}$$

whenever $2|x-x'|\le t^{1/s}\le\max_{1\le j\le m}|x-y_j|/2$.

It is known that condition (16) is weaker than, and indeed a consequence of, the Calderón-Zygmund kernel condition (5) from the proof of Proposition 2.1 in [4]. And also it is pointed out that Assumption 4 is weaker than the condition (5) for $K(x,y_1,\dots,y_m)$ in [6].

For T be an m-linear Calderón-Zygmund operator, $\vec{\omega}\in A_{\vec{P}}$ and $\nu_{\vec{\omega}}=\prod_{j=1}^m\omega_j^{p/p_j}$ with $1/p=1/p_1+\cdots+p_m$ and $\vec{b}\in BMO^m$, Lerner et al. [7] proved that T and $T_{\Sigma\vec{b}}$ bounded from $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)$ to $L^p(\nu_{\vec{\omega}})$ and Pérez et al. [2] extended the result to $T_{\Pi\vec{b}}$ when all $1<p_j<\infty$, in the case of the endpoint, that is, some $p_j=1$, weak type estimates have been established; for some details refer [2, 7]. To obtain the same results for the multilinear singular integral operators T in m-$GCZO(A,s,\eta,\epsilon)$ with kernel satisfying Assumption 4, some authors have done so much work. Duong et al. [5] obtained that T maps $L^{p_1}(\omega)\times\cdots\times L^{p_m}(\omega)$ to $L^p(\omega)$, where $\omega\in A_{p_0}$ with $p_0=\min(p_1,\dots,p_m)>1$. Grafakos et al. [8] proved that T maps $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)$ to $L^p(\nu_{\vec{\omega}})$ where all $p_j>1$ and $\vec{\omega}\in A_{\vec{P}}$, and maps $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)$ to $L^{p,\infty}(\nu_{\vec{\omega}})$ with some $p_j=1$. For $\vec{\omega}\in\prod_{j=1}^mA_{p_j}$ with $p_j>1$, $j=1,\dots,m$, Anh and Duong [6] established that $T_{\Sigma\vec{b}}$ are of boundedness from $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)$ to $L^p(\nu_{\vec{\omega}})$; after that, Chen and Wu [9] extended the results of Lerner et al. [7] and Pérez et al. [2] to the multilinear singular integral operators T in m-$GCZO(A,s,\eta,\epsilon)$ without the endpoint case.

Definition 5. Some multilinear maximal function used in Theorem 6 will be listed in the following, which are introduced by Lerner et al. [7] and Grafakos et al. [8]:

$$\begin{aligned}\mathscr{M}(\vec{f})(x)&=\sup_{Q\ni x}\prod_{j=1}^m\frac{1}{|Q|}\int_Q|f_j(y_j)|\,dy_j,\\\mathscr{M}_r(\vec{f})(x)&=\sup_{Q\ni x}\prod_{j=1}^m\frac{1}{|Q|}\left(\int_Q|f_j(y_j)|^r\,dy_j\right)^{1/r},\\\mathscr{M}_{L\log L}(\vec{f})(x)&=\sup_{Q\ni x}\prod_{j=1}^m\|f_j\|_{L\log L,Q}.\end{aligned} \tag{21}$$

The following relationship with the above three maximal functions is easy to check:

$$\mathscr{M}(\vec{f})(x)\le\mathscr{M}_{L\log L}(\vec{f})(x)\le C\mathscr{M}_r(\vec{f})(x). \tag{22}$$

Let $r>1$, $1\le l<m$, $\sigma=\{j_1,\dots,j_l\}\subset\{1,\dots,m\}$, and $\sigma'=\{1,\dots,m\}\setminus\sigma$. We define the following multilinear maximal functions:

$$\begin{aligned}&\mathscr{M}_\sigma(\vec{f})(x)\\&\quad=\sup_{Q\ni x}\sum_{k=0}^\infty2^{-knl}\prod_{j\in\sigma}\frac{1}{|Q|}\int_Q|f_j(y_j)|\,dy_j\\&\qquad\times\prod_{j\in\sigma'}\frac{1}{|2^kQ|}|f_j(y_j)|\,dy_j,\\&\mathscr{M}_\sigma(\vec{f})(x)\\&\quad=\sup_{Q\ni x}\sum_{k=0}^\infty2^{-knl}\prod_{j\in\sigma}\left(\frac{1}{|Q|}\int_Q|f_j(y_j)|\,dy_j\right)^{1/r}\\&\qquad\times\prod_{j\in\sigma'}\left(\frac{1}{|2^kQ|}|f_j(y_j)|^r\,dy_j\right)^{1/r},\\&\mathscr{M}_{\sigma,L\log L}(\vec{f})(x)\\&\quad=\sup_{Q\ni x}\sum_{k=0}^\infty2^{-knl}\prod_{j\in\sigma}\|f_j\|_{L\log L,Q}\prod_{j\in\sigma'}\|f_j\|_{L\log L,2^kQ}.\end{aligned} \tag{23}$$

We have that

$$\mathscr{M}_\sigma(\vec{f})(x)\le\mathscr{M}_{\sigma,L\log L}(\vec{f})(x)\le C\mathscr{M}_{\sigma,r}(\vec{f})(x). \tag{24}$$

The following statements are our main results.

Theorem 6. *Let $0<k<1$, $1\le p_1,\dots,p_m<\infty$, $1/p=1/p_1+\cdots+1/p_m$, and $\vec{\omega}\in\prod_{i=1}^mA_{p_i}$. Let $1\le j<m$, $\sigma=\{i_1,\dots,i_j\}\subset\{1,\dots,m\}$, and for some $r>1$ (r depending only on $\vec{\omega}$), if all $p_i>1$, then $\mathscr{M}_r$ and $\mathscr{M}_{\sigma,r}$ are bounded from $L^{p_1,k}(\omega_1)\times\cdots\times L^{p_m,k}(\omega_m)$ to $L^{p,k}(\nu_{\vec{\omega}})$, and or else, bounded from $L^{p_1,k}(\omega_1)\times\cdots\times L^{p_m,k}(\omega_m)$ to $WL^{p,k}(\nu_{\vec{\omega}})$.*

Corollary 7. *Under the same assumptions as in Theorem 6. $\mathcal{M}, \mathcal{M}_{L\log L}, \mathcal{M}_{\sigma}, \mathcal{M}_{\sigma,L\log L}$ are bounded from $L^{p_1,k}(\omega_1)\times\cdots\times L^{p_m,k}(\omega_m)$ to $L^{p,k}(\nu_{\vec{\omega}})$ or $WL^{p,k}(\nu_{\vec{\omega}})$.*

Theorem 8. *Assume that T is a multilinear operator in m-$GCZO(A,s,\eta,\epsilon)$ with kernel K satisfying Assumption 4. Let $0<k<1$, $1/m\le p<\infty$, $1\le p_j\le\infty$ with $1/p=1/p_1+\cdots+1/p_m$, and $\omega_j\in A_{p_j}$, $j=1,\ldots,m$. Then we have the following:*

(i) *when all $p_j>1$, there exists a constant C such that*

$$\left\|T(\vec{f})\right\|_{L^{p,k}(\nu_{\vec{\omega}})}\le C\left\|f_j\right\|_{L^{p_j,k}(\omega_j)}, \tag{25}$$

(ii) *when some $p_j=1$, there exists a constant C such that*

$$\left\|T(\vec{f})\right\|_{WL^{p,k}(\nu_{\vec{\omega}})}\le C\left\|f_j\right\|_{L^{p_j,k}(\omega_j)}, \tag{26}$$

where $\nu_{\vec{\omega}}=\prod_{j=1}^m\omega_j^{p/p_j}$.

Theorem 9. *Assume that T is a multilinear operator in m-$GCZO(A,s,\eta,\epsilon)$ with kernel K satisfying Assumption 4. Let $0<k<1$, $\vec{\omega}=(\omega_1,\ldots,\omega_m)\in\prod_{j=1}^m A_{p_j}$, and $\nu_{\vec{\omega}}=\prod_{j=1}^m\omega_j^{p/p_j}$ with $1/p=1/p_1+\cdots+p_m$ and $1<p_j<\infty$, $j=1,\ldots,m$ and $\vec{b}=(b_1,\ldots,b_m)\in BMO^m$. Then, there exists a constant C such that*

$$\begin{aligned}\left\|T_{\Sigma\vec{b}}(\vec{f})\right\|_{L^{p,k}(\nu_{\vec{\omega}})}&\le C\left\|f_j\right\|_{L^{p_j,k}(\omega_j)},\\ \left\|T_{\Pi\vec{b}}(\vec{f})\right\|_{L^{p,k}(\nu_{\vec{\omega}})}&\le C\left\|f_j\right\|_{L^{p_j,k}(\omega_j)}.\end{aligned} \tag{27}$$

Following [2], for positive integers m and j with $1\le j\le m$, we denote by C_j^m the family of all finite subsets $\sigma=\{\sigma(1),\ldots,\sigma(j)\}$ of $\{1,\ldots,m\}$ of j different elements, where we always take $\sigma(l)<\sigma(k)$ if $l<k$. For any $\sigma\in C_j^m$, the associated complementary sequence $\sigma'\in C_{m-j}^m$ is given by $\sigma'=\{1,\ldots,m\}\setminus\sigma$ with the convention $C_0^m=\emptyset$. Given an m-tuple of functions $\vec{b}$ and $\sigma\in C_j^m$, we also use the notation $\vec{b}_\sigma$ for the j-tuple obtained from $\vec{b}$ given by $(b_{\sigma(1)},\ldots,\sigma(j))$. Similar to $T_{\Pi\vec{b}}$, we define for T in m-$GCZO(A,s,\eta,\epsilon)$, $\sigma\in C_j^m$ and $\vec{b}_\sigma=(b_{\sigma(1)},\ldots,b_{\sigma(j)})$ in BMO^j, the jth order iterated commutator

$$T_{\Pi\vec{b}_\sigma}\left(\vec{f}\right)=\left[b_{\sigma(1)},\left[b_{\sigma(2)},\ldots,\left[b_{\sigma(j)},T\right]_{\sigma(j)}\cdots\right]_{\sigma(2)}\right]_{\sigma(1)}\left(\vec{f}\right); \tag{28}$$

that is, formally

$$\begin{aligned}T_{\Pi\vec{b}_\sigma}\left(\vec{f}\right)(x)\int_{(\mathbb{R}^n)^m}&\left(\prod_{i=1}^j\left(b_{\sigma(i)}(x)-b_{\sigma(i)}\left(y_{\sigma(i)}\right)\right)\right)\\ &\times K\left(x,y_1,\ldots,y_m\right)\prod_{i=1}^m f_i\left(y_i\right)d\vec{y}.\end{aligned} \tag{29}$$

Clearly, $T_{\Pi\vec{b}_\sigma}=T_{\Pi\vec{b}}$ when $\sigma=\{1,\ldots,m\}$, and $T_{\Pi\vec{b}_\sigma}=T_{b_j}^j$ when $\sigma=\{j\}$. We have the following general forms of Theorem 9 without the proof.

Theorem 10. *Assume that T is a multilinear operator in m-$GCZO(A,s,\eta,\epsilon)$ with kernel K satisfying Assumption 4. Let $1\le j\le m$, $\sigma\in C_j^m$, $\vec{\omega}=(\omega_1,\ldots,\omega_m)\in\prod_{i=1}^m A_{p_i}$, and $\nu_{\vec{\omega}}=\prod_{i=1}^m\omega_i^{p/p_i}$ with $1/p=1/p_1+\cdots+p_m$ and $1<p_i<\infty$, $i=1,\ldots,m$ and $\vec{b}_\sigma=(b_{\sigma(1)},\ldots,b_{\sigma(j)})\in BMO^j$. Then, there exists a constant C such that*

$$\left\|T_{\Pi\vec{b}_\sigma}(\vec{f})\right\|_{L^{p,k}(\nu_{\vec{\omega}})}\le C\prod_{i=1}^j\left\|b_{\sigma(i)}\right\|\prod_{i=1}^m\left\|f_i\right\|_{L^{p_i,k}(\omega_i)}. \tag{30}$$

2. Some Definitions and Results

In this section, we introduce some definitions and results used be later on.

Definition 11 (A_p weights). A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $1<p<\infty$; we call that a weight function ω that belongs to the class A_p, if there is a constant C such that, for any cube Q,

$$\left(\frac{1}{|Q|}\int_Q\omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q\omega(x)^{1-p'}dx\right)^{p-1}\le C, \tag{31}$$

and ω belongs to the class A_1, if there is a constant C such that, for any cube Q,

$$\frac{1}{|Q|}\int_Q\omega(x)\,dx\le C\inf_{x\in Q}\omega(x). \tag{32}$$

We denote $A_\infty=\cup_{p>1}A_p$.

Definition 12 (see [7]). For m exponents $p_1,\ldots,p_m\in[1,\infty)$, we often write p for the number given by $p=\sum_{j=1}^m p_j$ and denote $\vec{P}$ by the vector $\vec{P}=(p_1,\ldots,p_m)$. A multiple weight $\vec{\omega}=(\omega_1,\ldots,\omega_m)$ is said to satisfy the $A_{\vec{P}}$ condition if for

$$\nu_{\vec{\omega}}=\prod_{j=1}^m\omega^{p/p_j}, \tag{33}$$

it holds that

$$\sup_Q\left(\frac{1}{|Q|}\int_Q\nu_{\vec{\omega}}(x)\,dx\right)^{1/p}\prod_{j=1}^m\left(\frac{1}{|Q|}\int_Q\omega_j(x)^{1-p_j'}dx\right)^{1/p_j'}<\infty, \tag{34}$$

when $p_j=1$, $\left((1/|Q|)\int_Q\omega_j(x)^{1-p_j'}dx\right)^{1/p_j'}$ is understood as $(\inf_x\omega(x))^{-1}$.

As remarked in [7], $\prod_{j=1}^{m} A_{p_j}$ is strictly contained in $A_{\vec{P}}$; moreover, in general $\vec{\omega} \in A_{\vec{P}}$ does not imply $\omega_j \in L^1_{\mathrm{loc}}$ for any j, but instead

$$\vec{\omega} \in A_{\vec{P}} \Longleftrightarrow \begin{cases} \nu_{\vec{\omega}} \in A_{mp}, \\ \omega_j^{1-p_j'} \in A_{mp_j'}, \quad j = 1, \ldots, m, \end{cases} \tag{35}$$

where the condition $\omega_j^{1-p_j'} \in A_{mp_j'}$ in the case $p_j = 1$ is understood as $\omega_j^{1/m} \in A_1$.

Definition 13 (see [10]). Let $1 \le p < \infty$, $0 < k < 1$, and ω be a weight function on $\mathbb{R}^n$. The weighted Morrey space is define by

$$L^{p,k}(\omega) = \left\{ f \in L^p_{\mathrm{loc}} : \|f\|_{L^{p,k}(\omega)} < \infty \right\}, \tag{36}$$

where

$$\|f\|_{L^{p,k}(\omega)} = \sup_Q \left(\frac{1}{\omega(Q)^k} \int_Q |f(x)|^p \omega(x) \right)^{1/p}. \tag{37}$$

The weighted weak Morrey space is defined by

$$WL^{p,k}(\omega) = \left\{ f \text{ measurable} : \|f\|_{WL^{p,k}(\omega)} < \infty \right\}, \tag{38}$$

where

$$\|f\|_{WL^{p,k}(\omega)} = \sup_Q \sup_{\lambda > 0} \frac{\lambda}{\omega(Q)^{k/p}} \omega(\{x \in Q : |f|(x) > \lambda\})^{1/p}. \tag{39}$$

We say that a weight ω satisfies the doubling condition, denot ing $\omega \in \Delta_2$, if there is a constant $C > 0$ such that $\omega(2Q) \le C\omega(Q)$ holds for any cube Q. If $\omega \in A_p$ with $1 \le p < \infty$, we know that $\omega(\lambda Q) \le \lambda^{np}[\omega]_{A_p}\omega(Q)$ for all $\lambda > 1$, then $\omega \in \Delta_2$.

Lemma 14 (see [10]). *Suppose $\omega \in \Delta_2$, then there exists a constant $D > 1$ such that*

$$\omega(2Q) \ge D\omega(Q), \tag{40}$$

for any cube.

Lemma 15 (see [11]). *If $\omega_j \in A_\infty$, then for any cube Q, we have*

$$\int_Q \prod_{j=1}^{m} \omega_j^{\theta_j}(x)\, dx \ge \prod_{j=1}^{m} \left(\frac{\int_Q \omega_j(x)\, dx}{[\omega_j]_\infty} \right)^{\theta_j}, \tag{41}$$

where $\sum_{j=1}^{m} \theta_j = 1, 0 \le \theta_j \le 1$.

Lemma 16 (see [12]). *Suppose $\omega \in A_\infty$, then $\|b\|_{BMO(\omega)} \approx \|b\|_{BMO}$. Here*

$$\begin{aligned} BMO(\omega) = \Big\{ b : \|b\|_{BMO(\omega)} &= \sup_Q \frac{1}{\omega(Q)} \int_Q |b(x) - b_{Q,\omega}|\, \omega(x)\, dx < \infty \Big\}, \\ b_{Q,\omega} &= \frac{1}{\omega(Q)} \int_Q b(x)\, \omega(x)\, dx. \end{aligned} \tag{42}$$

From the fact $|b_{2^jQ} - b_Q| \le Cj\|b\|_{\mathrm{BMO}}$ and Lemma 16, we can deduce that $|b_{2^jQ,\omega} - b_{Q,\omega}| \le Cj\|b\|_{\mathrm{BMO}}$.

Lemma 17 (see [8]). *Assume that T is a multilinear operator in m-GCZO(A, s, η, ϵ) with kernel K satisfying Assumption 4. Let $1/m \le p < \infty$, $1 \le p_j \le \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$ and $\omega_j \in A_{p_j}$, $j = 1, \ldots, m$. Then we have the following:*

(i) *T extends to a bounded operators from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^p(\nu_{\vec{\omega}})$ if all the exponents p_j are strictly greater than 1;*

(ii) *T extends to a bounded operators from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^{p,\infty}(\nu_{\vec{\omega}})$ if some exponents p_j are equal to 1.*

In either case, the norm of T is bounded by $C(A + \|T\|_{L^{q_1} \times \cdots \times q_m} \to L^q)$, where C is a positive constant depending on A, s, η, ϵ, and $[w]_{A_{\vec{P}}}$.

Lemma 18 (see [6]). *Assume that T is a multilinear operator in m-GCZO(A, s, η, ϵ) with kernel K satisfying Assumption 4. Let $\vec{b} \in BMO^m$ with $\|\vec{b}\| = 1$ and $1/p = 1/p_1 + \cdots 1/p_m$ with $1 < p_j < \infty$, $j = 1, \ldots, m$. Then we have the following:*

(i) *there exists a constant C such that*

$$\left\| T_{\Sigma\vec{b}}(\vec{f}) \right\|_{L^p(\nu_{\vec{\omega}})} \le C \prod_{j=1}^{m} \left\| f_j \right\|_{L^{p_j}(M\omega_j)}; \tag{43}$$

(ii) *if $\omega_j \in A_{p_j}$, then there exists a constant C such that*

$$\left\| T_{\Sigma\vec{b}}(\vec{f}) \right\|_{L^p(\nu_{\vec{\omega}})} \le C \prod_{j=1}^{m} \left\| f_j \right\|_{L^{p_j}(\omega_j)}, \tag{44}$$

where $\nu_{\vec{\omega}} = \prod_{j=1}^{m} \omega_j^{p/p_j}$.

Lemma 19 (see [9]). *Assume that T is a multilinear operator in m-GCZO(A, s, η, ϵ) with kernel K satisfying Assumption 4. Let $\vec{b} \in BMO^m$ with $\|\vec{b}\| = 1$ and $1/p = 1/p_1 + \cdots 1/p_m$ with $1 < p_j < \infty$, $j = 1, \ldots, m$. If $\omega_j \in A_{\vec{P}}$ with $\vec{P} = (p_1, \ldots, p_m)$, then there exists a constant C such that*

$$\left\| T_{\Pi\vec{b}}(\vec{f}) \right\|_{L^p(\nu_{\vec{\omega}})} \le C \prod_{j=1}^{m} \left\| f_j \right\|_{L^{p_j}(\omega_j)}, \tag{45}$$

where $\nu_{\vec{\omega}} = \prod_{j=1}^{m} \omega_j^{p/p_j}$.

3. Proof of Theorems

Proof of Theorem 6. Here, we only prove the boundedness of $\mathscr{M}_{\sigma,r}$. From [9], there exists some $t \in (0,1)$ only depend on $\vec{\omega}$ such that

$$\mathscr{M}\left(\vec{f}\right)(x) \le C\prod_{j=1}^{m}\left\{M^{c}_{\nu_{\vec{\omega}}}\left(\left(\left|f_j\right|^{p_j}\omega_j/\nu_{\vec{\omega}}\right)^{t}\right)(x)\right\}^{1/tp_j}, \quad (46)$$

where $M^{c}_{\nu_{\vec{\omega}}}$ is the weighted centered maximal operator. Then by the Hölder inequality,

$$\begin{aligned}
&\left\|\mathscr{M}_{\sigma,r}(\vec{f})(x)\right\|_{L^{p,k}(\nu_{\vec{\omega}})}\\
&\quad\le C\left\|\prod_{i=1}^{m}\left\{M_{\nu_{\vec{\omega}}}\left(\left[\left|f_i\right|^{p_i}\omega_i/\nu_{\vec{\omega}}\right]^{t}\right)\right\}^{1/tp_i}\right\|_{L^{p,k}(\nu_{\vec{\omega}})}\\
&\quad\le C\prod_{i=1}^{m}\left\|\left\{M_{\nu_{\vec{\omega}}}\left(\left[\left|f_i\right|^{p_i}\omega_i/\nu_{\vec{\omega}}\right]^{t}\right)\right\}^{1/tp_i}\right\|_{L^{p_i,k}(\nu_{\vec{\omega}})}\\
&\quad\le C\prod_{i=1}^{m}\left\|\left\{M_{\nu_{\vec{\omega}}}\left(\left[\left|f_i\right|^{p_i}\omega_i/\nu_{\vec{\omega}}\right]^{t}\right)\right\}^{1/tp_i}\right\|^{1/tp_i}_{L^{1/t,k}(\nu_{\vec{\omega}})}\\
&\quad\le C\prod_{i=1}^{m}\left\|\left(\left|f_i\right|^{p_i}\omega_i/\nu_{\vec{\omega}}\right)^{t}\right\|^{1/tp_i}_{L^{1/t,k}(\nu_{\vec{\omega}})}\\
&\quad\le C\prod_{i=1}^{m}\left\|f_i\right\|_{L^{p_i,k}(\omega_i)}.
\end{aligned} \quad (47)$$

The weak version is a very similar process by the Hölder inequality for the weak spaces. We omit the details. □

Proof of Theorem 8. For any $B = B(x_B, r_B) \subset \mathbb{R}^n$, we split $f_i = f_i^0 + f_i^\infty$ where $f_i^0 = f_i\chi_{B*}$, $i = 1,2,\ldots,m$, and $B^* = 8B$; then

$$\begin{aligned}
\prod_{i=1}^{m} f_i(y_i) &= \prod_{i=1}^{m}\left(f_i^0(y_i) + f_i^\infty(y_i)\right)\\
&= \sum_{\alpha_1,\ldots,\alpha_m\in\{0,\infty\}}\prod_{i=1}^{m} f_i^{\alpha_i}(y_i)\\
&= \prod_{i=1}^{m} f_i^0(y_i) + \sum{}' f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m),
\end{aligned} \quad (48)$$

where each term of $\sum'$ contains at least one $\alpha_i \ne 0$. Write then

$$\begin{aligned}
&\frac{1}{\nu_{\vec{\omega}}(B)^{k/p}}\left(\int_B \left|T(f_1,\ldots,f_m)(x)\right|^p \nu_{\vec{\omega}}(x)\,dx\right)^{1/p}\\
&\quad\le \frac{1}{\nu_{\vec{\omega}}(B)^{k/p}}\left(\int_B \left|T\left(f_1^0,\ldots,f_m^0\right)(x)\right|^p \nu_{\vec{\omega}}(x)\,dx\right)^{1/p}\\
&\qquad+ \sum{}' \frac{1}{\nu_{\vec{\omega}}(B)^{k/p}}\left(\int_B \left|T\left(f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\right)(x)\right|^p \nu_{\vec{\omega}}(x)\,dx\right)^{1/p}\\
&\quad= I^{0,\ldots,0} + \sum{}' I^{\alpha_1,\ldots,\alpha_m}.
\end{aligned} \quad (49)$$

From Definition 12, Lemma 17, we can get

$$\begin{aligned}
I^{0,\ldots,0} &\le \frac{C}{\nu_{\vec{\omega}}(B)^{k/p}}\prod_{i=1}^{m}\left(\int_{B^*}\left|f_i^0(x)\right|^{p_i}\omega_i(x)\,dx\right)^{1/p_i}\\
&\le C\frac{\prod_{i=1}^{m}\omega_i(B^*)^{k/p_i}}{\nu_{\vec{\omega}}(B)^{k/p}}\prod_{i=1}^{m}\left\|f_i\right\|_{L^{p,k}(\omega_i)}\\
&\le C\prod_{i=1}^{m}\left\|f_i\right\|_{L^{p,k}(\omega_i)}.
\end{aligned} \quad (50)$$

The last inequality holds by Lemma 15. For $\sum' I^{\alpha_1,\ldots,\alpha_m}$, we first consider the case when $\alpha_1 = \cdots = \alpha_m = \infty$. Taking $t = (2r_B)^s$, since $x \in B$ and $y_i \in \mathbb{R}^n \setminus 8B$, we get

$$\left|y_i - x\right| > 7r_B > 2t^{1/s}, \quad \text{for all } j = 1,\ldots,m; \quad (51)$$

hence, $h(|y_i - x|/t^{1/s}) = 0$. By Assumption 4, we have that

$$\begin{aligned}
&\left|K(x,y_1,\ldots,y_m) - K_t^{(0)}(x,y_1,\ldots,y_m)\right|\\
&\quad\le \frac{At^{\epsilon/t}}{\left(\sum_{k=1}^{m}\left|x-y_k\right|\right)^{mn+\epsilon}} \le \frac{A}{\left(\sum_{k=1}^{m}\left|x-y_k\right|\right)^{mn}}.
\end{aligned} \quad (52)$$

For any $x \in B$, then by Assumption 4,

$$\begin{aligned}
&\left|T\left(f_1^\infty,\ldots,f_m^\infty\right)(x)\right|\\
&\quad\le \int_{(\mathbb{R}^n\setminus B^*)^m}\left|K(x,y_1,\ldots,y_m) - K_t^{(0)}(x,y_1,\ldots,y_m)\right|\\
&\qquad\times\prod_{i=1}^{m}\left|f_i^\infty(y_i)\right|d\vec{y}\\
&\qquad+\int_{(\mathbb{R}^n\setminus B^*)^m}\left|K_t^{(0)}(x,y_1,\ldots,y_m)\right|\prod_{i=1}^{m}\left|f_i^\infty(y_i)\right|d\vec{y}\\
&\quad\le C\int_{(\mathbb{R}^n\setminus B^*)^m}\frac{A}{\left(\left|x-y_1\right|+\cdots+\left|x-y_m\right|\right)^{mn}}\\
&\qquad\times\prod_{i=1}^{m}\left|f_i^\infty(y_i)\right|d\vec{y}\\
&\quad\le C\sum_{l=1}^{\infty}\prod_{i=1}^{m}\int_{8^{l+1}B\setminus 8^jB}\frac{\left|f_i(y_i)\right|}{\left|x-y_i\right|^n}dy_i\\
&\quad\le C\sum_{l=1}^{\infty}\prod_{i=1}^{m}\frac{1}{\left|8^{l+1}B\right|}\left(\int_{8^{l+1}B}\left|f_i(y_i)\right|^{p_i}\omega_i(y_i)\,dy_i\right)^{1/p_i}\\
&\qquad\times\left(\int_{8^{l+1}B}\omega_i(y_i)^{1-p_i'}dy_i\right)^{1/p_i'}\\
&\quad\le C\sum_{l=1}^{\infty}\prod_{i=1}^{m}\frac{\omega_i\left(8^{l+1}B\right)^{k/p_i}}{\left|8^{l+1}B\right|}\left\|f_i\right\|_{L^{p_i,k}(\omega_i)}\frac{\left|8^{l+1}B\right|}{\omega_i\left(8^{l+1}B\right)^{1/p_i}}\\
&\quad\le C\sum_{l=1}^{\infty}\nu_{\vec{\omega}}\left(8^{l+1}B\right)^{(k-1)/p}\prod_{i=1}^{m}\left\|f_i\right\|_{L^{p_i,k}(\omega_i)}.
\end{aligned} \quad (53)$$

Since $\nu_{\vec{\omega}} \in A_{mp}$, then there is a positive δ such that

$$\frac{\nu_\omega(B)}{\nu_\omega(8^{l+1}B)} \le C\left(\frac{|B|}{|8^{l+1}B|}\right)^{\delta}. \tag{54}$$

Hence

$$\begin{aligned} I^{\infty,\dots,\infty} &\le \nu_{\vec{\omega}}(B)^{(1-k)/p}\sup_{x\in B}\left|T\left(f_1^{\infty},\dots,f_m^{\infty}\right)(x)\right| \\ &\le C\sum_{l=1}^{\infty}\left(\frac{|B|}{|8^{l+1}B|}\right)^{\delta(1-k)/p}\prod_{i=1}^{m}\|f_i\|_{L^{p_i,k}(\omega_i)} \\ &\le C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,k}(\omega_i)}. \end{aligned} \tag{55}$$

It remains to estimate the terms with $\alpha_{i_1} = \cdots = \alpha_{i_j} = 0$ for some $\{i_1,\dots,i_j\} \subset \{1,\dots,m\}$ and $1 \le j < m$. We have

$$\begin{aligned} &\left|T\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(x)\right| \\ &\quad\le \int_{(\mathbb{R}^n\setminus B^*)^m}\Big|K(x,y_1,\dots,y_m) \\ &\qquad - K_t^{(0)}(x,y_1,\dots,y_m)\Big|\prod_{i=1}^{m}\left|f_i^{\alpha_i}(y_i)\right|d\vec{y} \\ &\qquad + \int_{(\mathbb{R}^n\setminus B^*)^m}\left|K_t^{(0)}(x,y_1,\dots,y_m)\right|\prod_{i=1}^{m}\left|f_i^{\alpha_i}(y_i)\right|d\vec{y} \\ &\quad\le C\prod_{i\in\{i_1,\dots,i_j\}}\int_{B^*}\left|f_i(y_i)\right|dy_i \\ &\qquad\times\left[\int_{(\mathbb{R}^n\setminus B^*)^{m-j}}\frac{t^{\epsilon/s}\prod_{i\notin\{i_1,\dots,i_j\}}\left|f_i(y_i)\right|dy_i}{\left(\sum_{i\notin\{i_1,\dots,i_j\}}|x-y_i|\right)^{mn+\epsilon}}\right. \\ &\qquad\left.+\int_{(\mathbb{R}^n\setminus B^*)^{m-j}}\frac{\prod_{i\notin\{i_1,\dots,i_j\}}\left|f_i(y_i)\right|dy_i}{\left(\sum_{i\notin\{i_1,\dots,i_j\}}|x-y_i|\right)^{mn}}\right] \\ &\quad\le C\frac{1}{|8^{j+1}B|^m}\prod_{i\in\{i_1,\dots,i_j\}}\int_{B^*}\left|f_i(y_i)\right|dy_i \\ &\qquad\times\sum_{l=1}^{\infty}\prod_{i\notin\{i_1,\dots,i_j\}}\int_{8^{j+1}B\setminus 8^jB}\left|f_i(y_i)\right|dy_i \\ &\quad\le C\sum_{l=1}^{\infty}\nu_{\vec{\omega}}\left(8^{l+1}B\right)^{(k-1)/p}\prod_{i=1}^{m}\|f_i\|_{L^{p_i,k}(\omega_i)}. \end{aligned} \tag{56}$$

Therefore, we also have

$$I^{\alpha_1,\dots,\alpha_m} \le C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,k}(\omega_i)}. \tag{57}$$

Combining the above estimates and then taking the supermum over all balls B in $\mathbb{R}^n$, we have proved the previous part of Theorem 8.

Next, we turn to complete the proof of the weak inequality. For any $\lambda > 0$, we can write

$$\begin{aligned} &\nu_{\vec{\omega}}\left(\left\{x\in B : \left|T\left(f_1,\dots,f_m\right)(x)\right| > \lambda\right\}\right)^{1/p} \\ &\quad\le \nu_{\vec{\omega}}\left(\left\{x\in B : \left|T\left(f_1^{\infty},\dots,f_m^{\infty}\right)(x)\right| > \lambda\right\}\right)^{1/p} \\ &\qquad + \sum{}' \nu_{\vec{\omega}}\left(\left\{x\in B : \left|T\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(x)\right| > \lambda\right\}\right)^{1/p} \\ &\quad= II^{0,\dots,0} + \sum{}' II^{\alpha_1,\dots,\alpha_m}. \end{aligned} \tag{58}$$

By Lemmas 17 and 15, we can easily check that

$$\begin{aligned} II^{0,\dots,0} &\le \frac{C}{\lambda}\prod_{i=1}^{m}\left(\int_{B^*} f_i^0(y_i)^{p_i}\omega_i(y_i)\,dy_i\right)^{1/p_i} \\ &\le \frac{C\nu_{\vec{\omega}}(B)^{k/p}}{\lambda}\prod_{i=1}^{m}\|f_i\|_{L^{p_i,k}(\omega_i)}. \end{aligned} \tag{59}$$

From the proof of (53) and (56), we have the following pointwise estimate:

$$\left|T\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(x)\right| \le C\sum_{l=1}^{\infty}\prod_{i=1}^{m}\frac{1}{|8^{l+1}B|}\int_{8^{l+1}B}\left|f_i(y_i)\right|dy_i. \tag{60}$$

Since at least one $p_i = 1$, we can assume that $\{i_1,\dots,i_j\} \subset \{1,\dots,m\}$ such that $p_{i_1} = \cdots = p_{i_j} = 1$ and others greater than 1. Then,

$$\begin{aligned} &\left|T\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(x)\right| \\ &\quad\le C\sum_{l=1}^{\infty}\prod_{i\in\{i_1,\dots,i_j\}}\frac{1}{|8^{l+1}B|}\int_{8^{l+1}B}\left|f_i(y_i)\right|\omega_i(y_i)\,dy_i \\ &\qquad\times\left(\inf_{y_i\in 8^{l+1}B}\omega_i(y_i)\right)^{-1} \\ &\qquad\times\prod_{i\notin\{i_1,\dots,i_j\}}\frac{1}{|8^{l+1}B|}\left(\int_{8^{l+1}B}\left|f_i(y_i)\right|^{p_i}\omega_i(y_i)\,dy_i\right)^{1/p_i} \\ &\qquad\times\left(\int_{8^{l+1}B}\omega_i(y_i)^{1-p_i'}dy_i\right)^{1/p_i'} \\ &\quad\le \frac{C}{\nu_{\vec{\omega}}(B)^{(1-k)/p}}\prod_{i=1}^{m}\|f_i\|_{L^{p_i,k}(\omega_i)}. \end{aligned} \tag{61}$$

Suppose that $\{x\in B : |T(f_1^{\alpha_1},\dots,f_m^{\alpha_m})(x)| > \lambda\} \ne \emptyset$; then we have that

$$\nu_{\vec{\omega}}(B)^{1/p} \le \frac{C\nu_{\vec{\omega}}(B)^{k/p}}{\lambda}\prod_{i=1}^{m}\|f_i\|_{L^{p_i,k}(\omega_i)}; \tag{62}$$

therefore,

$$II^{\alpha_1,\dots,\alpha_m} \le \nu_{\vec{\omega}}(B)^{1/p} \le \frac{C\nu_{\vec{\omega}}(B)^{k/p}}{\lambda}\prod_{i=1}^{m}\|f_i\|_{L^{p_i,k}(\omega_i)}. \tag{63}$$

Taking the supremum over all balls $B \subset \mathbb{R}^n$ and all $\lambda > 0$, we complete the proof of Theorem 6. □

Proof of Theorem 9. We will show the proof for $T_{\Pi\vec{b}}$ because the proof for $T_{\Sigma\vec{b}}$ is very similar but easier. Moreover, for simplicity of the expansion, we only present the case $m = 2$.

For any cube B, we also split f_i as $f_i = f_i^0 + f_i^\infty$ with $f_i^0 = f_i\chi_{B^*}$ and $f_i^\infty = f_i - f_i^0$. Then it remains only to verify the following inequalities:

$$\begin{aligned}
I &= \left(\frac{1}{\nu_{\vec{\omega}}(Q)^k}\int_Q \left|T_{\Pi\mathbf{b}}\left(f_1^0, f_2^0\right)(x)\right|^p \nu_{\vec{\omega}}(x)\,dx\right)^{1/p} \\
&\le C\prod_{j=1}^{2}\|b_j\|_{\mathrm{BMO}}\prod_{j=1}^{2}\|f_j\|_{L^{p_j,k}(\omega_j)}, \\
II &= \left(\frac{1}{\nu_{\vec{\omega}}(Q)^k}\int_Q \left|T_{\Pi\mathbf{b}}\left(f_1^0, f_2^\infty\right)(x)\right|^p \nu_{\vec{\omega}}(x)\,dx\right)^{1/p} \\
&\le C\prod_{j=1}^{2}\|b_j\|_{\mathrm{BMO}}\prod_{j=1}^{2}\|f_j\|_{L^{p_j,k}(\omega_j)}, \\
III &= \left(\frac{1}{\nu_{\vec{\omega}}(Q)^k}\int_Q \left|T_{\Pi\mathbf{b}}\left(f_1^\infty, f_2^0\right)(x)\right|^p \nu_{\vec{\omega}}(x)\,dx\right)^{1/p} \\
&\le C\prod_{j=1}^{2}\|b_j\|_{\mathrm{BMO}}\prod_{j=1}^{2}\|f_j\|_{L^{p_j,k}(\omega_j)}, \\
IV &= \left(\frac{1}{\nu_{\vec{\omega}}(Q)^k}\int_Q \left|T_{\Pi\mathbf{b}}\left(f_1^\infty, f_2^\infty\right)(x)\right|^p \nu_{\vec{\omega}}(x)\,dx\right)^{1/p} \\
&\le C\prod_{j=1}^{2}\|b_j\|_{\mathrm{BMO}}\prod_{j=1}^{2}\|f_j\|_{L^{p_j,k}(\omega_j)}.
\end{aligned} \tag{64}$$

From Lemma 19, Lemma 15, and Hölder's inequality, we can get

$$\begin{aligned}
I &\le C\frac{1}{\nu_{\vec{\omega}}(Q)^{k/p}}\prod_{j=1}^{2}\|b_j\|_{\mathrm{BMO}}\left(\int_{\mathbb{R}^n}\left|f_j^0(x)\right|^{p_j}\omega_j(x)\,dx\right)^{1/p_j} \\
&\le C\frac{1}{\nu_{\vec{\omega}}(Q)^{k/p}}\prod_{j=1}^{2}\left[\|b_j\|_{\mathrm{BMO}}\omega_j(2Q)^{k/p_j}\|f_j\|_{L^{p_j,k}(\omega_j)}\right] \\
&\le C\prod_{j=1}^{2}\left[\|b_j\|_{\mathrm{BMO}}\|f_j\|_{L^{p_j,k}(\omega_j)}\right].
\end{aligned} \tag{65}$$

Since II and III are symmetric, we only estimate II. Taking $\lambda_j = (b_j)_{B,\omega_j}$, $T_{\Pi\vec{b}}$ can be divided into four part:

$$\begin{aligned}
&T_{\Pi\mathbf{b}}\left(f_1^0, f_2^\infty\right)(x) \\
&\quad = \left(b_1(x)-\lambda_1\right)\left(b_2(x)-\lambda_2\right)T\left(f_1^0, f_2^\infty\right)(x) \\
&\qquad - \left(b_1(x)-\lambda_1\right)T\left(f_1^0, \left(b_2-\lambda_2\right)f_2^\infty\right)(x) \\
&\qquad - \left(b_2(x)-\lambda_2\right)T\left(\left(b_1-\lambda_1\right)f_1^0, f_2^\infty\right)(x) \\
&\qquad + T\left(\left(b_1-\lambda_1\right)f_1^0, \left(b_2-\lambda_2\right)f_2^\infty\right)(x) \\
&\quad = II_1 + II_2 + II_3 + II_4.
\end{aligned} \tag{66}$$

From the proof of Theorem 8 we know that, for any $x \in B$,

$$\left|T\left(f_1^0, f_2^\infty\right)(x)\right| \le C\sum_{l=1}^{\infty}\nu_{\vec{\omega}}\left(8^{l+1}B\right)^{(k-1)/p}\prod_{i=1}^{m}\|f_j\|_{L^{p_j,k}(\omega_j)}. \tag{67}$$

Applying (67), Hölder's inequality and Lemma 16, we have

$$\begin{aligned}
&\left(\frac{1}{\nu_{\vec{\omega}}(Q)^k}\int_Q |II_1|^p \nu_{\vec{\omega}}(x)\,dx\right)^{1/p} \\
&\quad \le \frac{1}{\nu_{\vec{\omega}}(Q)^{k/p}}\left(\int_Q \left|\left(b_1(x)-\lambda_1\right)\left(b_2(x)-\lambda_2\right)\right|^p \times \nu_{\vec{\omega}}(x)\,dx\right)^{1/p} \\
&\qquad \times \prod_{j=1}^{2}\|f_j\|_{L^{p_j,k}}\sum_{l=1}^{\infty}\nu_{\vec{\omega}}\left(2^{l+1}Q\right)^{(k-1)/p} \\
&\quad \le \frac{\nu_{\vec{\omega}}(Q)^{1/p}}{\nu_{\vec{\omega}}(Q)^{k/p}}\prod_{j=1}^{2}\left(\frac{1}{\nu_{\vec{\omega}}(Q)}\int_Q \left|\left(b_j(x)-\lambda_1\right)\right|^{2p}\nu_{\vec{\omega}}(x)dx\right)^{1/2p} \\
&\qquad \times \prod_{j=1}^{2}\|f_j\|_{L^{p_j,k}}\sum_{l=1}^{\infty}\nu_{\vec{\omega}}\left(2^{l+1}Q\right)^{(k-1)/p} \\
&\quad \le \prod_{j=1}^{2}\|b_j\|_{\mathrm{BMO}}\|f_j\|_{L^{p_j,k}(\omega_j)}.
\end{aligned} \tag{68}$$

The last inequality is obtained by the property of A_∞: there is a constant $\delta > 0$ such that

$$\frac{\nu_{\vec{\omega}}(Q)}{\nu_{\vec{\omega}}\left(2^{l+1}Q\right)} \le C\left(\frac{|Q|}{\left|2^{l+1}Q\right|}\right)^{\delta}. \tag{69}$$

For II_2, by the Assumption 4, Lemma 15, and Lemma 16, it follows that

$$\begin{aligned}
&\left|T\left(f_1^0,(b_2-\lambda_2)f_2^\infty\right)(x)\right| \\
&\quad\le \int_{(\mathbb{R}^n)^2}\left|K(x,y_1,y_2)-K_t^{(0)}(x,y_1,y_2)\right| \\
&\qquad\times\left|f_1^0(y_1)(b_2(y_2)-\lambda_2)f_2^\infty(y_2)\right|dy_1dy_2 \\
&\qquad+\int_{(\mathbb{R}^n)^2}\left|K_t^{(0)}(x,y_1,y_2)\right|\left|f_1^0(y_1)(b_2(y_2)-\lambda_2)\right. \\
&\qquad\qquad\left.\times f_2^\infty(y_2)\right|dy_1dy_2 \\
&\quad\le C\int_{8B}|f_1(y_1)|\,dy_1\sum_{l=1}^\infty\frac{1}{|8^lB|^2} \\
&\qquad\times\int_{2^{l+1}Q\setminus 2^lQ}|(b_2(y_2)-\lambda_2)f_2(y_2)|\,dy_2 \\
&\quad\le C\sum_{l=1}^\infty\frac{\|b_2\|_{\mathrm{BMO}}}{|8^{l+1}B|^2}\left(\int_{8^{l+1}B}|f_1(y_1)|^{p_1}\omega_j(y_1)\,dy_1\right)^{1/p_1} \\
&\qquad\times\left(\int_{2^{l+1}Q}\omega_1(y_1)^{1-p_1'}dy_j\right)^{1/p_1'} \\
&\quad\times\left(\int_{2^{l+1}Q}|f_2(y_2)|^{p_2}\omega_2(y_2)\,dy_2\right)^{1/p_2} \\
&\quad\times\left(\int_{2^{l+1}Q}|b_2(y_2)-\lambda_2|^{p_2'}\omega_2(y_2)^{-p_2'/p_2}dy_2\right)^{1/p_2'} \\
&\quad\le C\sum_{l=1}^\infty l\prod_{j=1}^2\frac{\|b_2\|_{\mathrm{BMO}}}{|8^{l+1}B|}\left(\int_{8^{l+1}B}|f_j(y_j)|^{p_j}\omega_j(y_j)\,dy_j\right)^{1/p_j} \\
&\qquad\times\left(\int_{2^{l+1}Q}\omega_j(y_j)^{1-p_j'}dy_j\right)^{1/p_j'} \\
&\quad\le C\|b_2\|_{\mathrm{BMO}}\prod_{j=1}^2\|f_j\|_{L^{p_j,k}(\omega_j)}\sum_{l=1}^\infty l\nu_{\vec{\omega}}\left(8^{l+1}B\right)^{(k-1)/p}.
\end{aligned}\tag{70}$$

Hölder's inequality and Lemma 16 tell us that

$$\begin{aligned}
&\left(\frac{1}{\nu_{\vec{\omega}}(Q)^k}\int_Q|II_2|^p\nu_{\vec{\omega}}(x)\,dx\right)^{1/p} \\
&\quad\le C\frac{1}{\nu_{\vec{\omega}}(Q)^{k/p}}\left(\int_Q|b_1(x)-\lambda_1|^p\nu_{\vec{\omega}}(x)\,dx\right)^{1/p} \\
&\qquad\times\prod_{j=1}^2\|f_j\|_{L^{p_j,k}}\sum_{l=1}^\infty l\nu_{\vec{\omega}}\left(2^{l+1}Q\right)^{(k-1)/p} \\
&\quad\le C\frac{\nu_{\vec{\omega}}(Q)^{1/p}}{\nu_{\vec{\omega}}(Q)^{k/p}}\prod_{j=1}^2\|f_j\|_{L^{p_j,k}}\sum_{l=1}^\infty l\nu_{\vec{\omega}}\left(2^{l+1}Q\right)^{(k-1)/p} \\
&\quad\le C\prod_{j=1}^2\|b_j\|_{\mathrm{BMO}}\|f_j\|_{L^{p_j,k}(\omega_j)}.
\end{aligned}\tag{71}$$

Similarly, we also have that

$$\left(\frac{1}{\nu_{\vec{\omega}}(Q)^k}\int_Q|II_3|^p\nu_{\vec{\omega}}(x)\,dx\right)^{1/p}\le C\prod_{j=1}^2\|b_j\|_{\mathrm{BMO}}\|f_j\|_{L^{p_j,k}(\omega_j)}.\tag{72}$$

By Assumption 4, Lemma 15, and Lemma 16, a similar way deduces that

$$\begin{aligned}
&\left|T\left((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty\right)(x)\right| \\
&\quad\le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}\prod_{j=1}^2\|f_j\|_{L^{p_j,k}(\omega_j)}\sum_{l=1}^\infty l^2\nu_{\vec{\omega}}\left(8^{l+1}B\right)^{(k-1)/p},
\end{aligned}\tag{73}$$

and so,

$$\left(\frac{1}{\nu_{\vec{\omega}}(Q)^k}\int_Q|II_4|^p\nu_{\vec{\omega}}(x)\,dx\right)^{1/p}\le C\prod_{j=1}^2\|b_j\|_{\mathrm{BMO}}\|f_j\|_{L^{p_j,k}(\omega_j)}.\tag{74}$$

Finally, we still decompose $T_{\Pi\mathbf{b}}(f_1^\infty,f_2^\infty)(x)$ into four terms:

$$\begin{aligned}
&T_{\Pi\mathbf{b}}\left(f_1^\infty,f_2^\infty\right)(x) \\
&\quad=(b_1(x)-\lambda_1)(b_2(x)-\lambda_2)T\left(f_1^\infty,f_2^\infty\right)(x) \\
&\qquad-(b_1(x)-\lambda_1)T\left(f_1^\infty,(b_2-\lambda_2)f_2^\infty\right)(x) \\
&\qquad-(b_2(x)-\lambda_2)T\left((b_1-\lambda_1)f_1^\infty,f_2^\infty\right)(x) \\
&\qquad+T\left((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty\right)(x) \\
&\quad=IV_1+IV_2+IV_3+IV_4.
\end{aligned}\tag{75}$$

Because each term of IV_j is completely analogous to II_j, $j=1,2,3,4$ with a bit difference, so we get the following estimate without details:

$$\left(\frac{1}{\nu_{\vec{\omega}}(Q)^k}\int_Q|IV|^p\nu_{\vec{\omega}}(x)\,dx\right)^{1/p}\le C\prod_{j=1}^2\|b_j\|_{\mathrm{BMO}}\|f_j\|_{L^{p_j,k}(\omega_j)}.\tag{76}$$

To this, we end the proof of Theorem 9. □

Conflict of Interests

The authors declare that they have no conflict of interests.

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