Research Article

Some Identities on the Generalized $q$-Bernoulli, $q$-Euler, and $q$-Genocchi Polynomials

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Mahmudov (2012, 2013) introduced and investigated some $q$-extensions of the $q$-Bernoulli polynomials $B_{n, q}^{(\alpha)}(x, y)$ of order $\alpha$, the $q$-Euler polynomials $E_{n, q}^{(\alpha)}(x, y)$ of order $\alpha$, and the $q$-Genocchi polynomials $G_{n, q}^{(\alpha)}(x, y)$ of order $\alpha$. In this paper, we give some identities for $B_{n, q}^{(\alpha)}(x, y)$, $E_{n, q}^{(\alpha)}(x, y)$, and $G_{n, q}^{(\alpha)}(x, y)$ and the recurrence relations between these polynomials. This is an analogous result to the $q$-extension of the Srivastava-Pintér addition theorem in Mahmudov (2013).

1. Introduction, Definitions, and Notations

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{C}$ denotes the set of complex numbers. The $q$-numbers and $q$-factorial are defined by

\[ [a]_q = \frac{1-q^a}{1-q}, \quad q \neq 1, \]

\[ [n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q, \]

respectively, where $[0]_q! = 1$, $n \in \mathbb{N}$, and $a \in \mathbb{C}$. The $q$-binomial coefficient is defined by

\[ \binom{n}{k}_q = \frac{(q : q)_n}{(q : q)_{n-k}(q : q)_k}, \]

where $(q : q)_n = (1 - q) \cdots (1 - q^n)$. The $q$-analogue of the function $(x + y)^n$ is defined by

\[ (x + y)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q q^{(k(k-1))/2} x^{n-k} y^k. \]

The $q$-binomial formula is known as

\[ (n; q)_a = (1 - a)_q^n = \prod_{j=0}^{n-1} (1 - q^j a) \]

\[ = \sum_{k=0}^{n} \binom{n}{k}_q q^{(k(k-1))/2} (-1)^k a^k. \]

The $q$-exponential functions are given by

\[ e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1 - (1 - q) q^k z)}, \quad 0 < |q| < 1, |z| < \frac{1}{|1 - q|}, \]
\[ E_q(z) = \sum_{n=0}^{\infty} \frac{q^{(n-1)/2} z^n}{[n]_q!} \]
\[ = \prod_{k=0}^{\infty} \left( 1 + (1-q) q^k z \right), \quad 0 < |q| < 1, \quad z \in C. \] (5)

From these forms, we easily see that \( e_q(z)E_q(-z) = 1 \). Moreover, \( D_q e_q(z) = e_q(z) \) and \( D_q F_q(z) = F_q(qz) \), where \( D_q \) is defined by
\[ D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1, \quad 0 \neq z \in C. \] (6)

The previous \( q \)-standard notation can be found in [1, 2]. Carlitz firstly extended the classical Bernoulli numbers and polynomials and Euler numbers and polynomials [3, 4]. There are numerous recent investigations on this subject by many other authors. Among them are Cenkci et al. [5, 6], Choi et al. [1], Cheon [7], Kim [8], Kurt [9], Kurt [10], Luo and Srivastava [11-13], Srivastava et al. [14, 15], Natalini and Bernardini [16], Tremblay et al. [17, 18], Gaboury and Kurt [19], Mahmudov [2, 20, 21], Araci et al. [22], and Kupershmidt [23].

Mahmudov defined and studied the properties of the following generalized \( q \)-Bernoulli polynomials \( G_{n,q}^{(\alpha)}(x,y) \) of order \( \alpha \) and \( q \)-Euler polynomials \( G_{n,q}^{(\alpha)}(x,y) \) of order \( \alpha \) as follows [2].

Let \( q \in C, \ \alpha \in \mathbb{N}, \) and \( 0 < |q| < 1 \). The \( q \)-Bernoulli numbers \( G_{n,q}^{(\alpha)} \) and polynomials \( G_{n,q}^{(\alpha)}(x,y) \) in \( x \) and \( y \) of order \( \alpha \) are defined by means of the generating functions:
\[ \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left( \frac{t}{e_q(t) - 1} \right)^\alpha, \quad |t| < 2\pi, \] (7)
\[ \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left( \frac{t}{e_q(t) - 1} \right)^\alpha e_q(tx)E_q(ty), \quad |t| < 2\pi. \] (8)

The \( q \)-Euler numbers \( G_{n,q}^{(\alpha)} \) and polynomials \( G_{n,q}^{(\alpha)}(x,y) \) in \( x \) and \( y \) of order \( \alpha \) are defined by means of the generating functions:
\[ \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left( \frac{2}{e_q(t) + 1} \right)^\alpha, \quad |t| < \pi, \] (9)
\[ \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left( \frac{2}{e_q(t) + 1} \right)^\alpha e_q(tx)E_q(ty), \quad |t| < \pi. \] (10)

The \( q \)-Genocchi numbers \( G_{n,q}^{(\alpha)} \) and polynomials \( G_{n,q}^{(\alpha)}(x,y) \) in \( x \) and \( y \) of order \( \alpha \) are defined by means of the generating functions:
\[ \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left( \frac{2t}{e_q(t) + 1} \right)^\alpha, \quad |t| < \pi, \] (11)
\[ \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left( \frac{2t}{e_q(t) + 1} \right)^\alpha e_q(tx)E_q(ty), \quad |t| < \pi. \] (12)

The familiar \( q \)-Stirling numbers \( S_{n,k}(n,k) \) of the second kind are defined by
\[ \frac{e_q(t) - 1}{q} \frac{k^\alpha}{[k]_q!} = \sum_{n=0}^{\infty} S_{n,k}(n,k) \frac{t^n}{[n]_q!}. \] (13)

It is obvious that
\[ G_{n,q}^{(1)}(x,y) := B_{n,q}(x,y), \quad G_{n,q}^{(1)}(x,y) := G_{n,q}(x,y), \]
\[ G_{n,q}^{(1)}(x,y) := G_{n,q}(x,y), \quad B_{n,q}(0,0) := B_{n,q}, \]
\[ G_{n,q}(0,0) := G_{n,q}, \quad G_{n,q}(0,0) := G_{n,q}^{(1)}, \]
\[ B_{n,q}^{(\alpha)} = B_{n,q}^{(\alpha)}(0,0), \quad G_{n,q}^{(\alpha)} = G_{n,q}^{(\alpha)}(0,0), \]
\[ \lim_{q \to 1} G_{n,q}^{(\alpha)}(x,y) = G_{n}^{(\alpha)}(x+y), \]
\[ \lim_{q \to 1} G_{n,q}^{(\alpha)}(x,y) = G_{n}^{(\alpha)}(x+y), \quad \lim_{q \to 1} G_{n,q}^{(\alpha)}(x,y) = G_{n}^{(\alpha)}(x+y), \]
\[ \lim_{q \to 1} G_{n,q}^{(\alpha)}(x,y) = G_{n}^{(\alpha)}(x+y). \] (14)

From (8) and (10), it is easy to check that
\[ G_{n,q}(x,y) = \sum_{k=0}^{n} \binom{n}{k} G_{n-k,q}(x,0) G_{k,q}^{(a-1)}(0, y), \]
\[ G_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} G_{n-k,q}^{(\alpha)}(x,0) G_{k,q}^{(a-1)}(0, y). \] (15)

In this work, we give some identities for the \( q \)-Bernoulli polynomials. Also, we give some relations between the \( q \)-Bernoulli polynomials and \( q \)-Euler polynomials and the \( q \)-Genocchi polynomials and \( q \)-Bernoulli polynomials. Furthermore, we give a different form of the analogue of the Srivastava-Pintér addition theorem. More precisely, we prove the following theorems.
Theorem 1. There are the following relations between the \(q\)-Bernoulli polynomials and \(q\)-Stirling numbers of the second kind:

\[
\mathcal{S}_{n,q}^{(\alpha)}(x, y) = \sum_{l=0}^{n-k} \left[ \frac{[l]_q! [n]_q!}{[n+k]_q!} \right]_{q} \sum_{p=0}^{n} \left[ \frac{p}{l} \right]_{q} S_{2,q}(n-p, j) \times \left. \chi^{p-l} y^q \right| \frac{j}{\chi^{p-l}}.
\]

(16)

where \(q \in \mathbb{C}, \alpha, n \in \mathbb{N}, \) and \(0 < |q| < 1\).

Theorem 2. The \(q\)-Stirling numbers of the second kind satisfy the following relations:

\[
\mathcal{S}_{n,q}^{(\alpha)}(x, y) = \sum_{j=0}^{\infty} \left[ -\alpha \right]_{j} \frac{1}{j!} [j]_{q}!
\]

\[
\times \sum_{p=1}^{n} \left[ \frac{n}{p} \right]_{q} S_{2,q}(n-p, j) \chi^{p-l} y^q \frac{j}{\chi^{p-l}}.
\]

(18)

Theorem 4. The polynomials \(B_{n,q}(x, y)\) and \(G_{n,q}(x, y)\) satisfy the following difference relationships:

\[
B_{n,q}(x, y) = \sum_{l=0}^{n} \left[ \frac{n+1}{l} \right]_{q} \frac{1}{[n+1]_q} \mathcal{B}_{l,q}^{(\alpha)}(x, y) \mathcal{B}_{n-1,q}^{(\alpha)}(x, y),
\]

(21)

\[
G_{n,q}(x, y) = -2 \sum_{l=0}^{n} \left[ \frac{n}{l} \right]_{q} \frac{1}{[n+1]_q} \mathcal{G}_{l,q}^{(\alpha)}(x, y) \mathcal{G}_{n-1,q}^{(\alpha)}(x, y),
\]

(22)

where \(q \in \mathbb{C}, \alpha, n \in \mathbb{N}, \) and \(0 < |q| < 1\).

Theorem 5. There is the following relation between the generalized \(q\)-Euler polynomials and generalized \(q\)-Bernoulli polynomials:

\[
\mathcal{G}_{n,q}^{(\alpha)}(x, y) = \sum_{s=0}^{n} \left[ \frac{n+1}{s} \right]_{q} \frac{1}{[n+1]_q} \mathcal{B}_{s,q}^{(\alpha)}(mx, 0) \times \mathcal{G}_{n-s,q}^{(\alpha)}(0, y) m^{s-n},
\]

(23)

where \(q \in \mathbb{C}, \alpha, n \in \mathbb{N}, \) and \(0 < |q| < 1\).

2. Proof of the Theorems

Lemma 6. The generalized \(q\)-Bernoulli polynomials, \(q\)-Euler polynomials, and \(q\)-Genocchi polynomials satisfy the following relations:

\[
\sum_{k=0}^{n} \left[ \frac{n}{k} \right]_{q} \mathcal{B}_{k,q}^{(\alpha)}(x, y) \mathcal{B}_{n-k,q}^{(-\alpha)} = (x+y)^n q^n,
\]

\[
\sum_{k=0}^{n} \left[ \frac{n}{k} \right]_{q} \mathcal{B}_{k,q}^{(\alpha)}(0, y) \mathcal{B}_{n-k,q}^{(-\alpha)} = q^{(n-\alpha)/2} y^n,
\]

\[
\mathcal{G}_{n,q}^{(\alpha)}(x, y) = \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_{q} \mathcal{G}_{k,q}^{(\alpha)}(x, y) \mathcal{G}_{n-k,q}^{(-\alpha)},
\]

(19)

where \(q \in \mathbb{C}, \alpha, n \in \mathbb{N}, \) and \(0 < |q| < 1\).
$$\mathcal{G}_n(x, y) = \sum_{n=0}^{\infty} \frac{\ell^n}{[n]_q!} \mathcal{G}_n(x, y)$$

$$\sum_{k=0}^{n} \frac{n}{k} \mathcal{G}_k(x, y) + \sum_{n=0}^{\infty} \frac{\ell^n}{[n]_q!}$$

Proof. The proof of this lemma can be found from (7)–(12).

Proof of Theorem 1. By (8) and (13) we have

$$\sum_{n=0}^{\infty} \mathcal{G}_n(x, y) \frac{\ell^n}{[n]_q!} = \mathcal{G}_n(x, y)$$

$$\sum_{n=0}^{\infty} \frac{\ell^n}{[n]_q!}$$

$$\mathcal{G}_n(x, y) = \sum_{n=0}^{\infty} \frac{\ell^n}{[n]_q!} [n]_q!$$

Equating the coefficients of $\ell^n/[n]_q!$, we obtain (16). Similarly, we have (17).

Proof of Theorem 3. It is obvious that

$$\frac{-2}{e_q(t) + 1} e_q(t) = \frac{2}{e_q(t) + 1} - \frac{2}{e_q(t)}$$

We write it as

$$\frac{-2}{e_q(t) + 1} e_q(t) = \frac{2}{e_q(t) + 1} e_q(tx) E_q(ty)$$

$$\frac{-2}{e_q(t) + 1} e_q(tx) E_q(ty) = \frac{2}{e_q(t) + 1} e_q(tx) E_q(ty)$$

$$- 2 e_q(tx) E_q(ty)$$

$$- 2 e_q(tx) E_q(ty)$$
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\[ - \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(x, y) \frac{t^n}{[n]_q!} \]

\[ = \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(x, y) \frac{t^n}{[n]_q!} \]

\[ \times \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} - 2 \sum_{n=0}^{\infty} (x + y)^n \frac{t^n}{[n]_q!}. \]

Using the Cauchy product and comparing the coefficients of \( (t^n/[n]_q!) \), we have

\[ \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_{k,q}(x, y) = 2(x + y)^n \mathcal{G}_{k,q}(x, y). \] \( \blacksquare \) \( (29) \)

Finally, we consider the interesting relationships between the \( q \)-Bernoulli polynomials and \( q \)-Genocchi polynomials and the \( q \)-Euler polynomials and \( q \)-Bernoulli polynomials. These relations are \( q \)-analogues to the Srivastava-Pintér addition theorems.

Proof of Theorem 4. It follows immediately that

\[ \sum_{n=0}^{\infty} \mathcal{B}_{n,q}(x, y) \frac{t^n}{[n]_q!} \]

\[ = \frac{1}{2} 2t e_q(tx) E_q(ty) + \frac{1}{t} \]

\[ + \sum_{n=0}^{\infty} \frac{n}{[n]_q!} \left( \frac{e_q(t) - 1}{e_q(t) + 1} \right) \frac{t^n}{[n]_q!} \]

\[ \times \mathcal{B}_{n,q}(x, y) \frac{t^n}{[n]_q!} \]

\[ = \frac{1}{2} 2t e_q(tx) E_q(ty) + \frac{1}{t} \]

\[ + \sum_{n=0}^{\infty} \frac{n}{[n]_q!} \left( \frac{e_q(t) - 1}{e_q(t) + 1} \right) \frac{t^n}{[n]_q!} \]

\[ \times \mathcal{B}_{n,q}(x, y) \frac{t^n}{[n]_q!} \]

\[ + \sum_{n=0}^{\infty} \left( -\frac{1}{2} \mathcal{G}_{n,q}(x, y) \right) \frac{t^n}{[n]_q!} \]

\[ + \sum_{n=0}^{\infty} \left( \frac{n+1}{[n+1]_q} \right) \frac{1}{q} \frac{t^{n+1}}{[n+1]_q!} \]

\[ \times \mathcal{B}_{l,q}(x, y) \mathcal{B}_{n+1-l,q} \frac{t^n}{[n]_q!}. \]

\[ \sum_{n=0}^{\infty} \mathcal{B}_{n,q}(x, y) \frac{t^n}{[n]_q!} \]

\[ = \frac{1}{t} \left( \frac{2t}{e_q(t) + 1} \left( e_q(t) - 1 \right) \right) \frac{t e_q(tx) E_q(ty)}{e_q(t) - 1} \]

\[ = \frac{1}{t} \left( \frac{2t - 2}{e_q(t) + 1} \right) \frac{t e_q(tx) E_q(ty)}{e_q(t) - 1} \]

\[ = \frac{1}{t} \left( -2 \sum_{n=0}^{\infty} \frac{1}{[n+1]_q!} \mathcal{B}_{l+1,q}(x, y) \frac{t^{l+1}}{[l+1]_q!} \sum_{n=0}^{\infty} \mathcal{B}_{n,q}(x, y) \frac{t^n}{[n]_q!} \right) \]

\[ = \sum_{n=1}^{\infty} \left( -2 \sum_{l \neq n} \frac{n}{[l]_q!} \mathcal{B}_{l,q}(x, y) \frac{t^n}{[n]_q!} \right) \]

\[ \times E_q(ty) \frac{e_q(t/m) - 1}{(t/m) e_q(t/m) - 1} e_q((t/m) mx) \]

\[ \times \mathcal{B}_{n-1,q} \frac{t^{n-1}}{[n-1]_q!} \]

\[ = \frac{1}{2} 2t e_q(tx) E_q(ty) + \frac{1}{t} \]

\[ + \sum_{n=0}^{\infty} \left( -\frac{1}{2} \mathcal{G}_{n,q}(x, y) \right) \frac{t^n}{[n]_q!} \]

\[ + \sum_{n=0}^{\infty} \left( \frac{n+1}{[n+1]_q} \right) \frac{1}{q} \frac{t^{n+1}}{[n+1]_q!} \]

\[ \times \mathcal{B}_{l,q}(x, y) \mathcal{B}_{n+1-l,q} \frac{t^n}{[n]_q!}. \]

Equating the coefficients of \( (t^n/[n]_q!) \), we have (21).

In a similar fashion, (12) yields

\[ \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(x, y) \frac{t^n}{[n]_q!} \]

\[ = \frac{1}{t} \left( \frac{2t}{e_q(t) + 1} \left( e_q(t) - 1 \right) \right) \frac{t e_q(tx) E_q(ty)}{e_q(t) - 1} \]

\[ = \frac{1}{t} \left( \frac{2t - 2}{e_q(t) + 1} \right) \frac{t e_q(tx) E_q(ty)}{e_q(t) - 1} \]

\[ = \frac{1}{t} \left( -2 \sum_{n=0}^{\infty} \frac{1}{[n+1]_q!} \mathcal{B}_{l+1,q}(x, y) \frac{t^{l+1}}{[l+1]_q!} \sum_{n=0}^{\infty} \mathcal{B}_{n,q}(x, y) \frac{t^n}{[n]_q!} \right) \]

\[ = \sum_{n=1}^{\infty} \left( -2 \sum_{l \neq n} \frac{n}{[l]_q!} \mathcal{B}_{l,q}(x, y) \frac{t^n}{[n]_q!} \right) \]

\[ \times E_q(ty) \frac{e_q(t/m) - 1}{(t/m) e_q(t/m) - 1} e_q((t/m) mx) \]

\[ \times \mathcal{B}_{n-1,q} \frac{t^{n-1}}{[n-1]_q!} \]

\[ = \frac{1}{2} 2t e_q(tx) E_q(ty) + \frac{1}{t} \]

\[ + \sum_{n=0}^{\infty} \left( -\frac{1}{2} \mathcal{G}_{n,q}(x, y) \right) \frac{t^n}{[n]_q!} \]

\[ + \sum_{n=0}^{\infty} \left( \frac{n+1}{[n+1]_q} \right) \frac{1}{q} \frac{t^{n+1}}{[n+1]_q!} \]

\[ \times \mathcal{B}_{l,q}(x, y) \mathcal{B}_{n+1-l,q} \frac{t^n}{[n]_q!}. \]

Comparing the coefficients of \( (t^n/[n]_q!) \), we have (22). \( \blacksquare \)

Proof of Theorem 5. By (10), we write

\[ \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(x, y) \frac{t^n}{[n]_q!} \]

\[ = \left( \frac{2}{e_q(t) + 1} \right)^{\alpha} \]

\[ \times E_q(ty) \frac{e_q(t/m) - 1}{(t/m) e_q(t/m) - 1} e_q((t/m) mx) \]

\[ \times \mathcal{B}_{n-1,q} \frac{t^{n-1}}{[n-1]_q!} \]
\[ m \sum_{n=0}^{\infty} \frac{\delta_n^0(x, y)}{[n]_q!} \left( \sum_{n=0}^{\infty} B_{n,q}(mx, 0) \right) \left( \sum_{n=0}^{\infty} E_{n,q}(\alpha) \right) \frac{t^n}{m^n [n]_q!} \]

By equating the coefficients of \( t^n/[n]_q! \), we get the theorem. \( \square \)

**Remark 7.** There are many different relationships which are analogues to the Srivastava-Pintér addition theorems at these polynomials.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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