Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 294892, 16 pages
http://dx.doi.org/10.1155/2013/294892

Research Article

Delay-Dependent Exponential Optimal H^∞ Synchronization for Nonidentical Chaotic Systems via Neural-Network-Based Approach

Feng-Hsiag Hsiao

Department of Electrical Engineering, National University of Tainan, 33, Section 2, Shu Lin Street, Tainan 700, Taiwan

Correspondence should be addressed to Feng-Hsiag Hsiao; fhhsiao@mail.nutn.edu.tw

Received 7 September 2012; Accepted 7 November 2012

Academic Editor: Sabri Arik

A novel approach is presented to realize the optimal H^∞ exponential synchronization of nonidentical multiple time-delay chaotic (MTDC) systems via fuzzy control scheme. A neural-network (NN) model is first constructed for the MTDC system. Then, a linear differential inclusion (LDI) state-space representation is established for the dynamics of the NN model. Based on this LDI state-space representation, a delay-dependent exponential stability criterion of the error system derived in terms of Lyapunov's direct method is proposed to guarantee that the trajectories of the slave system can approach those of the master system. Subsequently, the stability condition of this criterion is reformulated into a linear matrix inequality (LMI). According to the LMI, a fuzzy controller is synthesized not only to realize the exponential synchronization but also to achieve the optimal H^∞ performance by minimizing the disturbance attenuation level at the same time. Finally, a numerical example with simulations is given to demonstrate the effectiveness of our approach.

1. Introduction

The stability analysis and stabilization of time-delay systems are problems of considerable theoretical and practical significance and have attracted the interest of many investigators for several years. Furthermore, time delays often appear in various engineering systems [1], such as the structure control of tall buildings, hydraulics, or electronic networks. Notably, the introduction of a time-delay factor tends to complicate the analysis. Consequently, convenient methods to check stability have long been sought later. The stability criteria of time-delay systems so far have been approached from two main directions based on the dependence on the size of delay. One method is to contrive stability conditions which do not include information on the delay, while the other method takes time delay into account. The former case is often referred to as delay-independent criterion and generally gives good algebraic conditions. Nevertheless, the abandonment of information on the size of the time delay necessarily causes conservativeness of the criteria, especially when the delay is comparatively small. Hence, delay-dependent criteria are derived to deal with the stability problem in this study.

Moreover, time delays have gained increasing interest in chaotic systems, ever since chaotic phenomenon in time-delay systems was first found by Mackey and Glass [2]. Chaotic phenomena have been observed in numerous physical systems, which can lead to irregular performance and possibly catastrophic failures [3]. Chaos is a well-known nonlinear phenomenon, and it is the seemingly random behavior of a deterministic system that is characterized by sensitive dependence on initial conditions [4]. Besides, chaos is occasionally preferable but usually intrinsically unpredictable as it can restrict the operating range of many physical devices and reduce performance. Therefore, the ability to control chaos is of much practical importance. According to these properties, chaos has received a great deal of interest among scientists from various research fields [5, 6]. One of the research fields for communication, chaotic synchronization, has been investigated extensively.

The chaotic synchronization of identical systems with different initial conditions was first introduced by Pecora and Carroll in 1990 [7]. They are intended to control one chaotic system to follow another. Since the introduction of this concept, various synchronization approaches have been widely developed in the past two decades. Chaotic synchronization can be applied in the vast areas of physics and engineering science, especially in secure communication [8]. Consequently, chaotic synchronization has become a popular study [9, 10]. However, all of them are focused on synchronizing two identical chaotic systems with different initial conditions [11]. In fact, experimental and even more real systems are often not fully identical; in particular, there are mismatches in parameters of the systems [11]. Also, in many real world applications, there are no exactly two identical chaotic systems. As a result, the problem of chaos synchronization between two different uncertain chaotic systems is an important research issue [12]. For instance, He et al. [13] investigate synchronization of two nonidentical chaotic systems with time-varying delay and parameter mismatches via impulsive control. To synchronize nonidentical chaotic systems with unknown parameters, Li et al. [14] proposed an approach based on the invariance principle of differential equations, and employing a combination of feedback control and adaptive control. Li and Ge [15] presented a new fuzzy model to simulate and synchronize two totally different and complicated chaotic systems.

In general, some noise or disturbances always exist that may cause instability. The influence of the external disturbance will worsen the performance of chaotic systems. Therefore, how to reduce the effect of external disturbances in the synchronization process for chaotic systems is an important issue [16, 17]. The H^∞ control has been conferred for synchronization in chaotic systems over the last few years [16–20], and the H^∞ synchronization problem has been investigated extensively for time-delay chaotic systems (e.g., see [21–23]). Accordingly, the purpose of this study is to realize the exponential synchronization of nonidentical multiple time-delay chaotic (MTDC) systems and attenuate the effect of external disturbances on the control performance to a minimum level at the same time.

Neural-network-(NN-) based modeling has become an active research field in the past few years due to its unique merits in solving complex nonlinear system identification and control problems [24–29]. Neural networks consist of simple elements operating in parallel; these elements are inspired by biological nervous systems. As a result, we can train an NN to represent a particular function by adjusting the weights between elements. As in nature, the connections between elements largely determine the network function. Individuals can train a neural network to perform a particular function by adjusting the values of the connections (weights) between elements. Hence, the nonlinear systems can be approximated as close as desired by the NN models via repetitive training. Recently, numerous reports on the success of NN applications in control systems have appeared in the literature (see [30–35]). For instance, Limanond et al. [30] applied neural networks to the optimal etch time control design for a reactive ion etching process. Enns and Si [32] advanced an NN-based approximate dynamic programming control mechanism to helicopter flight control. Despite several promising empirical results and its nonlinear mapping approximation properties, the rigorous closed-loop stability results for systems using NN-based controllers are still difficult to establish. Therefore, an LDI state-space representation was introduced to deal with the stability analysis of NN models (see [36]).

In the past few years, significant research efforts have been devoted to fuzzy control, which has attracted a great deal of attention from both the academic and industrial communities, and there have been many successful applications. For example, Wang et al. [37] presented a new measurement system that comprises a model-based fuzzy logic controller, an arterial tonometer, and a micro syringe device for the noninvasive monitoring of the continuous blood pressure wave form in the radial artery. A good tracking performance control scheme, a hybrid fuzzy neural-network control for nonlinear motor-toggle servomechanisms, was given by Wai [38]; Hwang et al. [39] developed the trajectory tracking of a car-like mobile robot using network-based fuzzy decentralized sliding-mode control; a hybrid fuzzy-PI speed controller for permanent magnet synchronous motors was proposed in Sant [40]; Spatti et al. [41] introduced a fuzzy control strategy for voltage regulation in electric power distribution systems—this real-time controller would act on power transformers equipped with under-load tap changers.

In spite of the successes of fuzzy control, many basic problems remain to be solved. Stability analysis and systematic design are certainly among the most important issues for fuzzy control systems. Recently, significant research efforts have been devoted to these issues (see [42–45] and the references therein). However, all of them have neglected the modeling errors between the fuzzy models and the nonlinear systems. In fact, the existence of modeling errors may be a potential source of instability for control designs based on the assumption that the fuzzy model exactly matches the nonlinear plant [46]. In recent years, novel approaches to overcome the influence of modeling errors in the field of model-based fuzzy control for nonlinear systems have been proposed by Kiriakidis [46], Chen et al. [47, 48], and Cao et al. [49, 50].

Almost all the existing research works of synchronization method made use of fuzzy models to approximate the chaotic systems (see [3, 4, 28, 42] and the references therein). Although using fuzzy models to approximate the chaotic systems is more simple than the neural-networks (NNs), the NN models will approach the chaotic systems by iterative training and adjusting the weights. In other words, the modeling errors of NN models will be much less than those of fuzzy models. With a view to the abovementioned, a novel approach is proposed via the neural-network-(NN-) based technique to realize the optimal H^∞ exponential synchronization of nonidentical multiple time-delay chaotic (MTDC) systems such that the trajectories of the slave systems can approach those of the master systems and the effect of external disturbances on the control performance can be attenuated to a minimum level. First, the NN model is constructed for the chaotic systems with multiple time delays. Then, a linear differential inclusion (LDI) state-space representation is

established for the dynamics of the NN model. Next, in terms of Lyapunov's direct method, a delay-dependent criterion is derived to guarantee the exponential stability of the error system between the master system and slave system. Subsequently, the stability condition of this criterion is reformulated into a linear matrix inequality (LMI). According to the LMI, a fuzzy controller is synthesized not only to realize the exponential synchronization but also to achieve the optimal H^{∞} performance by minimizing the disturbance attenuation level at the same time.

The remainder of this paper is organized as follows. The system description is arranged in Section 2. In Section 3, a robustness design of fuzzy control and a delay-dependent stability criterion are proposed to realize the optimal H^{∞} exponential synchronization. The design algorithm is given in Section 4. In Section 5, the effectiveness of the proposed approach is illustrated by a numerical simulation. Finally, the conclusions are drawn in Section 6.

2. Problem Formulation

Consider two different multiple time-delay chaotic (MTDC) systems in master-slave configuration. The dynamics of the master system (N_m) and slave system (N_s) are described as follows:

$$N_m: \quad \dot{X}(t) = f(X(t)) + \sum_{k=1}^{g} H_k(X(t-\tau_k)), \tag{1}$$

$$N_s: \quad \dot{\widehat{X}}(t) = \widehat{f}(\widehat{X}(t)) + \sum_{k=1}^{g} \widehat{H}_k(\widehat{X}(t-\tau_k)) + BU(t) + D(t), \tag{2}$$

where $f(\cdot)$, $\widehat{f}(\cdot)$, $H_k(\cdot)$, and $\widehat{H}_k(\cdot)$ are the nonlinear vector-valued functions, $\tau_k (k = 1, 2, \ldots, g)$ are the time delays, $U(t)$ is the control input, and $D(t)$ denotes the external disturbance. Besides, $X(t)$ and $\widehat{X}(t)$ are the state vectors of N_m and N_s, respectively.

In this section, a neural-network (NN) model is first constructed for the MTDC system. The dynamics of the NN model are then converted into a linear differential inclusion (LDI) state-space representation. Finally, based on the LDI state-space representation, a fuzzy controller is synthesized to realize the synchronization of nonidentical MTDC systems.

2.1. Neural-Network (NN) Model. The MTDC system can be approximated by an NN model, as shown in Figure 1, that has S layers with $J^{\sigma} (\sigma = 1, 2, \ldots, S)$ neurons for each layer, in which $x_1(t) \sim x_{\delta}(t)$ are the state variables and $x_1(t-\tau_1) \sim x_1(t-\tau_g)$, $x_2(t-\tau_1) \sim x_{\delta}(t-\tau_g)$ are the state variables with delays.

To distinguish among these layers, the superscripts are used for identification. Specifically, the number of the layer is appended as a superscript to the names for each of these variables. Thus, the weight matrix for the σth layer is written as W^{σ}. Furthermore, it is assumed that $v_{\varsigma}^{\sigma}(t) (\varsigma = 1, 2, \ldots, J^{\sigma}; \sigma = 1, 2, \ldots, S)$ is the net input and $T(v_{\varsigma}^{\sigma}(t))$ is the transfer function of the neuron. Subsequently, the transfer function vector of the σth layer is defined as

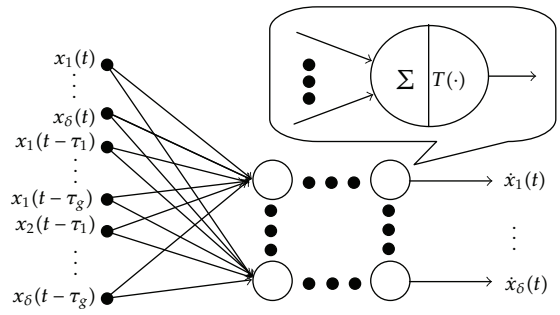


Figure 1: An NN model for N_d.

$$\Psi^{\sigma}(v_{\varsigma}^{\sigma}(t)) \equiv \left[T(v_1^{\sigma}(t))\, T(v_2^{\sigma}(t)) \cdots T(v_{J^{\sigma}}^{\sigma}(t))\right]^T, \quad \sigma = 1, 2, \ldots, S, \tag{3}$$

where $T(v_{\varsigma}^{\sigma}(t)) (\varsigma = 1, 2, \ldots, J^{\sigma})$ is the transfer function of the ςth neuron. The final output of NN model can then be inferred as follows:

$$\dot{X}(t) = \Psi^{S}\left(W^{S}\Psi^{S-1}\left(W^{S-1}\Psi^{S-2} \times \left(\cdots \Psi^{2}\left(W^{2}\Psi^{1}\left(W^{1}\Lambda(t)\right)\right)\cdots\right)\right)\right), \tag{4}$$

where $\Lambda^T(t) = [X^T(t) X^T(t-\tau_k)]$ with $X(t) = [x_1(t) x_2(t) \cdots x_{\delta}(t)]^T$,

$$X(t-\tau_k) = \left[x_1(t-\tau_1) \cdots x_1(t-\tau_g)\; x_2(t-\tau_1) \cdots x_{\delta}(t-\tau_g)\right]^T, \quad \text{for } k = 1, 2 \ldots, g. \tag{5}$$

2.2. Linear Differential Inclusion (LDI). To handle the synchronization problem of MTDC systems, this study establishes the following LDI state-space representation for the dynamics of the NN model, described as [36, 51]

$$\dot{O}(t) = A(a(t))\, O(t), \quad A(a(t)) = \sum_{i=1}^{\phi} h_i(a(t))\, \widetilde{A}_i, \tag{6}$$

where ϕ is a positive integer, $a(t)$ is a vector signifying the dependence of $h_i(\cdot)$ on its elements, $\widetilde{A}_i$ $(i = 1, 2, \ldots, \phi)$ are constant matrices, and $O(t) = [o_1(t) o_2(t) \cdots o_{\aleph}(t)]^T$. Moreover, it is assumed that $h_i(a(t)) \geq 0$ and $\sum_{i=1}^{\phi} h_i(a(t)) = 1$. According to the properties of LDI, without loss of generality, $h_i(t)$ can be replaced by $h_i(a(t))$. The following procedure represents the dynamics of the NN model (4) using the LDI state-space representation [36].

To begin with, notice that the output $T(v_\varsigma^\sigma(t))$ satisfies

$$\begin{aligned} g_{\varsigma 0}^\sigma v_\varsigma^\sigma(t) \le T\left(v_\varsigma^\sigma(t)\right) \le g_{\varsigma 1}^\sigma v_\varsigma^\sigma(t), \quad v_\varsigma^\sigma(t) \ge 0, \\ g_{\varsigma 1}^\sigma v_\varsigma^\sigma(t) \le T\left(v_\varsigma^\sigma(t)\right) \le g_{\varsigma 0}^\sigma v_\varsigma^\sigma(t), \quad v_\varsigma^\sigma(t) < 0, \end{aligned} \tag{7}$$

where $g_{\varsigma 0}^\sigma$ and $g_{\varsigma 1}^\sigma$ denote the minimum and maximum of the derivative of $T(v_\varsigma^\sigma(t))$, respectively, and are given in the following:

$$g_{\varsigma\varphi}^\sigma = \begin{cases} \min\limits_{v} \dfrac{dT\left(v_\varsigma^\sigma(t)\right)}{dv_\varsigma^\sigma(t)}, & \text{when } \varphi = 0, \\ \max\limits_{v} \dfrac{dT\left(v_\varsigma^\sigma(t)\right)}{dv_\varsigma^\sigma(t)}, & \text{when } \varphi = 1. \end{cases} \tag{8}$$

Subsequently, the min-max matrix G^σ of the σth layer is defined as follows:

$$\begin{aligned} G^\sigma &\equiv \operatorname{diag}\left[g_{\varsigma\varphi_\varsigma}^\sigma\right] \\ &= \begin{bmatrix} g_{1\varphi_1}^\sigma & 0 & 0 & \cdots & 0 \\ 0 & g_{2\varphi_2}^\sigma & 0 & \ddots & 0 \\ 0 & 0 & g_{3\varphi_3}^\sigma & 0 & \vdots \\ \vdots & \ddots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & g_{J^\sigma\varphi_J}^\sigma \end{bmatrix}. \end{aligned} \tag{9}$$

Besides, on the basis of the interpolation method, the transfer function $T(v_\varsigma^\sigma(t))$ can be represented as follows [36]:

$$\begin{aligned} T\left(v_\varsigma^\sigma(t)\right) &= \left(h_{\varsigma 0}^\sigma(t)\, g_{\varsigma 0}^\sigma + h_{\varsigma 1}^\sigma(t)\, g_{\varsigma 1}^\sigma\right) v_\varsigma^\sigma(t) \\ &= \left(\sum_{\varphi=0}^{1} h_{\varsigma\varphi}^\sigma(t)\, g_{\varsigma\varphi}^\sigma\right) v_\varsigma^\sigma(t), \end{aligned} \tag{10}$$

where the interpolation coefficients $h_{\varsigma\varphi}^\sigma(t) \in [0,1]$ and $\sum_{\varphi=0}^{1} h_{\varsigma\varphi}^\sigma(t) = 1$. Equations (3) and (10) show that

$$\begin{aligned} &\Psi^\sigma\left(v_\varsigma^\sigma(t)\right) \\ &\quad \equiv \left[T\left(v_1^\sigma(t)\right)\ T\left(v_2^\sigma(t)\right)\ \cdots\ T\left(v_{J^\sigma}^\sigma(t)\right)\right]^T \\ &\quad = \left[\left(\sum_{\varphi_1=0}^{1} h_{1\varphi_1}^\sigma(t)\, g_{1\varphi_1}^\sigma\right) v_1^\sigma(t)\ \left(\sum_{\varphi_2=0}^{1} h_{2\varphi_2}^\sigma(t)\, g_{2\varphi_2}^\sigma\right) v_2^\sigma(t)\ \cdots\ \left(\sum_{\varphi_J=0}^{1} h_{J^\sigma\varphi_J}^\sigma(t)\, g_{J^\sigma\varphi_J}^\sigma\right) v_{J^\sigma}^\sigma(t)\right]^T. \end{aligned} \tag{11}$$

Hence, the final output of the NN model (4) can be reformulated as follows:

$$\begin{aligned} &\dot{X}(t) \\ &\quad = \sum_{p=0}^{1} h_{\varsigma p}^S(t)\, G^S \\ &\qquad \times \Bigg(W^S \Bigg[\cdots \Bigg[\sum_{n=0}^{1} h_{\varsigma n}^2(t)\, G^2 \\ &\qquad\qquad \times \Bigg(W^2 \Bigg[\sum_{b=0}^{1} h_{\varsigma b}^1(t)\, G^1 \\ &\qquad\qquad\qquad \times \left(W^1 \Lambda(t)\right)\Bigg]\Bigg)\Bigg]\cdots\Bigg]\Bigg) \\ &\quad = \sum_{p=0}^{1} \cdots \sum_{n=0}^{1} \sum_{b=0}^{1} h_{\varsigma p}^S(t) \cdots h_{\varsigma n}^2(t)\, h_{\varsigma b}^1(t)\, G^S W^S \\ &\qquad \cdots G^2 W^2 G^1 W^1 \Lambda(t) \\ &\quad = \sum_{\Omega} h_{\varsigma\Omega}^\sigma(t)\, C_\Omega^\sigma \Lambda(t), \end{aligned} \tag{12}$$

where

$$\begin{aligned} \sum_{b=0}^{1} h_{\varsigma b}^1(t) &\equiv \sum_{b_1=0}^{1} h_{1b_1}^1(t) \sum_{b_2=0}^{1} h_{2b_2}^1(t) \cdots \sum_{b_J=0}^{1} h_{J^1 b_J}^1(t), \\ \sum_{n=0}^{1} h_{\varsigma n}^2(t) &\equiv \sum_{n_1=0}^{1} h_{1n_1}^2(t) \sum_{n_2=0}^{1} h_{2n_2}^2(t) \cdots \sum_{n_J=0}^{1} h_{J^2 n_J}^2(t), \\ &\vdots \\ \sum_{p=0}^{1} h_{\varsigma p}^S(t) &\equiv \sum_{p_1=0}^{1} h_{1p_1}^S(t) \sum_{p_2=0}^{1} h_{2p_2}^S(t) \cdots \sum_{p_J=0}^{1} h_{J^S p_J}^S(t), \\ \sum_{\Omega} h_{\varsigma\Omega}^\sigma(t) &\equiv \sum_{p=0}^{1} \cdots \sum_{n=0}^{1} \sum_{b=0}^{1} h_{\varsigma p}^S(t) \cdots h_{\varsigma n}^2(t)\, h_{\varsigma b}^1(t), \\ &\qquad \varsigma = 1, 2, \ldots, J^\sigma, \\ C_\Omega^\sigma &\equiv G^S W^S \cdots G^2 W^2 G^1 W^1, \end{aligned} \tag{13}$$

and b_ς, n_ς, p_ς ($\varsigma = 1, 2, \ldots, J$) represent the variables φ of the ςth neuron of the first, second, and Sth layer, respectively.

Finally, based on (6), the dynamics of the NN model (12) can be rewritten as the following LDI state-space representation:

$$\dot{X}(t) = \sum_{i=1}^{\phi} h_i(t) C_i \Lambda(t), \tag{14}$$

where $h_i(t) \geq 0$, $\sum_{i=1}^{\phi} h_i(t) = 1$, ϕ is a positive integer and C_i is a constant matrix with appropriate dimension associated with C_{Ω}^{σ}. Furthermore, the LDI state-space representation (14) can be rearranged as follows:

$$\dot{X}(t) = \sum_{i=1}^{\phi} h_i(t) \left\{ A_i X(t) + \sum_{k=1}^{g} \overline{A}_{ik} X(t-\tau_k) \right\}, \tag{15}$$

where A_i and $\overline{A}_{ik}$ are the partitions of C_i corresponding to the partitions of $\Lambda^T(t)$.

From the abovementioned, the NN models of the master and slave chaotic systems are described by the following LDI state-space representations (16) and (17), respectively:

$$\text{master}: \quad \dot{X}(t) = \sum_{i=1}^{\phi} h_i(t) \left\{ A_i X(t) + \sum_{k=1}^{g} \overline{A}_{ik} X(t-\tau_k) \right\}, \tag{16}$$

$$\text{slave}: \quad \dot{\widehat{X}}(t) = \sum_{j=1}^{\phi} \widehat{h}_j(t) \left[\widehat{A}_j \widehat{X}(t) + \sum_{k=1}^{g} \widehat{\overline{A}}_{jk} \widehat{X}(t-\tau_k) \right] + BU(t). \tag{17}$$

2.3. Fuzzy Controller. On the basis of the state-feedback control scheme, a fuzzy controller is utilized to make the slave system synchronize with the master system. The fuzzy controller is in the following form:

Control Rule l : IF $e_1(t)$ is M_{l1} and $\cdots$ and $e_{\delta}(t)$ is $M_{l\delta}$,

$$\text{THEN} \quad U(t) = -K_l E(t), \tag{18}$$

where $l = 1, 2, \ldots, \rho$, and ρ is the number of IF-THEN rules of the fuzzy controller and $M_{l\eta}$ $(\eta = 1, 2, \ldots, \delta)$ are the fuzzy sets. Therefore, the final output of this fuzzy controller can be inferred as follows:

$$U(t) = \frac{-\sum_{l=1}^{\rho} w_l(t) K_l E(t)}{\sum_{l=1}^{\rho} w_l(t)} = -\sum_{l=1}^{\rho} \overline{h}_l(t) K_l E(t), \tag{19}$$

with $w_l(t) \equiv \prod_{\eta=1}^{\delta} M_{l\eta}(e_{\eta}(t))$, $M_{l\eta}(e_{\eta}(t))$ is the grade of membership of $e_{\eta}(t)$ in $M_{l\eta}$.

3. Stability Analysis and Chaotic Synchronization via Fuzzy Control

In this section, the synchronization of nonidentical multiple time-delay chaotic (MTDC) systems is examined under the influence of modeling error. The exponential synchronization scheme of the multiple time-delay chaotic systems is described as follows.

3.1. Error Systems. From (1) and (2), the synchronization error is defined as $E(t) \equiv \widehat{X}(t) - X(t) = [e_1(t), e_2(t), \ldots, e_{\delta}(t)]^T$, and then the dynamics of the error system under the fuzzy control (19) can be described as follows:

$$\begin{aligned}
\dot{E}(t) &= \widehat{\Psi} + D(t) - \Psi \\
&\quad + \sum_{i=1}^{\phi}\sum_{j=1}^{\phi}\sum_{l=1}^{\rho} h_i(t) \widehat{h}_j(t) \overline{h}_l(t) \\
&\quad \times \left\{ G_{il} E(t) + \left(\widehat{A}_j - A_i\right) \widehat{X}(t) \right. \\
&\qquad + \sum_{k=1}^{g} \left(\widehat{\overline{A}}_{jk} - \overline{A}_{ik}\right) \widehat{X}(t-\tau_k) \\
&\qquad \left. + \sum_{k=1}^{g} \overline{A}_{ik} E(t-\tau_k) \right\} \\
&\quad - \sum_{i=1}^{\phi}\sum_{j=1}^{\phi}\sum_{l=1}^{\rho} h_i(t) \widehat{h}_j(t) \overline{h}_l(t) \\
&\quad \times \left\{ G_{il} E(t) + \left(\widehat{A}_j - A_i\right) \widehat{X}(t) \right. \\
&\qquad + \sum_{k=1}^{g} \left(\widehat{\overline{A}}_{jk} - \overline{A}_{ik}\right) \widehat{X}(t-\tau_k) \\
&\qquad \left. + \sum_{k=1}^{g} \overline{A}_{ik} E(t-\tau_k) \right\} \\
&= \sum_{i=1}^{\phi}\sum_{l=1}^{\rho} h_i(t) \overline{h}_l(t) \left\{ G_{il} E(t) + \sum_{k=1}^{g} \overline{A}_{ik} E(t-\tau_k) \right\} \\
&\quad + D(t) + \Phi(t),
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
G_{il} &\equiv A_i - BK_l, \\
\widehat{\Psi} &\equiv \widehat{f}\left(\widehat{X}(t)\right) + \sum_{k=1}^{g} \widehat{H}_k\left(\widehat{X}(t-\tau_k)\right) + U(t), \\
\Psi &\equiv f(X(t)) + \sum_{k=1}^{g} H_k\left(X(t-\tau_k)\right),
\end{aligned} \tag{21}$$

with

$$U(t) = -\sum_{l=1}^{\rho} \overline{h}_l(t) K_l E(t),$$

$$\begin{aligned}
\Phi(t) &\equiv \widehat{\Psi} - \Psi \\
&\quad - \left\{ \sum_{i=1}^{\phi}\sum_{l=1}^{\rho} h_i(t) \overline{h}_l(t) \left[G_{il} E(t) + \sum_{k=1}^{g} \overline{A}_{ik} E(t-\tau_k) \right] \right\}.
\end{aligned} \tag{22}$$

Suppose that there exists a bounding matrix ΘR_{il} such that

$$\|\Phi(t)\| \leq \left\| \sum_{i=1}^{\phi}\sum_{l=1}^{\rho} h_i(t)\,\overline{h}_l(t)\,\Theta R_{i\ell} E(t) \right\| \tag{23}$$

for the trajectory $E(t)$, and the bounding matrix ΘR_{il} can be described as follows:

$$\Theta R_{il} = \varepsilon_{il} R, \tag{24}$$

where R is the specified structured bounding matrix and $\|\varepsilon_{il}\| \leq 1$, for $i = 1, 2, \ldots, \phi$; $l = 1, 2, \ldots, \rho$. Equations (23) and (24) show that

$$\begin{aligned} \Phi^T(t)\,\Phi(t) &\leq \sum_{i=1}^{\phi}\sum_{l=1}^{\rho} h_i(t)\,\overline{h}_l(t)\,\|RE(t)\|\,\|\varepsilon_{il}\| \\ &\quad \times \sum_{i=1}^{\phi}\sum_{l=1}^{\rho} h_i(t)\,\overline{h}_l(t)\,\|\varepsilon_{il}\|\,\|RE(t)\| \\ &\leq [RE(t)]^T [RE(t)]. \end{aligned} \tag{25}$$

Namely, $\Phi(t)$ is bounded by the specified structured bounding matrix R.

Remark 1 (see [47]). The following simple example describes the procedures for determining ε_{il} and R. First, assume that the possible bounds for all elements in ΘR_{il} are

$$\Theta R_{il} = \begin{bmatrix} \Theta r_{il}^{11} & \Theta r_{il}^{12} & \Theta r_{il}^{13} \\ \Theta r_{il}^{21} & \Theta r_{il}^{22} & \Theta r_{il}^{23} \\ \Theta r_{il}^{31} & \Theta r_{il}^{32} & \Theta r_{il}^{33} \end{bmatrix}, \tag{26}$$

where $-r^{qs} \leq \Delta r_{il}^{qs} \leq r^{qs}$ for some r_{il}^{qs} with $q, s = 1, 2, 3$; $i = 1, 2, \ldots, \phi$, and $l = 1, 2, \ldots, \rho$.

A possible depiction for the bounding matrix ΘR_{il} is

$$\Theta R_{il} = \begin{bmatrix} \varepsilon_{il}^{11} & 0 & 0 \\ 0 & \varepsilon_{il}^{22} & 0 \\ 0 & 0 & \varepsilon_{il}^{33} \end{bmatrix} \begin{bmatrix} r^{11} & r^{12} & r^{13} \\ r^{21} & r^{22} & r^{23} \\ r^{31} & r^{32} & r^{33} \end{bmatrix} = \varepsilon_{il} R, \tag{27}$$

where $-1 \leq \varepsilon_{il}^{qq} \leq 1$ for $q = 1, 2, 3$. Notice that ε_{il} can be chosen by other forms as long as $\|\varepsilon_{il}\| \leq 1$. The validity of (23) is then checked in the simulation. If it is not satisfied, we can expand the bounds for all elements in ΘR_{il} and repeat the design procedure until (23) holds.

3.2. Delay-Dependent Stability Criterion for Exponential H^∞ Synchronization. In this subsection, a delay-dependent criterion is proposed to guarantee the exponential stability of the error system described in (20). Moreover, in general, some noises or disturbances always exist that may cause instability. The influence of the external disturbance $D(t)$ will worsen the performance of chaotic systems. To reduce the effect of the external disturbance, an optimal H^∞ scheme is used to design the fuzzy control so that the effect of external disturbance on control performance can be attenuated to a minimum level. In other words, the fuzzy controller (19) realizes exponential synchronization and at the same time achieves the optimal H^∞ control performance in this study.

Before examination of the stability of the error system, some definitions and a lemma are given follows.

Lemma 2 (see [52]). *For the real matrices A and B with appropriate dimension,*

$$A^T B + B^T A \leq \lambda A^T A + \lambda^{-1} B^T B, \tag{28}$$

where λ is a positive constant.

Definition 3 (see [51]). The slave system (2) can exponentially synchronize with the master system (1) (i.e., the error system (20) is exponentially stable) if there exist two positive numbers α and β such that the synchronization error satisfies

$$\|E(t)\| \leq \alpha \exp\left(-\beta\left(t - t_0\right)\right), \quad \forall t \geq 0, \tag{29}$$

where the positive number β is called the exponential convergence rate.

Definition 4 (see [19–23]). The master system (1) and slave system (2) are said to be exponential H^∞ synchronization if the following conditions are satisfied:

(i) with zero disturbance (i.e., $D(t) = 0$), the error system (20) with the fuzzy controller (19) is exponentially stable;

(ii) under the zero initial conditions (i.e., $E(t) = 0$ for $t \in [-\tau_{\max}, 0]$, in which $\tau_{\max}$ is the maximal value of τ_k's) and a given constant $\kappa > 0$, the following condition holds:

$$\begin{aligned} \Theta(E(t), \partial(t)) &= \int_0^\infty E^T(t)\,E(t)\,dt - \kappa^2 \int_0^\infty D^T(t)\,D(t)\,dt \\ &\leq 0, \end{aligned} \tag{30}$$

where the parameter κ is called the H^∞ norm bound or the disturbance attenuation level. If the minimum κ is found (i.e., the error system can reject the external disturbance as strong as possible) to satisfy the previous conditions, the fuzzy controller (19) is an optimal H^∞ synchronizer [18].

Theorem 5. *For given positive constants a and n, if there exist two symmetric positive definite matrices P, ψ_k and positive constants ξ, κ so that the following inequalities hold, then the exponential H^∞ synchronization with the disturbance attenuation κ is guaranteed via the fuzzy controller* (19) *consider.*

$$\begin{aligned} \Delta_{il} &\equiv \sum_{k=1}^{g} \tau_k P G_{il} + \sum_{k=1}^{g} \tau_k G_{il}^T P + \sum_{k=1}^{g} \psi_k + ngR^T R + I \\ &\quad + \sum_{k=1}^{g} \tau_k^2 P^2 \left(\xi^{-1} + n^{-1} + ga^{-1}\right) < 0, \end{aligned} \tag{31a}$$

$$\nabla_{ik} \equiv ga\overline{A}_{ik}^T\overline{A}_{ik} - \psi_k < 0, \tag{31b}$$

$$\kappa > \sqrt{\xi g}, \tag{31c}$$

where $G_{il} \equiv A_i - BK_l$, *for* $i = 1, 2, \ldots, \phi$; $k = 1, 2, \ldots, g$, *and* $l = 1, 2, \ldots, \rho$.

Proof. Let the Lyapunov function for the error system (20) be defined as

$$V(t) = \sum_{k=1}^{g} E^T(t)\tau_k PE(t) + \sum_{k=1}^{g}\int_0^{\tau_k} E^T(t-\pi)\psi_k E(t-\pi)\,d\pi, \tag{32}$$

where the weighting matrices $P = P^T > 0$ and $\psi_k = \psi_k^T > 0$. We then evaluate the time derivative of $V(t)$ on the trajectories of (20) to obtain

$$\begin{aligned}
\dot{V}(t) &= \sum_{k=1}^{g}\tau_k\left[\dot{E}^T(t)PE(t) + E^T(t)P\dot{E}(t)\right] \\
&\quad + \sum_{k=1}^{g}\left[E^T(t)\psi_k E(t) - E^T(t-\tau_k)\psi_k E(t-\tau_k)\right] \\
&= \sum_{k=1}^{g}\tau_k\left\{\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)\left[G_{il}E(t) + \sum_{d=1}^{g}\overline{A}_{id}E(t-\tau_d)\right]\right. \\
&\qquad \left. + D(t) + \Phi(t)\right\}^T PE(t) + \sum_{k=1}^{g}\tau_k E^T(t)P \\
&\quad \times\left\{\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)\left[G_{il}E(t) + \sum_{d=1}^{g}\overline{A}_{id}E(t-\tau_d)\right.\right. \\
&\qquad \left.\left. + D(t) + \Phi(t)\right]\right\} \\
&\quad + \sum_{k=1}^{g}\left[E^T(t)\psi_k E(t) - E^T(t-\tau_k)\psi_k E(t-\tau_k)\right] \\
&= \sum_{k=1}^{g}\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)E^T(t)\left[\tau_k G_{il}^T P + \tau_k PG_{il} + \psi_k\right]E(t) \\
&\quad + \sum_{k=1}^{g}\sum_{i=1}^{\phi}\sum_{d=1}^{g}h_i(t)\left[E^T(t-\tau_d)\tau_k\overline{A}_{id}^T PE(t)\right. \\
&\qquad \left. + E^T(t)\tau_k P\overline{A}_{id}E(t-\tau_d)\right] \\
&\quad + \sum_{k=1}^{g}\tau_k\left[D^T(t)\tau_k PE(t) + E^T(t)\tau_k PD(t)\right. \\
&\qquad \left. + \Phi^T(t)\tau_k PE(t) + E^T(t)\tau_k P\Phi(t)\right] \\
&\quad - \sum_{k=1}^{g}\left[E^T(t-\tau_k)\psi_k E(t-\tau_k)\right].
\end{aligned} \tag{33}$$

According to Lemma 2 and (33), we have

$$\begin{aligned}
\dot{V}(t) &\le \sum_{k=1}^{g}\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)E^T(t)\left[\tau_k G_{il}^T P + \tau_k PG_{il} + \psi_k\right]E(t) \\
&\quad + \sum_{k=1}^{g}\sum_{i=1}^{\phi}\sum_{d=1}^{g}h_i(t)\left[aE^T(t-\tau_d)\overline{A}_{id}^T\overline{A}_{id}E(t-\tau_d)\right. \\
&\qquad \left. + a^{-1}E^T(t)\tau_k^2P^2E(t)\right] \\
&\quad + \sum_{k=1}^{g}\left[\xi D^T(t)D(t) + \xi^{-1}E^T(t)\tau_k^2P^2E(t)\right. \\
&\qquad \left. + n\Phi^T(t)\Phi(t) + n^{-1}E^T(t)\tau_k^2P^2E(t)\right] \\
&\quad - \sum_{k=1}^{g}\left[E^T(t-\tau_k)\psi_k E(t-\tau_k)\right]
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\le \sum_{k=1}^{g}\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)E^T(t) \\
&\quad \times\left[\tau_k G_{il}^T P + \tau_k PG_{il} + \psi_k\right]E(t) \\
&\quad + \sum_{k=1}^{g}\sum_{i=1}^{\phi}\sum_{d=1}^{g}h_i(t)\left[aE^T(t-\tau_d)\overline{A}_{id}^T\overline{A}_{id}E(t-\tau_d)\right. \\
&\qquad \left. + a^{-1}E^T(t)\tau_k^2P^2E(t)\right] \\
&\quad + \sum_{k=1}^{g}\left[\xi D^T(t)D(t) + \xi^{-1}E^T(t)\tau_k^2P^2E(t)\right. \\
&\qquad \left. + nE^T(t)R^TRE(t) + n^{-1}E^T(t)\tau_k^2P^2E(t)\right] \\
&\quad - \sum_{k=1}^{g}\left[E^T(t-\tau_k)\psi_k E(t-\tau_k)\right] \quad (\text{by } (25))
\end{aligned} \tag{35}$$

$$\begin{aligned}
&= \sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)E^T(t) \\
&\quad \times\left[\sum_{k=1}^{g}\tau_k PG_{il} + \sum_{k=1}^{g}\tau_k G_{il}^T P\right. \\
&\qquad + \sum_{k=1}^{g}\psi_k + ngR^TR \\
&\qquad \left. + \sum_{k=1}^{g}\tau_k P^2\left(\xi^{-1} + n^{-1} + ga^{-1}\right)\right]E(t) \\
&\quad + \sum_{k=1}^{g}\sum_{i=1}^{\phi}h_i(t)E^T(t-\tau_k)\left[ga\overline{A}_{ik}^T\overline{A}_{ik} - \psi_k\right] \\
&\quad \times E(t-\tau_k) + \xi gD^T(t)D(t).
\end{aligned} \tag{36}$$

From (36), we have

$$\begin{aligned}
&\dot{V}(t)+E^T(t)E(t)-\kappa^2D^T(t)D(t)\\
&\quad\le\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)E^T(t)\Delta_{il}E(t)\\
&\qquad+\sum_{i=1}^{\phi}\sum_{k=1}^{g}h_i(t)E^T(t-\tau_k)\nabla_{ik}E(t-\tau_k)\\
&\qquad+\left(\xi g-\kappa^2\right)D^T(t)D(t)\\
&\quad\le\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)\lambda_{\max}(\Delta_{il})E^T(t)E(t)\\
&\qquad+\sum_{i=1}^{\phi}\sum_{k=1}^{g}h_i(t)\lambda_{\max}(\nabla_{ik})E^T(t-\tau_k)E(t-\tau_k)\\
&\qquad+\left(\xi g-\kappa^2\right)D^T(t)D(t)<0,
\end{aligned}\tag{37}$$

where

$$\begin{aligned}
\Delta_{il}&\equiv\sum_{k=1}^{g}\tau_kPG_{il}+\sum_{k=1}^{g}\tau_kG_{il}^TP+\sum_{k=1}^{g}\psi_k+ngR^TR+I\\
&\quad+\sum_{k=1}^{g}\tau_k^2P^2\left(\xi^{-1}+n^{-1}+ga^{-1}\right)\quad(\text{see (31a)}),\\
\nabla_{ik}&\equiv ga\overline{A}_{ik}^T\overline{A}_{ik}-\psi_k\quad(\text{see (31b)}).
\end{aligned}\tag{38}$$

Integrating (37) from $t=0$ to $t=\infty$, the following inequality is obtained as

$$V(\infty)-V(0)+\int_0^{\infty}E^T(t)E(t)\,dt-\kappa^2\int_0^{\infty}D^T(t)D(t)\,dt\le0.\tag{39}$$

With zero initial conditions (i.e., $E(t)\equiv0$ for $t\in[-\tau_{\max},0]$), we have

$$\int_0^{\infty}E^T(t)E(t)\,dt\le\kappa^2\int_0^{\infty}D^T(t)D(t)\,dt.\tag{40}$$

That is, (30) and the H^∞ control performance are achieved with a prescribed attenuation κ.

Since

$$\begin{aligned}
&\sum_{k=1}^{g}\tau_k\lambda_{\min}(P)E^T(t)E(t)\\
&\quad\le\sum_{k=1}^{g}\tau_kE^T(t)PE(t)\\
&\quad=V(t)-\sum_{k=1}^{g}\int_0^{\tau_k}E^T(t-\pi)\psi_kE(t-\pi)\,d\pi\\
&\quad<V(t)
\end{aligned}\tag{41}$$

(from (32)), we can get the following inequality from (37):

$$\begin{aligned}
&\dot{V}(t)+E^T(t)E(t)-\kappa^2D^T(t)D(t)\\
&\quad<\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)\frac{\lambda_{\max}(\Delta_{il})}{\sum_{k=1}^{g}\tau_k\lambda_{\min}(P)}V(t)<0.
\end{aligned}\tag{42}$$

Then, we can easily obtain

$$V(t)|_{\partial(t)=0}<V(t_0)\exp\overline{\beta}(t-t_0),\tag{43}$$

where $\overline{\beta}=\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)[\lambda_{\max}(\Delta_{il})/\sum_{k=1}^{g}\tau_k\lambda_{\min}(P)]<0$.

Equations (32) and (43) show that

$$\begin{aligned}
&\sum_{k=1}^{g}\tau_k\lambda_{\min}(P)E^T(t)E(t)\\
&\quad\le\sum_{k=1}^{g}E^T(t)\tau_kPE(t)\\
&\quad<V(t_0)\exp\overline{\beta}(t-t_0)\\
&\qquad-\sum_{k=1}^{g}\int_0^{\tau_k}E^T(t-\pi)\psi_kE(t-\pi)\,d\pi\\
&\quad<V(t_0)\exp\overline{\beta}(t-t_0).
\end{aligned}\tag{44}$$

That is, $\|E(t)\|^2\le(V(t_0)/\sum_{k=1}^{g}\tau_k\lambda_{\min}(P))\exp\overline{\beta}(t-t_0)$. Therefore, we conclude that

$$\begin{aligned}
&\|E(t)\|\le\alpha\exp\left(-\beta(t-t_0)\right),\\
&\text{with }\alpha\equiv\sqrt{\frac{V(t_0)}{\sum_{k=1}^{g}\tau_k\lambda_{\min}(P)}}>0,\ \beta\equiv-\frac{1}{2}\overline{\beta}>0.
\end{aligned}\tag{45}$$

Hence, on basis of the Definition 3, the error system (20) with the fuzzy controller (19) is exponentially stable for $D(t)=0$. □

Corollary 6. *Equations* (31a) *and* (31b) *can be reformulated into LMIs via the following procedure.*

By introducing the new variables $Q=P^{-1}$, $F_l=K_lQ$, and $\overline{\psi}_k=Q\psi_kQ$, (31a) *and* (31b) *can be rewritten as follows:*

$$\begin{aligned}
&\sum_{k=1}^{g}\tau_k\left\{A_iQ-BF_l+QA_i^T-F_l^TB^T\right\}\\
&\quad+\sum_{k=1}^{g}\overline{\psi}_k+ngQR^TRQ+QIQ\\
&\quad+\sum_{k=1}^{g}\tau_k^2\left(\xi^{-1}+n^{-1}+ga^{-1}\right)I<0,
\end{aligned}\tag{46a}$$

$$ga\overline{A}_{ik}^TQ\overline{A}_{ik}-\overline{\psi}_k<0,\tag{46b}$$

for $i=1,2,\ldots,\phi$; $k=1,2,\cdots,g$ and $l=1,2,\cdots,\rho$. According to Schur's complement [36], it is easy to show that the linear

matrix inequalities in (46a) *and* (46b) *are equivalent to the following LMIs in* (47a) *and* (47b)*:*

$$\begin{bmatrix} \Xi & QR^T & Q \\ RQ^T & -(ng)^{-1}I & 0 \\ Q & 0 & -I \end{bmatrix} < 0, \tag{47a}$$

$$\begin{bmatrix} -\overline{\psi}_k & Q\overline{A}_{ik}^T \\ \overline{A}_{ik}Q & -(ga)^{-1}I \end{bmatrix} < 0, \tag{47b}$$

where

$$\begin{aligned} \Xi \equiv & \sum_{k=1}^{g} \tau_k A_i Q - \sum_{k=1}^{g} \tau_k B F_l \\ & + \sum_{k=1}^{g} \tau_k Q A_i^T - \sum_{k=1}^{g} \tau_k F_l^T B^T \\ & + \sum_{k=1}^{g} \overline{\psi}_k + \sum_{k=1}^{g} \tau_k^2 \left(\xi^{-1} + n^{-1} + ga^{-1}\right) I. \end{aligned} \tag{48}$$

Hence, Theorem 5 can be transformed into an LMI problem, and efficient interior-point algorithms are now available in Matlab LMI Solver to solve this problem.

Corollary 7 (see [53]). *In order to verify the feasibility of solving the inequalities in* (47a) *and* (47b) *using LMI Solver (Matlab), the interior-point optimization techniques are utilized to compute feasible solutions. Such techniques require that the system of LMI is constrained to be strictly feasible; that is, the feasible set has a nonempty interior. For feasibility problems, the LMI Solver by feasp (feasp is the syntax used to test feasibility of a system of LMIs in MATLAB) is shown as follows:*

$$\textit{find } x \textit{ such that the LMI } L(x) < 0, \tag{49a}$$

(in this study, (49a) *can be represented as* (47a) *and* (47b)*) and*

$$\textit{minimize } t \textit{ subject to } \; L(x) < t \times I. \tag{49b}$$

From the abovementioned, the LMI constraint is always strictly feasible in x, t and the original LMI (49a) is feasible if and only if the global minimum $t_{\min}$ (the global minimum $t_{\min}$ is the scalar value returned as the output argument by feasp) of (49b) satisfies $t_{\min} < 0$. In other words, if $t_{\min} < 0$ will satisfy (47a) and (47b) then the stability conditions (31a) and (31b) in Theorem 5 can be met. Then, the obtained fuzzy controller (19) can exponentially stabilize the error system, and the H^∞ control performance is achieved at the same time.

Corollary 8. *In order to achieve optimal H^∞ exponential synchronization, the fuzzy control design is formulated as the following constrained optimization problem:*

$$\begin{aligned} & \textit{minimize} \quad \kappa > \sqrt{\xi g} \\ & \textit{subject to} \quad Q = Q^T > 0, \\ & \overline{\psi}_k = \overline{\psi}_k^T > 0, \quad (47a) \textit{ and } (47b). \end{aligned} \tag{50}$$

More details to search the minimum κ are given as follows.

The positive constant ξ is minimized by the mincx function of Matlab LMI toolbox. Therefore, the minimum disturbance attenuation level $\kappa_{\min} > \sqrt{\xi_{\min} g}$ can be obtained.

Remark 9. In order to reduce the computational burden, this study sets the positive constants a and n as unity.

Remark 10. It is an important issue to reduce the effect of external disturbances in the synchronization process. The H^∞ norm bound κ is generally chosen as a positive small value less than unity for attenuation of disturbance. A smaller κ is desirable as this yields better performance. However, a smaller κ will result in a smaller ξ, making the stability conditions (31a) more difficult to satisfy.

Remark 11. According to (25), the modeling error $\Phi(t)$ is assumed to be bounded by the specified structured bounding matrix R, and then a larger $\Phi(t)$ results in a larger R. Since the matrices Δ_{il} must be negative definite to meet the stability condition (31a), a larger R will make Theorem 5 more difficult to satisfy.

4. Algorithm

The complete design procedure can be summarized as follows.

Problem 1. Given two different multiple time-delay chaotic systems with different initial conditions, the problem is centered on how to synthesize a fuzzy controller to realize the optimal H^∞ exponential synchronization.

We can solve this problem based on the following steps.

Step 1. Construct the neural-network (NN) models of the master system (1) and the slave system (2), respectively. According to the interpolation method, the NN models are then converted into LDI state-space representations.

Step 2. On the basis of the state-feedback control scheme, a fuzzy controller (19) is synthesized to exponentially stabilize the error system.

Step 3. Define the synchronization error $E(t) = \widehat{X}(t) - X(t)$, and then the dynamics of the error system (20) can be obtained.

Step 4. Based on Corollary 8, the positive constant ξ is minimized by the mincx function of Matlab LMI toolbox, and then we have the minimum disturbance attenuation level.

Step 5. The matrices Q, F_l, and $\overline{\psi}_k$ can be obtained with the minimum disturbance attenuation $\kappa_{\min}$.

5. Numerical Example

The following example illustrates the effectiveness of the previous algorithm.

Problem 2. The purpose of this example is to synthesize a fuzzy controller to achieve optimal H^{∞} exponential synchronization. Consider the modified multiple time-delay Genesio and Lorenz chaotic systems in master-slave configuration, described as follows:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t), \\ \dot{x}_3(t) &= -6x_1(t) - 2.92x_2(t-0.015) \\ &\quad - 1.2x_3(t) + x_1^2(t-0.13) \end{aligned} \tag{51}$$

$$\begin{aligned} \dot{\hat{x}}_1(t) &= 10\left(\hat{x}_2(t) - \hat{x}_1(t)\right) + D(t) + u_1(t), \\ \dot{\hat{x}}_2(t) &= 28\hat{x}_1(t) - \hat{x}_2(t-0.13) \\ &\quad - \hat{x}_1(t)\hat{x}_3(t) + D(t) + u_2(t), \\ \dot{\hat{x}}_3(t) &= \hat{x}_1(t)\hat{x}_2(t) - \left(\frac{8}{3}\right)\hat{x}_3(t-0.015) + D(t) + u_3(t), \end{aligned} \tag{52}$$

where $[x_1(t)\ x_2(t)\ x_3(t)]^T$ and $[\hat{x}_1(t)\ \hat{x}_2(t)\ \hat{x}_3(t)]^T$ are the state vectors of master and slave systems, respectively. Let the different initial conditions of master and slave systems be $[x_1(0) = -0.5x_2(0) = 2x_3(0) = 6]$ and $[\hat{x}_1(0) = 0.2\hat{x}_2(0) = -1.5\hat{x}_3(0) = 5]$, and the external disturbance $D(t) = 0.5\sin(2.3t)$.

Figures 2(a) and 2(b) show the chaotic behaviors of the master (51) and slave (52) systems, respectively.

Solution 1. We can solve the previous problem based on the following steps.

Step 1. Establish the NN models for master and slave systems via back propagation algorithm, respectively. First, the NN model to approximate the master chaotic system is constructed by 7–3, and the transfer functions of the hidden layer are chosen as follows:

$$T\left(v_{\varsigma}^{\sigma}(t)\right) = \left\{\frac{2}{\left[1+\exp\left(-v_{\varsigma}^{\sigma}(t)/0.5\right)\right]} - 1\right\}, \quad \text{for } \sigma = 1. \tag{53}$$

On the other hand, the transfer functions of the output layer are chosen as follows:

$$T\left(v_{\varsigma}^{\sigma}(t)\right) = v_{\varsigma}^{\sigma}(t), \quad \text{for } \sigma = 2. \tag{54}$$

After training, we can obtain the following connection weights (the indices in $W_{\varsigma\vartheta}^{\sigma}$ state that the weight of the σth layer in the NN model represents the connection to the ςth neuron from the ϑth source):

$$W^1 = \left[W_{\varsigma\vartheta}^1\right] = 10^{-3} \times \begin{bmatrix} -1.03122 & 5.94314 & -20.9809 & 0.13627 & 507.458 & 868.021 & 588.569 & 0.2062 & 651.633 \\ 8.37089 & 26.8407 & 21.6151 & 0.00088 & -239.108 & -740.187 & -377.569 & -0.01391 & 76.6848 \\ 501.958 & -3.80717 & 132.938 & 0.80211 & 135.643 & 137.647 & 57.0662 & 0.06242 & 992.269 \\ 1963.99 & -273.63 & 359.637 & 8.01727 & -848.291 & -61.2187 & -668.702 & 5.75107 & -843.648 \\ -2.69396 & -2.90578 & -10.7761 & 0.02579 & -892.099 & -976.195 & 203.963 & 0.06003 & -114.643 \\ -770.561 & 146.747 & -194.79 & 1.70179 & 61.5951 & -325.754 & -474.057 & 0.70796 & -786.694 \\ -495.801 & 7.53321 & -127.132 & -0.59639 & 558.334 & -675.635 & 308.158 & -0.00742 & 923.796 \end{bmatrix},$$

$$W^2 = \left[W_{\varsigma\vartheta}^2\right] = 10^2 \times \begin{bmatrix} 0.22075 & 0.37482 & -0.15363 & -0.00174 & 0.2211 & -0.00355 & -0.14954 \\ -0.12996 & -0.05835 & -0.00991 & 0.00027 & -0.84887 & -0.00075 & -0.40655 \\ 11.2915 & -7.58331 & 4.85989 & -0.05542 & -35.5864 & -0.22732 & 5.1942 \end{bmatrix}. \tag{55}$$

Then, the net inputs of the σth ($\sigma = 1, 2$) layer are as follows (the symbol v_{ς}^{σ} denotes the net input of the ςth neuron of the σth layer in the NN model, and the indices σ and ς shown in $h_{\varsigma\varphi}^{\sigma}$ ($\varphi = 1, 2$) indicate the same thing):

$$\begin{aligned} v_{\varsigma}^1(t) &= W_{\varsigma1}^1 x_1(t) + W_{\varsigma2}^1 x_2(t) + W_{\varsigma3}^1 x_3(t) + W_{\varsigma4}^1 x_1(t-0.13) \\ &\quad + W_{\varsigma5}^1 \cdot 0 + W_{\varsigma6}^1 \cdot 0 + W_{\varsigma7}^1 \cdot 0 + W_{\varsigma8}^1 x_2(t-0.015) \\ &\quad + W_{\varsigma9}^1 \cdot 0, \quad \varsigma = 1,2,3,4,5,6,7, \end{aligned} \tag{56a}$$

$$\begin{aligned} v_{\varsigma}^2(t) &= W_{\varsigma1}^2 T\left(v_1^1(t)\right) + W_{\varsigma2}^2 T\left(v_2^1(t)\right) \\ &\quad + W_{\varsigma3}^2 T\left(v_3^1(t)\right) + W_{\varsigma4}^2 T\left(v_4^1(t)\right) + W_{\varsigma5}^2 T\left(v_5^1(t)\right) \\ &\quad + W_{\varsigma6}^2 T\left(v_6^1(t)\right) + W_{\varsigma7}^2 T\left(v_7^1(t)\right), \quad \varsigma = 1,2,3, \end{aligned} \tag{56b}$$

$$\dot{X}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} T\left(v_1^2(t)\right) \\ T\left(v_2^2(t)\right) \\ T\left(v_3^2(t)\right) \end{bmatrix}. \tag{57}$$

Based on (8), the minimum and maximum of the derivative of each transfer function shown in (53) and (54) can be obtained as follows:

$$\begin{aligned} g_{\varsigma0}^1 &= 0, \qquad g_{\varsigma0}^2 = 1, \\ g_{\varsigma1}^1 &= g_{\varsigma1}^2 = 1, \quad \text{for } \varsigma = 1,2,\ldots,J^{\sigma}. \end{aligned} \tag{58}$$

In order to simplify the notation, we let $g_{\varsigma0}^1 = g_0^1$, $g_{\varsigma1}^1 = g_1^1$, $g_{\varsigma0}^2 = g_0^2$ and $g_{\varsigma1}^2 = g_1^2$. Then, according to the interpolation

method, we have

$$\begin{aligned}
\dot{x}_1(t) &= \sum_{d=0}^{1} h_{1d}^2(t)\, g_d^2 \sum_{\varsigma=1}^{7} W_{1\varsigma}^2 T\left(v_\varsigma^1(t)\right) \\
&= \sum_{d=0}^{1} h_{1d}^2(t)\, g_d^2 \sum_{\varsigma=1}^{7} W_{1\varsigma}^2 \left(h_{\varsigma 0}^1(t)\, g_0^1 + h_{\varsigma 1}^1(t)\, g_1^1\right) v_\varsigma^1(t) \\
&= \sum_{d=0}^{1} h_{1d}^2(t)\, g_d^2 \\
&\quad \times \sum_{s=0}^{1}\sum_{p=0}^{1}\sum_{r=0}^{1}\sum_{o=0}^{1}\sum_{c=0}^{1}\sum_{l=0}^{1}\sum_{k=0}^{1} h_{1s}^1(t)\, h_{2p}^1(t)\, h_{3r}^1(t) \\
&\qquad \times h_{4o}^1(t)\, h_{5c}^1(t)\, h_{6l}^1(t)\, h_{7k}^1(t) \\
&\quad \cdot \left(g_s^1 W_{\varsigma 1}^2 v_1^1(t) + g_p^1 W_{\varsigma 2}^2 v_2^1(t) + g_r^1 W_{\varsigma 3}^2 v_3^1(t)\right. \\
&\qquad + g_o^1 W_{\varsigma 4}^2 v_4^1(t) + g_c^1 W_{\varsigma 5}^2 v_5^1(t) + g_l^1 W_{\varsigma 6}^2 v_6^1(t) \\
&\qquad \left. + g_k^1 W_{\varsigma 7}^2 v_7^1(t)\right), \\
\dot{x}_2(t) &= \sum_{e=0}^{1} h_{2e}^2(t)\, g_e^2 \sum_{\varsigma=1}^{7} W_{2\varsigma}^2 T\left(v_\varsigma^1(t)\right) \\
&= \sum_{e=0}^{1} h_{2e}^2(t)\, g_e^2 \sum_{\varsigma=1}^{7} W_{1\varsigma}^2 \left(h_{\varsigma 0}^1(t)\, g_0^1 + h_{\varsigma 1}^1(t)\, g_1^1\right) v_\varsigma^1(t) \\
&= \sum_{e=0}^{1} h_{2e}^2(t)\, g_e^2 \\
&\quad \times \sum_{s=0}^{1}\sum_{p=0}^{1}\sum_{r=0}^{1}\sum_{o=0}^{1}\sum_{c=0}^{1}\sum_{l=0}^{1}\sum_{k=0}^{1} h_{1s}^1(t)\, h_{2p}^1(t)\, h_{3r}^1(t) \\
&\qquad \times h_{4o}^1(t)\, h_{5c}^1(t)\, h_{6l}^1(t)\, h_{7k}^1(t) \\
&\quad \cdot \left(g_s^1 W_{\varsigma 1}^2 v_1^1(t) + g_p^1 W_{\varsigma 2}^2 v_2^1(t) + g_r^1 W_{\varsigma 3}^2 v_3^1(t)\right. \\
&\qquad + g_o^1 W_{\varsigma 4}^2 v_4^1(t) + g_c^1 W_{\varsigma 5}^2 v_5^1(t) \\
&\qquad \left. + g_l^1 W_{\varsigma 6}^2 v_6^1(t) + g_k^1 W_{\varsigma 7}^2 v_7^1(t)\right), \\
\dot{x}_3(t) &= \sum_{f=0}^{1} h_{3f}^2(t)\, g_f^2 \sum_{\varsigma=1}^{7} W_{2\varsigma}^2 T\left(v_\varsigma^1(t)\right) \\
&= \sum_{f=0}^{1} h_{3f}^2(t)\, g_f^2 \sum_{\varsigma=1}^{7} W_{1\varsigma}^2 \left(h_{\varsigma 0}^1(t)\, g_0^1 + h_{\varsigma 1}^1(t)\, g_1^1\right) v_\varsigma^1(t) \\
&= \sum_{f=0}^{1} h_{3f}^2(t)\, g_f^2 \\
&\quad \times \sum_{s=0}^{1}\sum_{p=0}^{1}\sum_{r=0}^{1}\sum_{o=0}^{1}\sum_{c=0}^{1}\sum_{l=0}^{1}\sum_{k=0}^{1} h_{1s}^1(t)\, h_{2p}^1(t) \\
&\qquad \times h_{3r}^1(t)\, h_{4o}^1(t)\, h_{5c}^1(t)\, h_{6l}^1(t)\, h_{7k}^1(t) \\
&\quad \cdot \left(g_s^1 W_{\varsigma 1}^2 v_1^1(t) + g_p^1 W_{\varsigma 2}^2 v_2^1(t)\right. \\
&\qquad + g_r^1 W_{\varsigma 3}^2 v_3^1(t) + g_o^1 W_{\varsigma 4}^2 v_4^1(t) + g_c^1 W_{\varsigma 5}^2 v_5^1(t) \\
&\qquad \left. + g_l^1 W_{\varsigma 6}^2 v_6^1(t) + g_k^1 W_{\varsigma 7}^2 v_7^1(t)\right).
\end{aligned} \tag{59}$$

On the basis of (9), let

$$G^1 = \begin{bmatrix} g_s^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_p^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_r^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_o^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_c^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_l^1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_k^1 \end{bmatrix},$$
$$G^2 = \begin{bmatrix} g_d^2 & 0 & 0 \\ 0 & g_e^2 & 0 \\ 0 & 0 & g_f^2 \end{bmatrix}, \tag{60}$$

then, $E_{defsproclk} \equiv G^2 W^2 G^1 W^1 = [\Upsilon_{\mathfrak{R}\aleph}]_{3\times 9}$, $\mathfrak{R} = 1, 2, 3$; $\aleph = 1, 2 \ldots, 9$.

Plugging (56a) and (56b) into (59) leads to

$$\begin{aligned}
\dot{X}(t) &= \sum_{d=0}^{1}\sum_{e=0}^{1}\sum_{f=0}^{1}\sum_{s=0}^{1}\sum_{p=0}^{1}\sum_{r=0}^{1}\sum_{o=0}^{1}\sum_{c=0}^{1}\sum_{l=0}^{1}\sum_{k=0}^{1} h_{1d}^2(t) \\
&\quad \times h_{2e}^2(t)\, h_{3f}^2(t)\, h_{1s}^1(t)\, h_{2p}^1(t)\, h_{3r}^1(t)\, h_{4o}^1(t) \\
&\quad \times h_{5c}^1(t)\, h_{6l}^1(t)\, h_{7k}^1(t) \left\{A_{defsproclk} X(t)\right. \\
&\qquad + \overline{A}_{defsproclk1} X(t - 0.13) \\
&\qquad \left. + \overline{A}_{defsproclk2} X(t - 0.015)\right\},
\end{aligned} \tag{61}$$

where $X(t) = [x_1(t)\, x_2(t)\, x_3(t)]^T$, $X(t - 0.13) = [x_1(t - 0.13)\, 0\, 0]^T$, $X(t - 0.015) = [0\, x_2(t - 0.015)\, 0]^T$,

$$A_{defsproclk} = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} \\ \Upsilon_{21} & \Upsilon_{22} & \Upsilon_{23} \\ \Upsilon_{31} & \Upsilon_{32} & \Upsilon_{33} \end{bmatrix},$$
$$\overline{A}_{defsproclk1} = \begin{bmatrix} \Upsilon_{14} & \Upsilon_{15} & \Upsilon_{16} \\ \Upsilon_{24} & \Upsilon_{25} & \Upsilon_{26} \\ \Upsilon_{34} & \Upsilon_{35} & \Upsilon_{36} \end{bmatrix},$$
$$\overline{A}_{defsproclk2} = \begin{bmatrix} \Upsilon_{17} & \Upsilon_{18} & \Upsilon_{19} \\ \Upsilon_{27} & \Upsilon_{28} & \Upsilon_{29} \\ \Upsilon_{37} & \Upsilon_{38} & \Upsilon_{39} \end{bmatrix}. \tag{62}$$

Next, by renumbering the matrices shown in (61), the NN model of master system can be rewritten as the following LDI state-space representation:

$$\dot{X}(t) = \sum_{i=1}^{1024} h_i(t) \left\{A_i X(t) + \sum_{k=1}^{2} \overline{A}_{ik} X(t - \tau_k)\right\}, \tag{63}$$

where $\tau_1 = 0.13, \tau_2 = 0.015$,

$$A_1 = A_{0000000000}, \ldots, \quad A_{1023} = A_{1111111110},$$
$$A_{1024} = A_{1111111111},$$
$$\overline{A}_{11} = A_{0000000000\,1}, \ldots, \quad \overline{A}_{1023\,1} = A_{1111111110\,1},$$
$$\overline{A}_{1024\,1} = A_{1111111111\,1},$$
$$\overline{A}_{12} = A_{0000000000\,2}, \ldots, \quad \overline{A}_{1023\,2} = A_{1111111110\,2},$$
$$\overline{A}_{1024\,2} = A_{1111111111\,2}. \tag{64}$$

Similarly, the connection weights of the NN model for the slave system are obtained as follows:

$$\widehat{W}^1 = \left[\widehat{W}^1_{\varsigma\vartheta}\right] = 10^{-3} \times \begin{bmatrix} 152.414 & -108.845 & 5.89316 & -2.46010 & 141.365 & -68.6751 & 379.275 & 0.92354 & -435.589 \\ 25.9179 & -1.25408 & 0.23133 & -0.03779 & 143.659 & -441.921 & -736.338 & 0.04626 & 951.915 \\ 16.4571 & -32.4571 & -0.07041 & -0.03095 & -427.963 & 350.75 & -752.998 & 0.05790 & -927.148 \\ 185.863 & -168.015 & 63.4102 & 0.77781 & 398.267 & 807.329 & -618.194 & 2.26166 & -347.51 \\ 30.1031 & -6.65419 & 20.3791 & -0.18987 & 592.515 & 817.051 & -708.535 & 0.22644 & 946.027 \\ -29.6884 & 2.93641 & -19.0384 & 0.14757 & -116.821 & 494.393 & 170.087 & -0.19531 & -269.934 \\ -30.3835 & -2.8972 & 14.6662 & -0.05355 & -107.568 & -478.976 & -853.276 & 0.04245 & -381.7 \end{bmatrix},$$

$$\widehat{W}^2 = \left[\widehat{W}^2_{\varsigma\vartheta}\right] = 10^{-2} \times \begin{bmatrix} -0.02461 & -2.39353 & -2.95221 & 0.00325 & -0.54634 & -0.78523 & -0.23624 \\ 0.36735 & -35.2973 & 3.22363 & -0.06269 & -32.8677 & -48.5154 & -16.6456 \\ -2.20513 & 8.93374 & -5.80935 & -1.25235 & 164.85 & 164.933 & -10.6894 \end{bmatrix}. \tag{65}$$

Step 2. The procedures of constructing the NN model for the slave system are similar to those for that of the master system, and then we have the NN model of the slave system as follows:

$$\dot{\widehat{X}}(t) = \sum_{j=1}^{1024} \widehat{h}_j(t) \left\{ \widehat{A}_j \widehat{X}(t) + \sum_{k=1}^{2} \widehat{\overline{A}}_{jk} \widehat{X}(t-\tau_k) \right\} + BU(t), \tag{66}$$

where $\widehat{X}(t) = [\widehat{x}_1(t) \quad \widehat{x}_2(t) \quad \widehat{x}_3(t)]^T$, $\widehat{X}(t-0.13) = [0 \quad \widehat{x}_2(t-0.13) \quad 0]^T$, $\widehat{X}(t-0.015) = [0 \quad 0 \quad \widehat{x}_3(t-0.015)]^T$ and B is identity matrix. The responses of $\dot{X}(t)$ and $\dot{\widehat{X}}(t)$ for original systems and NN models are shown in Figures 3(a) and 3(b).

Step 3. In order to synchronize the master and slave systems, a fuzzy controller is synthesized as follows:

$$\begin{aligned} &\text{Control Rule 1: IF } e_1(t) \text{ is } M_1, \text{ THEN } U(t) = -K_1 E(t), \\ &\text{Control Rule 2: IF } e_1(t) \text{ is } M_2, \text{ THEN } U(t) = -K_2 E(t), \end{aligned} \tag{67}$$

where M_1 and M_2 are the membership functions for each e_1 (see Figure 4) as follows:

$$M_1(e_1(t)) = \frac{1}{2}\left(1 + \frac{e_1(t)}{q}\right), \tag{68a}$$

$$M_2(e_1(t)) = \frac{1}{2}\left(1 - \frac{e_1(t)}{q}\right). \tag{68b}$$

Based on (19), we have the overall fuzzy controller

$$U(t) = -\frac{\sum_{l=1}^{2} w_l(t) K_l E(t)}{\sum_{l=1}^{2} w_l(t)} = -\sum_{l=1}^{2} \overline{h}_l(t) K_l E(t), \tag{69}$$

with $w_l(t) \equiv M_l(e_1(t))$, $\overline{h}_l(t) \equiv w_l(t) / \sum_{l=1}^{2} w_l(t)$.

Based on (20), the dynamics of the error system are obtained as follows:

$$\dot{E}(t) = \sum_{i=1}^{1024} \sum_{k=1}^{2} \sum_{l=1}^{2} h_i(t) \overline{h}_l(t) \times \left\{ G_{il} E(t) + \overline{A}_{ik} E(t-\tau_k) \right\} + D(t) + \Phi(t), \tag{70}$$

where $G_{il} \equiv A_i - BK_l$, $\widehat{\Psi} \equiv f(\widehat{X}(t)) + \sum_{k=1}^{2} H_k(\widehat{X}(t-\tau_k)) + U(t)$, with $U(t) = -\sum_{l=1}^{2} h_l(t) K_l E(t)$, $\Psi \equiv \widehat{f}(X(t)) + \sum_{k=1}^{2} \widehat{H}_k(X(t-\tau_k))$, $\Phi(t) \equiv \widehat{\Psi} - \Psi - \{\sum_{i=1}^{1024} \sum_{k=1}^{2} \sum_{l=1}^{2} h_i(t) \overline{h}_l(t) [G_{il} E(t) + \overline{A}_{ik} E(t-\tau_k)\}$.

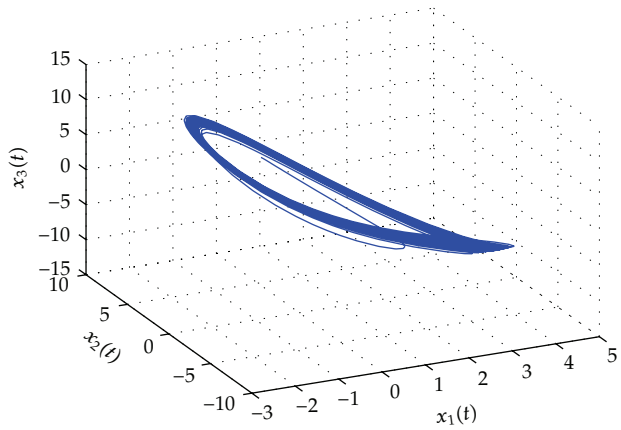
Step 4. According to (55) and (61)–(70), the LMIs in (47a) and (47b) can be solved via the Matlab LMI toolbox. In accordance with Remark 1, the specified structured bounding matrix R and ε_{il} are set as

$$R = \begin{bmatrix} 18000 & 0 & 0 \\ 0 & 18000 & 0 \\ 0 & 0 & 18000 \end{bmatrix}, \qquad \varepsilon_{il} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{71}$$

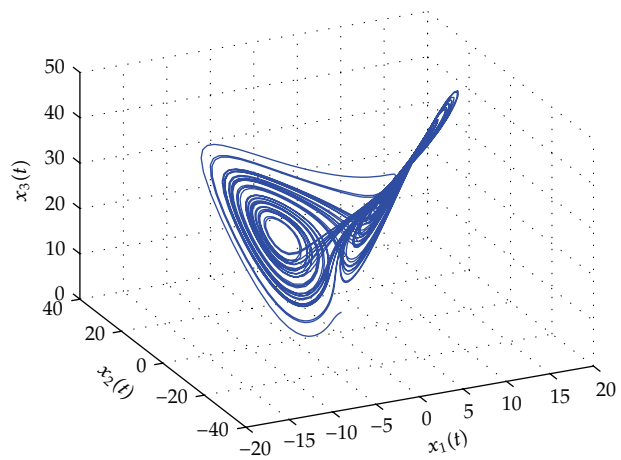
Based on Corollary 8, the positive constant ξ is minimized by the mincx function of Matlab LMI toolbox $\xi_{\min} = 0.0000125$, and then we have the minimum disturbance attenuation level $\rho_{\min} = 0.006$.

Step 5. The common solutions Q, F_1, F_2, $\overline{\psi}_1$, and $\overline{\psi}_2$ of the stability conditions (31a) and (31b) can be obtained with the best value *tmin* of LMI Solver (Matlab) as -2.202477×10^{-7} as follows:

$$Q = 10^{-6} \times \begin{bmatrix} 0.7456 & 0 & 0 \\ 0 & 0.7454 & -0.0001 \\ 0 & -0.0001 & 0.7452 \end{bmatrix}, \tag{72}$$

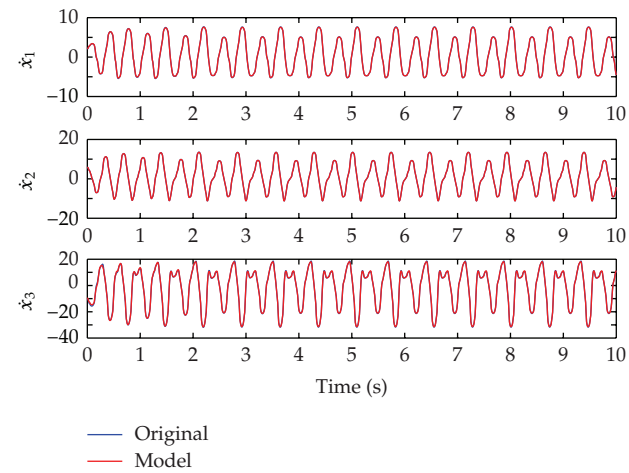


(a)

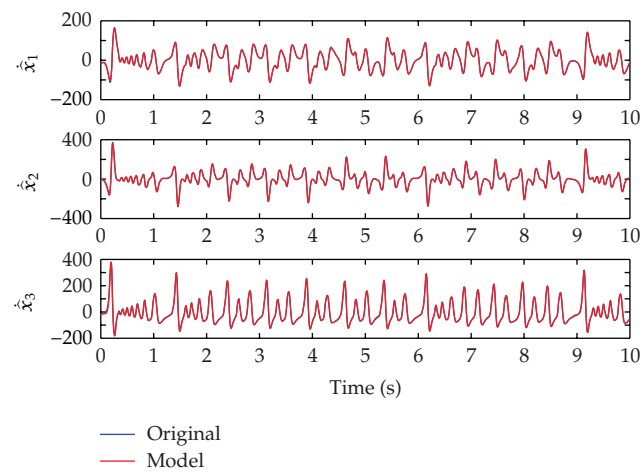


(b)

FIGURE 2: (a) Chaotic behavior of the master system (51). (b) Chaotic behavior of the slave system (52) without control.



(a)



(b)

FIGURE 3: (a) The responses of $\dot{X}(t)$ for original system and NN model. (b) The responses of $\dot{\widehat{X}}(t)$ for original system and NN model.

$$F_1 = \begin{bmatrix} 0.0071 & 0 & 0 \\ 0 & 0.0071 & 0 \\ 0 & 0 & 0.0071 \end{bmatrix}, \tag{73a}$$

$$F_2 = \begin{bmatrix} 0.0071 & 0 & 0 \\ 0 & 0.0071 & 0 \\ 0 & 0 & 0.0071 \end{bmatrix}. \tag{73b}$$

In addition, the resulting controller gains are

$$K_1 = 10^3 \times \begin{bmatrix} 9.4701 & -0.0004 & -0.0055 \\ 0.0004 & 9.4701 & 0.0014 \\ 0.0055 & -0.0014 & 9.4701 \end{bmatrix},$$

$$K_2 = 10^3 \times \begin{bmatrix} 9.4701 & 0.0001 & 0.0013 \\ -0.0001 & 9.4701 & -0.0053 \\ -0.0013 & 0.0053 & 9.4701 \end{bmatrix}, \tag{74}$$

$$\overline{\psi}_1 = \overline{\psi}_2 = \begin{bmatrix} 1.0344 & 0 & -0.0032 \\ 0 & 1.0345 & 0.0002 \\ -0.0032 & 0.0002 & 1.0345 \end{bmatrix}.$$

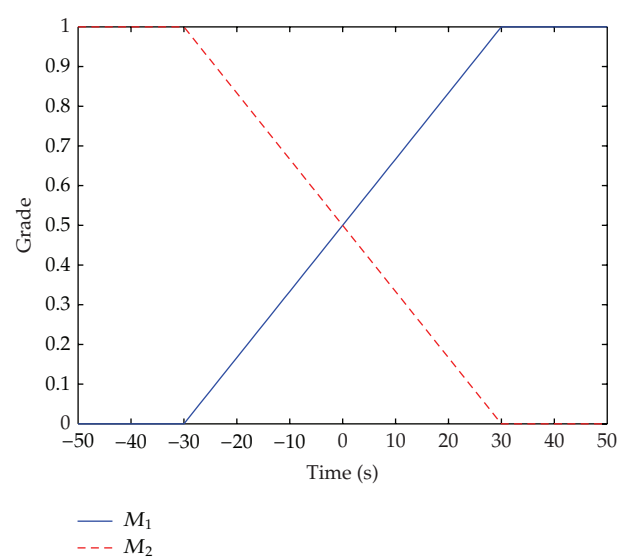


FIGURE 4: Membership functions of the fuzzy controller.

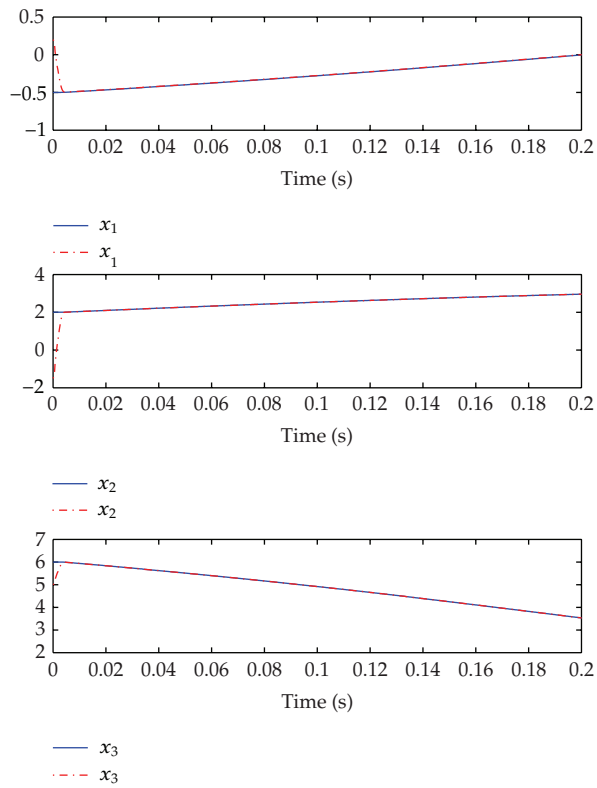


FIGURE 5: State responses of both master and slave systems.

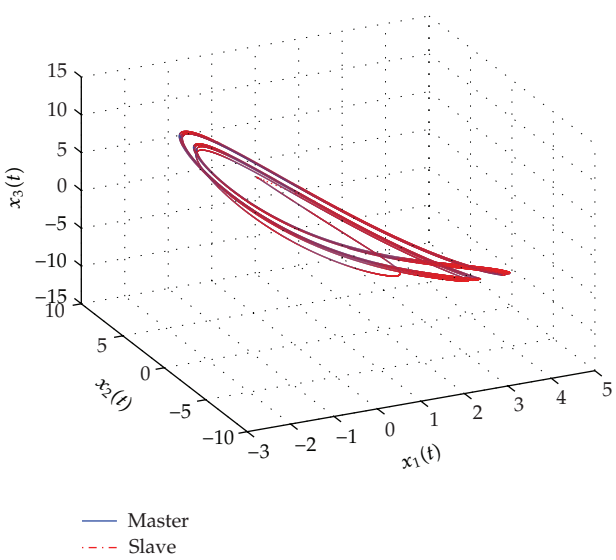


FIGURE 6: The chaotic behaviors of the master and slave systems.

Figure 5 displays the state responses of both master and slave systems. The chaotic behaviors of the master and slave systems are shown in Figure 6. Besides, Figure 7 illustrates the synchronization errors (e_1, e_2, and e_3) which converge to zero. Moreover, the assumption of $\|\Phi(t)\| \leq \|\sum_{i=1}^{1024}\sum_{l=1}^{2} h_i(t)h_l(t)\Theta R_{il}E(t)\|$ is satisfied from the illustration shown in Figure 8.

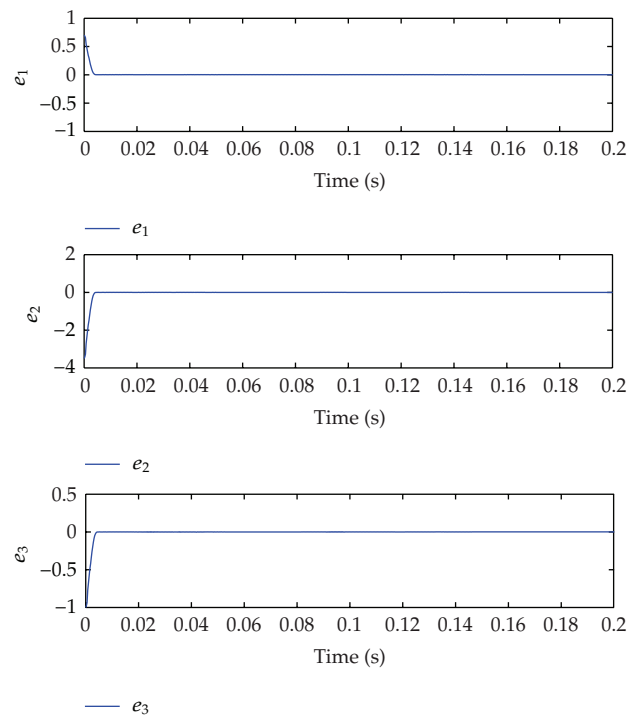


FIGURE 7: State responses of the error system.

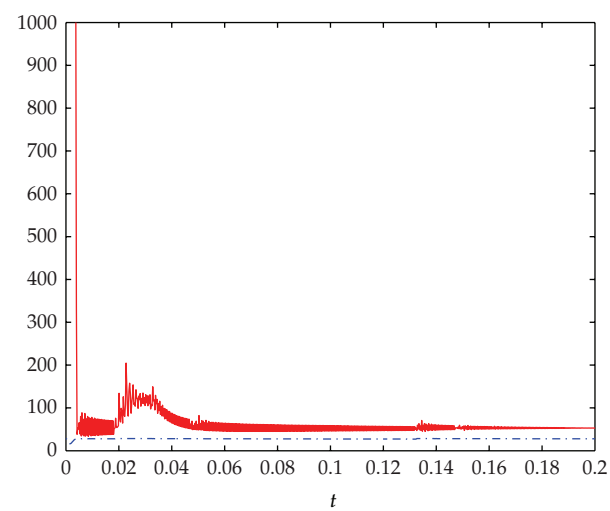


FIGURE 8: Plots of $\|\Phi(t)\|$ (blue line) and $\|\sum_{i=1}^{1024}\sum_{l=1}^{2} h_i(t)h_l(t)\Theta R_{il}E(t)\|$ (red line).

6. Conclusion

This study proposes a novel approach not only to realize the exponential synchronization of nonidentical multiple time-delay chaotic (MTDC) systems but also to achieve the optimal H^{∞} performance at the same time. First, a neural-network (NN) model is employed to approximate the MTDC system. Then, a linear differential inclusion (LDI) state-space representation is established for the dynamics of the NN model. Next, in terms of Lyapunov's direct method, a delay-dependent stability criterion is derived to ensure that the

slave system can exponentially synchronize with the master system. Subsequently, the stability condition of this criterion is reformulated into a linear matrix inequality (LMI). On the basis of the Lyapunov stability theory and LMI approach, a fuzzy controller is synthesized to realize the exponential H^∞ synchronization of the chaotic master-slave systems and reduce the H^∞ norm from disturbance to synchronization error at the lowest level. Finally, the simulation results demonstrate that the exponential H^∞ synchronization of two different MTDC systems can be achieved by the designed fuzzy controller. algorithm, respectively. First, the NN model to approximate the master chaotic

References

[1] K. R. Lee, J. H. Kim, E. T. Jeung, and H. B. Park, "Output feedback robust H∞ control of uncertain fuzzy dynamic systems with time-varying delay," *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 6, pp. 657–664, 2000.

[2] M. C. Mackey and L. Glass, "Oscillation and chaos in physiological control systems," *Science*, vol. 197, no. 4300, pp. 287–289, 1977.

[3] Z.-R. Tsai, Y.-Z. Chang, J.-D. Hwang, and J. Lee, "Robust fuzzy stabilization of dithered chaotic systems using island-based random optimization algorithm," *Information Sciences*, vol. 178, no. 4, pp. 1171–1188, 2008.

[4] C. Hu, H. Jiang, and Z. Teng, "General impulsive control of chaotic systems based on a TS fuzzy model," *Fuzzy Sets and Systems*, vol. 174, pp. 66–82, 2011.

[5] G. Poddar, K. Chakrabarty, and S. Banerjee, "Control of chaos in DC-DC converters," *IEEE Transactions on Circuits and Systems I*, vol. 45, no. 6, pp. 672–676, 1998.

[6] S. L. Lin and P. C. Tung, "A new method for chaos control in communication systems," *Chaos, Solitons and Fractals*, vol. 42, no. 5, pp. 3234–3241, 2009.

[7] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Physical Review Letters*, vol. 64, no. 8, pp. 821–824, 1990.

[8] H.-T. Yau and J.-J. Yan, "Chaos synchronization of different chaotic systems subjected to input nonlinearity," *Applied Mathematics and Computation*, vol. 197, no. 2, pp. 775–788, 2008.

[9] H. K. Lam, W.-K. Ling, H. H.-C. Iu, and S. S. H. Ling, "Synchronization of chaotic systems using time-delayed fuzzy state-feedback controller," *IEEE Transactions on Circuits and Systems*, vol. 55, no. 3, pp. 893–903, 2008.

[10] M. Liu, "Optimal exponential synchronization of general chaotic delayed neural networks: an LMI approach," *Neural Networks*, vol. 22, no. 7, pp. 949–957, 2009.

[11] H.-H. Chen, G.-J. Sheu, Y.-L. Lin, and C.-S. Chen, "Chaos synchronization between two different chaotic systems via nonlinear feedback control," *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal. Series A*, vol. 70, no. 12, pp. 4393–4401, 2009.

[12] M. Pourmahmood, S. Khanmohammadi, and G. Alizadeh, "Synchronization of two different uncertain chaotic systems with unknown parameters using a robust adaptive sliding mode controller," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 7, pp. 2853–2868, 2011.

[13] W. He, F. Qian, J. Cao, and Q.-L. Han, "Impulsive synchronization of two nonidentical chaotic systems with time-varying delay," *Physics Letters A*, vol. 375, no. 3, pp. 498–504, 2011.

[14] S. Li, W. Xu, and R. Li, "Synchronization of two different chaotic systems with unknown parameters," *Physics Letters A*, vol. 361, no. 1-2, pp. 98–102, 2007.

[15] S. Y. Li and Z. M. Ge, "Fuzzy modeling and synchronization of two totally different chaotic systems via novel fuzzy model," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 41, no. 4, pp. 1015–1026, 2011.

[16] Y.-Y. Hou, T.-L. Liao, and J.-J. Yan, "H^∞ synchronization of chaotic systems using output feedback control design," *Physica A. Statistical Mechanics and its Applications*, vol. 379, no. 1, pp. 81–89, 2007.

[17] D. Qi, M. Liu, M. Qiu, and S. Zhang, "Exponential H∞ synchronization of general discrete-time chaotic neural networks with or without time delays," *IEEE Transactions on Neural Networks*, vol. 21, no. 8, pp. 1358–1365, 2010.

[18] C. M. Lin, Y. F. Peng, and M. H. Lin, "CMAC-based adaptive backstepping synchronization of uncertain chaotic systems," *Chaos, Solitons and Fractals*, vol. 42, no. 2, pp. 981–988, 2009.

[19] C. K. Ahn, S.-T. Jung, S.-K. Kang, and S.-C. Joo, "Adaptive H^∞ synchronization for uncertain chaotic systems with external disturbance," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 8, pp. 2168–2177, 2010.

[20] C. F. Chuang, W. J. Wang, and Y. J. Chen, "H∞ synchronization of fuzzy model based chen chaotic systems," in *2010 IEEE International Conference on Control Applications, CCA 2010*, pp. 1199–1204, jpn, September 2010.

[21] H. R. Karimi and H. Gao, "New delay-dependent exponential H∞ synchronization for uncertain neural networks with mixed time delays," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 40, no. 1, pp. 173–185, 2010.

[22] H. R. Karimi and P. Maass, "Delay-range-dependent exponential H^∞ synchronization of a class of delayed neural networks," *Chaos, Solitons and Fractals*, vol. 41, no. 3, pp. 1125–1135, 2009.

[23] B.-S. Chen, C.-H. Chiang, and S. K. Nguang, "Robust H^∞ synchronization design of nonlinear coupled network via fuzzy interpolation method," *IEEE Transactions on Circuits and Systems. I. Regular Papers*, vol. 58, no. 2, pp. 349–362, 2011.

[24] S.-J. Wu, H.-H. Chiang, H.-T. Lin, and T.-T. Lee, "Neural-network-based optimal fuzzy controller design for nonlinear systems," *Fuzzy Sets and Systems*, vol. 154, no. 2, pp. 182–207, 2005.

[25] J. T. Tsai, J. H. Chou, and T. K. Liu, "Tuning the structure and parameters of a neural network by using hybrid Taguchi-genetic algorithm," *IEEE Transactions on Neural Networks*, vol. 17, no. 1, pp. 69–80, 2006.

[26] A. Savran, "Multifeedback-layer neural network," *IEEE Transactions on Neural Networks*, vol. 18, no. 2, pp. 373–384, 2007.

[27] A. Alessandri, C. Cervellera, and M. Sanguineti, "Design of asymptotic estimators: an approach based on neural networks and nonlinear programming," *IEEE Transactions on Neural Networks*, vol. 18, no. 1, pp. 86–96, 2007.

[28] F. H. Hsiao, S. D. Xu, C. Y. Lin, and Z. R. Tsai, "Robustness design of fuzzy control for nonlinear multiple time-delay large-scale systems via neural-network-based approach," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 38, no. 1, pp. 244–251, 2008.

[29] P. M. Patre, S. Bhasin, Z. D. Wilcox, and W. E. Dixon, "Composite adaptation for neural network-based controllers," *Institute of Electrical and Electronics Engineers. Transactions on Automatic Control*, vol. 55, no. 4, pp. 944–950, 2010.

[30] S. Limanond, J. Si, and Y. L. Tseng, "Production data based optimal etch time control design for a reactive ion etching process," *IEEE Transactions on Semiconductor Manufacturing*, vol. 12, no. 1, pp. 139–147, 1999.

[31] Z. Su and K. Khorasani, "A neural-network-based controller for a single-link flexible manipulator using the inverse dynamics approach," *IEEE Transactions on Industrial Electronics*, vol. 48, no. 6, pp. 1074–1086, 2001.

[32] R. Enns and J. Si, "Helicopter trimming and tracking control using direct neural dynamic programming," *IEEE Transactions on Neural Networks*, vol. 14, no. 4, pp. 929–939, 2003.

[33] D. H. Shin and Y. Kim, "Reconfigurable flight control system design using adaptive neural networks," *IEEE Transactions on Control Systems Technology*, vol. 12, no. 1, pp. 87–100, 2004.

[34] F. J. Lin, H. J. Shieh, P. H. Shieh, and P. H. Shen, "An adaptive recurrent-neural-network motion controller for X-Y table in CNC machine," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 36, no. 2, pp. 286–299, 2006.

[35] H. C. Liaw, B. Shirinzadeh, and J. Smith, "Robust neural network motion tracking control of piezoelectric actuation systems for micro/nanomanipulation," *IEEE Transactions on Neural Networks*, vol. 20, no. 2, pp. 356–367, 2009.

[36] S. Limanond and J. Si, "Neural-network-based control design: an LMI approach," *IEEE Transactions on Neural Networks*, vol. 9, no. 6, pp. 1422–1429, 1998.

[37] J. J. Wang, C. T. Lin, S. H. Liu, and Z. C. Wen, "Model-based synthetic fuzzy logic controller for indirect blood pressure measurement," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 32, no. 3, pp. 306–315, 2002.

[38] R. J. Wai, "Hybrid fuzzy neural-network control for nonlinear motor-toggle servomechanism," *IEEE Transactions on Control Systems Technology*, vol. 10, no. 4, pp. 519–532, 2002.

[39] C. L. Hwang, L. J. Chang, and Y. S. Yu, "Network-based fuzzy decentralized sliding-mode control for car-like mobile robots," *IEEE Transactions on Industrial Electronics*, vol. 54, no. 1, pp. 574–585, 2007.

[40] A. V. Sant, "PM synchronous motor speed control using hybrid fuzzy-PI with novel switching functions," *IEEE Transactions on Magnetics*, vol. 45, no. 10, pp. 4672–4675, 2009.

[41] D. H. Spatti, I. N. Da Silva, W. F. Usida, and R. A. Flauzino, "Fuzzy control system for voltage regulation in power transformers," *IEEE Latin America Transactions*, vol. 8, no. 1, pp. 51–57, 2010.

[42] W. J. Wang and H. R. Lin, "Fuzzy control design for the trajectory tracking on uncertain nonlinear systems," *IEEE Transactions on Fuzzy Systems*, vol. 7, no. 1, pp. 53–62, 1999.

[43] S. J. Wu and C. T. Lin, "Discrete-time optimal fuzzy controller design: global concept approach," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 1, pp. 21–38, 2002.

[44] C. H. Sun and W. J. Wang, "An improved stability criterion for T-S fuzzy discrete systems via vertex expression," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 36, no. 3, pp. 672–678, 2006.

[45] H. K. Lam and L. D. Seneviratne, "Stability analysis of interval type-2 fuzzy-model-based control systems," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 38, no. 3, pp. 617–628, 2008.

[46] K. Kiriakidis, "Fuzzy model-based control of complex plants," *IEEE Transactions on Fuzzy Systems*, vol. 6, no. 4, pp. 517–529, 1998.

[47] B. S. Chen, C. S. Tseng, and H. J. Uang, "Robustness design of nonlinear dynamic systems via fuzzy linear control," *IEEE Transactions on Fuzzy Systems*, vol. 7, no. 5, pp. 571–585, 1999.

[48] B. S. Chen, C. S. Tseng, and H. J. Uang, "Mixed H2/H∞ fuzzy output feedback control design for nonlinear dynamic systems: an LMI approach," *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 3, pp. 249–265, 2000.

[49] Y. Y. Cao and P. M. Frank, "Robust H∞ disturbance attenuation for a class of uncertain discrete-time fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 4, pp. 406–415, 2000.

[50] Y. Y. Cao and Z. Lin, "Robust stability analysis and fuzzy-scheduling control for nonlinear systems subject to actuator saturation," *IEEE Transactions on Fuzzy Systems*, vol. 11, no. 1, pp. 57–67, 2003.

[51] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa, USA, 1994.

[52] Y. J. Sun, "Exponential synchronization between two classes of chaotic systems," *Chaos, Solitons and Fractals*, vol. 39, no. 5, pp. 2363–2368, 2009.

[53] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox User's Guide*, The MathWorks, Inc., Natick, Mass, USA, 1995.