Research Article

Strong Convergence Properties and Strong Stability for Weighted Sums of AANA Random Variables

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The Khintchine-Kolmogorov-type convergence theorem and three-series theorem for AANA random variables are established. By using these convergence theorems, we obtain convergence results for AANA sequences, which extend the corresponding ones for independent sequences and NA sequences. In addition, we study the strong stability for weighted sums of AANA random variables and obtain some new results, which extend some earlier ones for NA random variables.

1. Introduction

Firstly, let us recall some definitions.

Definition 1 (cf. Wu [1]). A sequence \( \{X_n, n \geq 1\} \) of random variables is said to be stochastically dominated by a random variable \( X \) if there exists a constant \( C \) such that

\[
P(\{|X_n| > x\}) \leq CP(\{|X| > x\}) \tag{1}
\]

for all \( x \geq 0 \) and \( n \geq 1 \).

Definition 2 (cf. Chow and Teicher [2]). A sequence \( \{Y_n, n \geq 1\} \) of random variables is said to be strongly stable if there exist two constant sequences \( \{b_n, n \geq 1\} \) and \( \{d_n, n \geq 1\} \) with \( 0 < b_n \uparrow \infty \) such that

\[
b_n^{-1}Y_n - d_n \rightarrow 0 \quad \text{a.s.} \tag{2}
\]

Definition 3 (cf. Wu [1]). A function \( l(x) > 0, x > 0 \), is said to be slowly varying if

\[
\lim_{x \rightarrow \infty} \frac{l(\lambda x)}{l(x)} = 1 \tag{3}
\]

for each \( \lambda > 0 \).

Definition 4 (cf. Wu [1]). A real-valued function \( l(x) \), positive and measurable on \( (0, \infty) \), is said to be quasimonotonically increasing function if there exist \( x_0 > 0 \) and constant \( C > 0 \) with \( \forall t \geq x \geq x_0 \) such that

\[
l(t) \geq Ct \tag{4}
\]

Definition 5 (cf. Joag-Dev and Proschan [3]). A finite collection of random variables \( X_1, X_2, \ldots, X_n \) is said to be negatively associated (NA, in short) if for every pair of disjoint subsets \( A_1, A_2 \) of \( \{1, 2, \ldots, n\} \),

\[
\text{Cov}\{f(X_i : i \in A_1), g(X_j : j \in A_2)\} \leq 0, \tag{5}
\]

whenever \( f \) and \( g \) are coordinatewise nondecreasing such that this covariance exists. An infinite sequence \( \{X_n, n \geq 1\} \) is NA if every finite subcollection is NA.

Definition 6 (cf. Chandra and Ghosal [4]). A sequence \( \{X_n, n \geq 1\} \) of random variables is called asymptotically almost negatively associated (AANA) if there exists a non-negative sequence \( q(n) \rightarrow 0 \) as \( n \rightarrow \infty \) such that

\[
\text{Cov}\{f(X_n), g(X_{n+1}, X_{n+2}, \ldots, X_{n+k})\} \\
\leq q(n) \left[\text{Var}(f(X_n)) \text{Var}(g(X_{n+1}, X_{n+2}, \ldots, X_{n+k}))\right]^{1/2} \tag{6}
\]

(5)
for all $n, k \geq 1$ and for all coordinatewise nondecreasing continuous functions $f$ and $g$ whenever the variances exist.

Obviously, the family of AANA sequences contains NA (in particular, independent) sequences (taking $q(n) = 0$, $n \geq 1$) and some more sequences of random variables which do not much deviates from being NA. An example of an AANA sequence which is not NA was introduced by Chandra and Ghosal [5].

Since the concept of AANA random variables was introduced by Chandra and Ghosal [4], many applications have been found. For example, Chandra and Ghosal [4] derived the Kolmogorov-type inequality and the strong law of large numbers of Marcinkiewicz-Zygmund, Chandra and Ghosal [5] obtained the almost sure convergence of weighted averages, Ko et al. [6] studied the Hájek-Rényi type inequality, and Wang et al. [7] established the law of the iterated logarithm for product sums. Yuan and An [8] established Rosenthal-type inequalities for maximum partial sums of AANA sequences. Wang et al. [9] studied some convergence properties for AANA sequence. Wang et al. [10] generalized and improved the results of Ko et al. [6] and studied the large deviation and Marcinkiewicz-type strong law of large numbers for AANA sequences. Yang et al. [11] investigated the complete convergence of moving average process for AANA sequence. Hu et al. [12] and She and Wu [13, 14] studied strong convergence property for weighted sums of AANA sequence. Wang et al. [15,16] and Shen et al. [17] obtained some results on complete convergence for AANA sequence, and so forth.

In this paper, we mainly study convergence results for AANA random variables, and the strong stability for weighted sums of AANA random variables, which extend the corresponding ones for independent sequences and NA sequences without necessarily adding extra conditions. The techniques used in the paper are the truncated method, the Khintchine-Kolmogorov-type convergence theorem and three series theorem for AANA random variables.

Throughout this paper, let $I(A)$ be the indicator function of the set $A$, and $X_n^{(c)} = -cI(X_n < -c) + X_nI(|X_n| \leq c) + cI(X_n > c)$ for some $c > 0$. $a_n = O(b_n)$ denotes that there exists a positive constant $C$ such that $|a_n/b_n| \leq C$. The symbol $C$ represents a positive constant which may be different in various places. The main results of this paper depend on the following lemmas.

**Lemma 7** (cf. Yuan and An [8]). Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$, and let $f_1, f_2, \ldots$ be all nondecreasing (or nonincreasing) continuous functions; then $\{f_n(X_n), n \geq 1\}$ is still a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$.

**Lemma 8** (cf. Wang et al. [9]). Let $1 < p \leq 2$ and $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$. Assume that $EX_n = 0$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} q^2(n) < \infty$; then there exists a positive constant $C_p$ depending only on $p$ such that

$$E \left( \max_{1 \leq j \leq n} \left| \frac{\sum_{i=1}^{j} X_i}{\sum_{i=1}^{j} |X_i|^p} \right| \right) \leq C_p \sum_{i=1}^{n} E|X_i|^p$$

for all $n \geq 1$, where $C_p = 2p^2(2^{2-p} p + 6p^p(\sum_{n=1}^{\infty} q^2(n))^{(p-1)})$.

By Lemmas 7 and 8, we can get the following Khintchine-Kolmogorov-type convergence theorem and three series theorem for AANA sequences, which can be applied to prove the main results of the paper. The proofs are standard, so we omit them.

**Corollary 9** (Khintchine-Kolmogorov-type convergence theorem). Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$ and $\sum_{n=1}^{\infty} q^2(n) < \infty$. If

$$\sum_{n=1}^{\infty} \text{Var} X_n < \infty,$$

then $\sum_{n=1}^{\infty} (X_n - EX_n)$ converges almost surely.

**Corollary 10** (three-series theorem for AANA random variables). Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$ and $\sum_{n=1}^{\infty} q^2(n) < \infty$. Assume that for some $c > 0$,

$$\sum_{n=1}^{\infty} P(|X_n| > c) < \infty, \quad \sum_{n=1}^{\infty} EX_n^{(c)}$$

converges,

$$\sum_{n=1}^{\infty} \text{Var} X_n^{(c)} < \infty.$$

Then, $\sum_{n=1}^{\infty} X_n$ converges almost surely.

**Remark 11.** Since NA implies AANA, Corollaries 9 and 10 extend corresponding results for NA random variables (see Matula [18]) to AANA random variables without adding any extra condition.

**Lemma 12** (cf. Wu [19] or Shen [20]). Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable $X$. For any $\alpha > 0$ and $b > 0$, the following two statements hold:

$$E|X_n|^{\alpha} I(|X_n| \leq b) \leq C_1 E|X|^{\alpha} I(|X| \leq b) + b^\alpha P(|X| > b),$$

$$E|X_n|^{\alpha} I(|X_n| > b) \leq C_2 E|X|^{\alpha} I(|X| > b),$$

where $C_1$ and $C_2$ are positive constants.

**Lemma 13** (cf. Wu [1]). Let $h(x) > 0$ be a slowly varying function; then for any $\delta > 0$, $x^\delta h(x)$ is a quasimonotonically increasing function and $x^{-\delta} h(x)$ is a quasimonotonically decreasing function.
2. Strong Convergence Properties of Weighted Sums for AANA Sequence

Theorem 14. Let \( \{X_n, n \geq 1\} \) be a sequence of AANA random variables with \( \sum_{n=1}^{\infty} q^r(n) < \infty \). Assume that \( \{g_n(x), n \geq 1\} \) is a sequence of even functions defined on \( R \). For each \( n \geq 1 \), \( g_n(x) \) is a positive and nondecreasing function in \( (0, \infty) \) and satisfies one of the following conditions:

(i) for some \( 0 < r \leq 1 \), \( x' / g_n(x) \) is a nondecreasing function in \( (0, \infty) \);

(ii) for some \( 1 < r \leq 2 \), \( x / g_n(x) \) and \( g_n(x) / x' \) are non-increasing functions in \( (0, \infty) \);

Furthermore, assume that \( EX_n = 0 \) for each \( n \geq 1 \).

For any positive number sequence \( \{a_n, n \geq 1\} \) with \( a_n \uparrow \infty \) such that

\[
\sum_{n=1}^{\infty} \frac{Eg_n(X_n)}{g_n(a_n)} < \infty,
\]

then \( \sum_{n=1}^{\infty} X_n/a_n \) converges a.s., and

\[
a_n^{-1} \sum_{i=1}^{n} X_i \rightarrow 0 \quad a.s., \quad \text{as } n \rightarrow \infty.
\]

Proof. For each \( n \geq 1 \), denote

\[
X^{(a_n)}_n = -a_n I(X_n < -a_n) + X_n I(|X_n| \leq a_n) + a_n I(X_n > a_n).
\]

By Lemma 7, we can see that, for fixed \( n \geq 1 \), \( \{X^{(a_n)}_n\} \) is still a sequence of AANA random variables. So by Corollary 10 in order to prove (II), we need only to prove the convergence of three series of (8), where \( c = 1 \).

Firstly, we prove that \( \sum_{n=1}^{\infty} P(|X_n/a_n| > 1) < \infty \) under condition (i) or (ii).

For each \( n \geq 1 \), if \( g_n(x) \) satisfies condition (i), noting that \( \{g_n(x), n \geq 1\} \) is a sequence of positive and non-decreasing even function in \( (0, +\infty) \). Combining Markov's inequality with (10), it follows that

\[
\sum_{n=1}^{\infty} P \left( \frac{X_n}{a_n} > 1 \right) \leq \sum_{n=1}^{\infty} P(g_n(X_n) > g_n(a_n)) \leq \sum_{n=1}^{\infty} \frac{Eg_n(X_n)}{g_n(a_n)} < \infty.
\]

If \( g_n(x) \) satisfies condition (ii), it is easy to prove that (13) also holds.

Secondly, we will show \( \sum_{n=1}^{\infty} \text{Var}(X^{(a_n)}_n/a_n) < \infty \).

If \( g_n(x) \) satisfies (i), when \( |x| \leq a_n \), we have \( \left( |x|'/g_n(x) \right) \leq \left( a_n'/g_n(a_n) \right) \), which implies that

\[
\frac{|x|^r}{a_n^r} \leq \frac{g_n(x)}{g_n(a_n)} \quad \frac{x^2}{a_n^2} \leq \left( \frac{g_n(x)}{g_n(a_n)} \right)^{2/r}.
\]

Note that \( \{g_n(x), n \geq 1\} \) is a sequence of positive and non-decreasing functions in \( (0, r\infty) \), so \( 0 \leq (g_n(x)/g_n(a_n)) \leq 1 \) when \( |x| \leq a_n \). Consequently,

\[
x^2 \leq \left( \frac{g_n(x)}{g_n(a_n)} \right)^{2/r} \leq \frac{g_n(x)}{g_n(a_n)}, \quad \text{for } 0 < r \leq 1.
\]

On the other hand, if \( g_n(x) \) satisfies condition (ii), then we can also get that

\[
x^2 \leq \left( \frac{g_n(x)}{g_n(a_n)} \right)^{2/r} \leq \frac{g_n(x)}{g_n(a_n)}, \quad \text{for } 1 < r \leq 2.
\]

Therefore, whether even function \( g_n(x) \) satisfies condition (i) or (ii), we can obtain

\[
\text{Var} \left( \frac{X^{(a_n)}_n}{a_n} \right) \leq \frac{Eg_n(X_n)}{g_n(a_n)},
\]

Finally, we prove that \( \sum_{n=1}^{\infty} E[X^{(a_n)}_n/a_n] < \infty \).

If \( g_n(x) \) satisfies condition (i), when \( |x| \leq a_n \), we have \( (|x|'/a_n) \leq (|x|'/a_n') \), for \( 0 < r \leq 1 \). It follows that

\[
E \left( \frac{X^{(a_n)}_n}{a_n} \right) \leq E \left( \frac{|X_n|}{a_n} I(|X_n| \leq a_n) \right) + EI(|X_n| > a_n)
\]

\[
\leq E \left( \frac{|X_n|^r}{a_n^r} I(|X_n| \leq a_n) \right) + E \left( \frac{g_n(X_n)}{g_n(a_n)} I(|X_n| > a_n) \right)
\]

\[
= E \left( \frac{g_n(X_n)}{g_n(a_n)} \right).
\]

(19)
If \( g_n(x) \) satisfies condition (ii), then by the fact that \( E X_n = 0 \) and \( x/g_n(x) \) is a nonincreasing function in \((0, \infty)\), we get
\[
\left| E \left( \frac{X_n^{a_n}}{a_n} \right) \right| \leq E \left( \frac{X_n}{a_n} I(|X_n| \leq a_n) \right) + E I(|X_n| > a_n)
\]
\[
\leq E \left( \frac{X_n}{a_n} I(|X_n| > a_n) \right) + E I(|X_n| > a_n)
\]
\[
\leq E \left( \frac{g_n(X_n)}{g_n(a_n)} I(|X_n| > a_n) \right) + E \left( \frac{g_n(X_n)}{g_n(a_n)} I(|X_n| > a_n) \right)
\]
\[
\leq 2 E g_n(X_n) \frac{X_n}{g_n(a_n)} I(|X_n| > a_n)
\]
\[
\leq 2 E g_n(X_n) \frac{X_n}{g_n(a_n)}
\]
where \( L(x) \) is a slowly varying function. Let \( \{a_n, n \geq 1\} \) and \( \{b_n, n \geq 1\} \) be sequences of positive constants satisfying \( 0 < b_n \uparrow \infty \). Denote \( c_n = b_n/a_n \) for each \( n \geq 1 \).

Assume that
\[
\sum_{n=1}^{\infty} P(|X_n| > c_n) < \infty ;
\]
then
\[
b_n^{-1} \sum_{k=1}^{n} a_k X_k \rightarrow 0 \quad a.s., \quad as \; n \rightarrow \infty.
\]

**Proof.** Since (23) and (24) imply that \( \chi_k \geq 1 \) for all sufficiently large \( k \). Without loss of generality, we assume \( \chi_k \geq 1 \) for all \( k \geq 1 \).

By Borel-Cantelli Lemma, it is easily seen that (24) implies that
\[
\sum_{k=1}^{n} a_k X_k I(|X_k| > \chi_k) = o(b_n) \quad a.s.
\]

Denote
\[
Y_k = -\chi_k I(X_k < -\chi_k) + X_k I(|X_k| \leq \chi_k) + \chi_k I(X_k > \chi_k), \quad k \geq 1;
\]

thus, \( \{Y_k, k \geq 1\} \) is still AANA from Lemma 7. It is easy to check that
\[
\sum_{k=1}^{n} a_k X_k = \sum_{k=1}^{n} a_k (Y_k - E Y_k) + \sum_{k=1}^{n} a_k E Y_k
\]
\[
+ \sum_{k=1}^{n} \chi_k (I(X_k < -\chi_k) - I(Y_k > \chi_k))
\]
\[
+ \sum_{k=1}^{n} \chi_k X_k I(|X_k| > \chi_k)
\]

In order to show that \( b_n^{-1} \sum_{k=1}^{n} a_k X_k \rightarrow 0 \) a.s., we only need to show that the first three terms above are \( o(b_n) \) or \( o(b_n) \) a.s.

By \( C_3 \) inequality, Theorem 1b in [22, page 281] (or see Adler [23]) and (24), we can get
\[
\sum_{k=1}^{\infty} Var \left( \frac{Y_k}{\chi_k} \right)
\]
\[
\leq 3 \sum_{k=1}^{\infty} \chi_k^{-2} E Y_k^2
\]
\[
\leq 3 \sum_{k=1}^{\infty} \chi_k^{-2} E \left[ \chi_k^2 I(|X_k| > \chi_k) + X_k^2 I(|X_k| \leq \chi_k) \right]
\]
\[
= 3 \sum_{k=1}^{\infty} P(|X_k| > \chi_k) + 3 \sum_{k=1}^{\infty} \chi_k^{-2} E X_k^2 I(|X_k| \leq \chi_k)
\]
\[
\leq C + 6 \sum_{k=1}^{\infty} \chi_k^{-2} \int_0^{c_k} t P(|X_k| > t) dt
\]

Abstract and Applied Analysis

\[ \leq C + C \sum_{k=1}^{\infty} L(c_k) c_k^{-\alpha} \]

\[ \leq C + C \sum_{k=1}^{\infty} P(|X_k| > c_k) < \infty. \]

(29)

It follows from Corollary 9 and Kronecker’s lemma that

\[ \sum_{k=1}^{n} a_k (Y_k - E Y_k) = o(b_n) \quad \text{a.s.} \]

(30)

By (24) again,

\[ \sum_{k=1}^{\infty} E \left( \frac{a_k c_k (I(X_k < -c_k) - I(X_k > c_k))}{b_k} \right) \]

\[ \leq \sum_{k=1}^{\infty} E (I(X_k < -c_k) + I(X_k > c_k)) \]

\[ = \sum_{k=1}^{\infty} P(|X_k| > c_k) < \infty, \]

which implies that

\[ \sum_{k=1}^{\infty} \frac{a_k c_k (I(X_k < -c_k) - I(X_k > c_k))}{b_k} \]

converges a.s. (32)

By Kronecker’s lemma, it follows that

\[ \sum_{k=1}^{n} a_k c_k (I(X_k < -c_k) - I(X_k > c_k)) = o(b_n) \quad \text{a.s.} \]

(33)

By Theorem 1b in [22, page 281] (or see Adler [23]) and (24) again, we have

\[ \sum_{j=1}^{\infty} \left| \frac{a_j E Y_j}{b_j} \right| \]

\[ \leq \sum_{j=1}^{\infty} c_j^{-1} \left[ k P(|X_k| > c_k) + E |X_k| I(|X_k| > c_k) \right] \]

\[ \leq \sum_{j=1}^{\infty} c_j^{-1} \left[ k P(|X_k| > c_k) + 2 \sum_{j=1}^{\infty} c_j^{-1} E |X_k| I(|X_k| > c_k) \right] \]

\[ \leq \sum_{j=1}^{\infty} c_j^{-1} \int_{c_j}^{\infty} L(t) t^{-\alpha} dt \]

\[ \leq C + C \sum_{k=1}^{\infty} L(c_k) c_k^{-\alpha} \]

\[ \leq C + C \sum_{k=1}^{\infty} P(|X_k| > c_k) < \infty, \]

which implies that

\[ \sum_{k=1}^{\infty} \frac{a_k E Y_k}{b_k} \]

converges. (35)

By Kronecker’s Lemma, it follows that

\[ \sum_{k=1}^{n} a_k E Y_k = o(b_n). \]

(36)

Hence, the desired result (25) follows from (26)–(36) immediately.

\[ \square \]

Remark 18. Theorem 17 generalizes and extends the corresponding one for NA random variables (see Wang et al. [24]) to AANA random variables.

Theorem 19. Let \( 1 < r < 2 \) and \( \{X_n, n \geq 1\} \) a sequence of mean zero AANA random variables with \( \sum_{n=1}^{\infty} q^2(n) < \infty \), which is stochastically dominated by a random variable \( X \). Let \( \{a_n, n \geq 1\} \) be a sequence of positive constants satisfying \( A_n = \sum_{k=1}^{n} a_k \uparrow \infty \). Denote \( c_n = A_n/a_n \) for each \( n \geq 1 \). Assume that

\[ E|X|^r < \infty, \]

\[ N(n) = \text{Card} \{ i : c_i \leq n \} = O(n^r), \quad n \geq 1; \]

then

\[ A_n^{-1} \sum_{k=1}^{n} a_k X_k \rightarrow 0 \quad \text{a.s., as} \quad n \rightarrow \infty. \]

(37)

Proof. Let \( N(0) = 0 \) and denote

\[ X_n^{(c_n)} = -c_n I(X_n < -c_n) + X_n I(|X_n| \leq c_n) + c_n I(X_n > c_n), \quad n \geq 1. \]

(38)

It follows by (37) that

\[ a_n \sum_{j=1}^{n} P(X_j \neq X_j^{(c_n)}) \]

\[ = \sum_{j=1}^{n} P(|X_j| > c_j) = \sum_{j=1}^{n} \sum_{1 \leq j \leq k \leq c_j} P(|X_j| > c_j) \]

\[ \leq C \sum_{j=1}^{n} (N(j) - N(j - 1)) P(|X| > j - 1) \]

\[ = C \sum_{j=1}^{n} (N(j) - N(j - 1)) \sum_{n=j}^{\infty} P(n-1 < |X| \leq n) \]

\[ = C \sum_{n=1}^{\infty} \sum_{j=1}^{n} (N(j) - N(j - 1)) P(n-1 < |X| \leq n) \]

\[ \leq C \sum_{n=1}^{\infty} n' P(n-1 < |X| \leq n) \leq CE|X|^r < \infty. \]
By the equality above and Borel-Cantelli lemma, we can get
\[ P(X_i \neq X^{(c)}_i, \text{i.o.}) = 0. \]
Therefore, in order to prove (38), we only need to prove that
\[ A_n^{-1} \sum_{i=1}^n a_i X^{(c)}_i \to 0 \quad \text{a.s., } n \to \infty. \] (41)

By the inequality, Lemma 12, and (37) again,
\[ \sum_{k=1}^\infty \text{Var} \left( \frac{a_k X^{(c)}_k}{A_k} \right) \]
\[ \leq \sum_{k=1}^\infty c_k^{-2} E(\sum_{i=1}^k X^{(c)}_i)^2 \]
\[ \leq 3 \sum_{k=1}^\infty c_k^{-2} E \left[ \sum_{i=1}^k |X_i| > c_k \right] + \sum_{k=1}^\infty c_k^{-2} E X^2 I(|X| \leq c_k) \]
\[ \leq C \sum_{k=1}^\infty P(|X| > c_k) + C \sum_{k=1}^\infty c_k^{-2} E X I(|X| \leq c_k) \]
\[ \leq C \sum_{k=1}^\infty \sum_{j=1}^\infty \sum_{c_k < j \leq c_i} c_k^{-2} E X^2 I(|X| \leq c_k) \]
\[ \leq C \sum_{k=1}^\infty \sum_{j=1}^\infty (N(j) - N(j-1))(j-1)^{-2} \]
\[ \times \sum_{k=1}^j E X^2 I(k-1 < |X| \leq k) \]
\[ \leq C \sum_{k=1}^\infty \sum_{j=1}^\infty (N(j) - N(j-1))(j-1)^{-2} \]
\[ \times \sum_{k=1}^j E X^2 I(k-1 < |X| \leq k) \]
\[ \leq C \sum_{k=1}^\infty \sum_{j=1}^\infty \sum_{k=j}^\infty N(j)(j-1)^{-2} - j^{-2} \]
\[ \times \sum_{k=1}^\infty E X^2 I(k-1 < |X| \leq k) \]
\[ \leq C \sum_{k=1}^\infty \sum_{j=1}^\infty \sum_{k=j}^\infty j^{-3} X^2 I(k-1 < |X| \leq k) \]
\[ \leq C \sum_{k=1}^\infty \sum_{j=1}^\infty j^{-3} E X^2 I(k-1 < |X| \leq k) \]
\[ = C \sum_{k=1}^\infty E X^2 I(k-1 < |X| \leq k) \]
\[ \leq C \sum_{k=1}^\infty |X|^2 < \infty. \]

Hence, by the inequality above, Corollary 9 and Kronecker’s lemma, we have
\[ A_n^{-1} \sum_{i=1}^n a_i \left( X^{(c)}_i - E X^{(c)}_i \right) \to 0 \quad \text{a.s., } n \to \infty. \] (43)

In order to prove (41), it suffices to prove that
\[ A_n^{-1} \sum_{i=1}^n a_i E X^{(c)}_i \to 0 \quad n \to \infty. \] (44)

Notice that \( E X_n = 0 \) for each \( n \geq 1 \), we have
\[ |E X_n I(|X_n| \leq c_n)| \leq |E X_n I(|X_n| > c_n)|. \] (45)

It follows from Lemma 12 and (37) that,
\[ \leq C + \sum_{k=1}^{\infty} E|X|^r \mathbb{I}(k < |X| \leq k+1) \]
\[ \leq C + CE|X|^r < \infty. \]  
(46)

By Kronecker’s lemma, we can get (44) immediately. The proof is complete. \( \square \)

3. Strong Stability for Weighted Sums of AANA Sequence

**Theorem 20.** Let \( \{X_n, n \geq 1\} \) be a sequence of AANA random variables with \( \sum_{n=1}^{\infty} \mathbb{E}^2(n) < \infty \), which is stochastically dominated by a random variable \( X \). Let \( \{a_n, n \geq 1\} \) and \( \{b_n, n \geq 1\} \) be two sequences of positive numbers with \( c_n = b_n/a_n \) and \( b_n \uparrow \infty \). Denote \( N(x) = \text{Card}\{n : c_n < x\} \), \( x > 0 \). If the following conditions are satisfied:

(i) \( \mathbb{E}N(|X|) < \infty; \)

(ii) \( \int_0^{\infty} t^{-1} \mathbb{P}(|X| > t) \left( \int_t^{\infty} y^{-(p+1)} N(y) \, dy \right) \, dt < \infty \), for some \( p \in [1, 2] \),

then there exist \( d_n \in \mathbb{R} \), \( n = 1, 2, \ldots \), such that

\[ b_n^{-1} \sum_{i=1}^{n} a_i X_i - d_n \longrightarrow 0 \quad \text{a.s.} \]  
(47)

**Proof.** For each \( i \geq 1 \), denote

\[ X_i^{(c_i)} = -c_i I(X_i < -c_i) + X_i I(|X_i| \leq c_i) + c_i I(X_i > c_i). \]  
(48)

By Definition 1 and conditions (i), we can obtain

\[ \sum_{i=1}^{\infty} \mathbb{P}(X_i \neq X_i^{(c_i)}) \]
\[ = \sum_{i=1}^{\infty} \mathbb{P}(|X_i| > c_i) \leq C \sum_{i=1}^{\infty} \mathbb{P}(|X| > c_i) \]
\[ \leq C \sum_{i=1}^{\infty} \int_{0}^{\infty} \mathbb{I}(c_i \leq t) \, d\mathbb{P}(|X| \leq t) \]
\[ = C \int_{0}^{\infty} N(t) \, d\mathbb{P}(|X| \leq t) \]
\[ = \text{CEN}(|X|) < \infty. \]  
(49)

By Borel-Cantelli lemma for any sequence \( \{d_n, n \geq 1\} \subset \mathbb{R} \), with probability 1, the sequences \( \{b_n^{-1} \sum_{i=1}^{n} a_i X_i - d_n\} \) and \( \{b_n^{-1} \sum_{i=1}^{n} a_i X_i^{(c_i)} - d_n\} \) converge on the same set and to the same limit. We will prove that \( b_n^{-1} \sum_{i=1}^{n} a_i (X_i^{(c_i)} - EX_i^{(c_i)}) \rightarrow 0 \) a.s., which implies (6) with \( d_n = b_n^{-1} \sum_{i=1}^{n} a_i EX_i^{(c_i)} \). According to Lemma 7, \( \{a_i (X_i^{(c_i)} - EX_i^{(c_i)}), i \geq 1\} \) is a sequence of AANA random variables with mean zero. By C inequality and Lemma 12, we have

\[ \sum_{n=1}^{\infty} \mathbb{E} \left[ \left| b_n \left( X_n^{(c_n)} - EX_n^{(c_n)} \right) \right|^p \right]^\frac{1}{p} \]
\[ \leq C \sum_{n=1}^{\infty} c_n^p \mathbb{E} \left[ (|X_n|^p |X_n| \leq c_n) \right] \]
\[ + \sum_{n=1}^{\infty} c_n^p \mathbb{P}(|X_n| > c_n) \]
\[ \leq C \sum_{n=1}^{\infty} c_n^p \mathbb{P}(|X| > c_n) \]
\[ + \sum_{n=1}^{\infty} \mathbb{P}(|X| > c_n) \]
\[ = \text{CEN}(|X|) + C \sum_{n=1}^{\infty} \mathbb{P}(|X| > c_n) \]  
(50)

Notice that

\[ \sum_{n=1}^{\infty} c_n^{-p} \int_0^{c_n} t^{p-1} \mathbb{P}(|X| > t) \, dt \]
\[ = \int_0^{\infty} t^{p-1} \mathbb{P}(|X| > t) \sum_{n : c_n \geq t} c_n^{-p} \, dt \]
\[ \leq \int_0^{\infty} t^{p-1} \mathbb{P}(|X| > t) \left( \int_t^{\infty} y^{-(p+1)} N(y) \, dy \right) \, dt, \]  
(51)

where the last inequality follows from the fact that for \( t > 0 \)

\[ \sum_{n : c_n \geq t} c_n^{-p} \]
\[ = \lim_{u \to \infty} \sum_{n : c_n \geq t} c_n^{-p} = \lim_{u \to \infty} \int_0^{u} y^{-p} dN(y) \]
\[ = \lim_{u \to \infty} \left( u^{-p} N(u) - t^{-p} N(t) + \int_t^{u} y^{-(p+1)} N(y) \, dy \right), \]
\[ \leq \lim_{u \to \infty} \left( \int_t^{u} y^{-(p+1)} N(y) \, dy \right) \]
\[ \leq \int_t^{\infty} y^{-(p+1)} N(y) \, dy. \]  
(52)

By (50), (51) and condition (i), (ii), we can get that

\[ \sum_{n=1}^{\infty} \mathbb{E} \left[ \left| b_n \left( X_n^{(c_n)} - EX_n^{(c_n)} \right) \right|^p \right]^\frac{1}{p} < \infty. \]  
(53)
Therefore, it follows from (53) and Corollary 15 that
\[ b^{-1}_n \sum_{i=1}^{n} a_i (X_i^{(c_i)} - EX_i^{(c_i)}) \rightarrow 0, \text{ a.s.} \]  
(54)

The proof is complete. 

**Corollary 21.** Suppose that the conditions of Theorem 20 are satisfied and $EX_n = 0$ for each $n \geq 1$. If $\int_1^\infty \frac{E(N(|X|/s))}{s} ds < \infty$, then $b^{-1}_n \sum_{i=1}^{n} a_i X_i \rightarrow 0$ a.s.

**Proof.** According to the proof of Theorem 20, we need only to prove that
\[ b^{-1}_n \sum_{i=1}^{n} a_i EX_i^{(c_i)} \rightarrow 0, \text{ as } n \rightarrow \infty. \]  
(55)

Notice that $EX_n = 0$ for each $n \geq 1$; then
\begin{align*}
\left|EX_n^{(c_i)}\right| &= \left| -E(X_i I(|X_i| > c_i)) - E(c_i I(X_i < -c_i)) + E(c_i I(X_i > c_i)) \right| \\
&= E((X_i - c_i)I(X_i > c_i) + E((X_i + c_i)I(X_i < -c_i)) \\
&\leq C (|X_i| + c_i) I(|X_i| > c_i), \\
\sum_{i=1}^{n} a_i \frac{|EX_i^{(c_i)}|}{b_i} &\leq \sum_{i=1}^{n} c_i^{-1} E(|X_i| + c_i) I(|X_i| > c_i) \\
&= \sum_{i=1}^{n} E(|X_i| > c_i) + \sum_{i=1}^{n} c_i^{-1} E(|X_i| I(|X_i| > c_i)) \\
&\leq C E(N(|X|)) + \sum_{i=1}^{n} c_i^{-1} \int_{c_i}^{\infty} P(|X_i| > t) dt \\
&= C E(N(|X|)) + \int_{1}^{\infty} C E\left(\frac{|X|}{s}\right) ds < \infty.
\end{align*}

By Kronecker's lemma, we can get (55) immediately. 

**Theorem 22.** Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mean zero and $\sum_{n=1}^{\infty} q^2(n) < \infty$, which is stochastically dominated by a random variable $X$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be two sequences of positive numbers with $c_n = b_n/a_n$ and $b_n \uparrow \infty$. Denote $N(x) = \text{Card}\{n : c_n \leq x\}, x > 0$. If the following conditions are satisfied:

(i) $E(N(|X|)) < \infty$;
(ii) $\int_1^\infty E(N(|X|/s)) ds < \infty$;
(iii) $\max_{1 \leq j \leq n} c_j^{-p} \sum_{i=1}^{n} c_i^{-p} = O(n)$, for some $p \in [1, 2]$,

then
\[ b^{-1}_n \sum_{i=1}^{n} a_i X_i \rightarrow 0 \text{ a.s.} \]  
(58)

**Proof.** By (49), condition (i), and Borel-Cantelli lemma, it suffices to prove $b^{-1}_n \sum_{i=1}^{\infty} a_i X_i^{(c_i)} \rightarrow 0$, a.s. So we need only to prove
\begin{align*}
&b^{-1}_n \sum_{i=1}^{n} a_i (X_i^{(c_i)} - EX_i^{(c_i)}) \rightarrow 0, \text{ a.s.,} \\
&\quad \text{as } n \rightarrow \infty. \tag{59}
\end{align*}

We can get (60) from the proof of Corollary 21. In the following, we prove (59). Put $\varepsilon_0 = 0$ and $\varepsilon_n = \max_{1 \leq j \leq n} c_j$ for $n \geq 1$. According to Lemma 7, $\{a_i(X_i^{(c_i)} - EX_i^{(c_i)}), i \geq 1\}$ is a sequence of AANA random variables with mean zero. By $C_r$ inequality and Lemma 12,
\begin{align*}
&\sum_{n=1}^{\infty} \frac{E[a_n (X_n^{(c_n)} - EX_n^{(c_n)})]^p}{b_n^p} \\
&\quad \leq C \sum_{n=1}^{\infty} \varepsilon_n^{-p} E(|X_n|^p I(|X_n| \leq c_n)) \\
&\quad + C \sum_{n=1}^{\infty} P(|X_n| > c_n) \\
&\quad \leq C \sum_{n=1}^{\infty} \varepsilon_n^{-p} E(|X|^p I(|X| \leq c_n)) \\
&\quad + C \sum_{n=1}^{\infty} P(|X| > c_n) \\
&\quad \leq C E(N(|X|)) + C \sum_{n=1}^{\infty} \varepsilon_n^{-p} E(|X|^p I(|X| \leq c_n)).
\end{align*}

It is easy to see that
\begin{align*}
&\sum_{n=1}^{\infty} \varepsilon_n^{-p} E(|X|^p I(|X| \leq c_n)) \\
&\quad \leq \sum_{n=1}^{\infty} \varepsilon_n^{-p} E(|X|^p I(|X| \leq \varepsilon_n)) \\
&\quad = \sum_{n=1}^{\infty} \varepsilon_n^{-p} \sum_{j=1}^{n} E(|X|^p I(\varepsilon_{j-1} < |X| \leq \varepsilon_j)) \\
&\quad \leq \sum_{j=1}^{\infty} P(\varepsilon_{j-1} < |X| \leq \varepsilon_j) \varepsilon_j^{-p} \sum_{n=j}^{\infty} \varepsilon_n^{-p} \\
&\quad \leq \sum_{j=1}^{\infty} P(\varepsilon_{j-1} < |X| \leq \varepsilon_j) \\
&\quad \leq C \sum_{n=1}^{\infty} P(|X| > \varepsilon_n) \leq C \left(1 + \sum_{n=1}^{\infty} P(|X| > \varepsilon_n)\right) \\
&\quad \leq C (1 + E(N(|X|))) < \infty.
\end{align*}
Therefore,
\[
\sum_{n=1}^{\infty} \frac{E\left[|a_n(X_n^{(c)}) - EX_n^{(c)}|\right]^p}{b_n^p} < \infty
\] (63)
follows from condition (i), (61) and (62). By Corollary 15 and (63), we can obtain (59) immediately. The proof is complete.

\textbf{Theorem 23.} Let \(\{X_n, n \geq 1\}\) be a sequence of AANA random variables with \(\sum_{n=1}^{\infty} q^2(n) < \infty\), which is stochastically dominated by a random variable \(X\). Let \(\{a_n, n \geq 1\}\) and \(\{b_n, n \geq 1\}\) be two sequences of positive numbers with \(c_n = b_n/a_n\) and \(b_n \uparrow \infty\). Define \(N(x) = \text{Card}\{n : c_n \leq x\}\), \(R(x) = \int_x^{\infty} N(y)y^{-3}dy, x > 0\). If

(i) \(N(x) < \infty\) for any \(x > 0\);

(ii) \(R(1) = \int_1^{\infty} N(y)y^{-3}dy < \infty\);

(iii) \(E(X^2R(|X|)) < \infty\),

then there exist \(d_n \in \mathbb{R}, n = 1, 2, \ldots\), such that

\[
b_n^{-1}\sum_{i=1}^{n} a_i X_i - d_n \rightarrow 0 \quad \text{a.s.} \quad (64)
\]

\textbf{Proof.} According to Lemma 7, \(\{X_n^{(c)}, i \geq 1\}\) and \(\{X_i^{(c)}/c_i, i \geq 1\}\) are sequences of AANA random variables. Since \(N(x)\) is nondecreasing, then for any \(x > 0\)

\[
R(x) \geq N(x) \int_x^{\infty} y^{-3}dy = \frac{1}{2}x^{-2}N(x),
\] (65)

which implies that \(EN(|X|) \leq 2E(X^2R(|X|)) < \infty\). Therefore,

\[
\sum_{i=1}^{\infty} P\left(X_i \neq X_i^{(c)}\right) = \sum_{i=1}^{\infty} P\left(|X_i| > c_i\right)
\leq C \sum_{i=1}^{\infty} P\left(|X| > c_i\right) \leq \text{CEN} \left(|X|\right) < \infty.
\] (66)

By Borel-Cantelli lemma for any sequence \(\{d_n, n \geq 1\} \subset \mathbb{R},\) with probability 1, the sequences \(b_n^{-1}\sum_{i=1}^{n} a_i X_i - d_n\) and \(b_n^{-1}\sum_{i=1}^{n} a_i (X_i^{(c)} - EX_i^{(c)})\) converge on the same set and to the same limit. We will prove that \(b_n^{-1}\sum_{i=1}^{n} a_i (X_i^{(c)} - EX_i^{(c)}) \rightarrow 0 \text{ a.s.,}\) which implies the theorem with \(d_n = b_n^{-1}\sum_{i=1}^{n} a_i EX_i^{(c)}\).

By \(C_r\) inequality and Lemma 12,

\[
\sum_{n=1}^{\infty} \frac{\text{Var} \left(\frac{a_nX_n^{(c)}}{b_n}\right)}{b_n^2} \leq \sum_{n=1}^{\infty} c_n^{-2}E \left(X^2(\text{C}_n)^2\right)
\leq 3 \sum_{n=1}^{\infty} P\left(|X_n| > c_n\right) + 3 \sum_{n=1}^{\infty} c_n^{-2}E \left(X^2I(|X_n| \leq c_n)\right)
\leq C \sum_{n=1}^{\infty} P\left(|X| > c_n\right)
\leq C E\left(X^2I(|X| \leq c_n)\right)
\leq \text{CEN} \left(|X|\right) + CI,
\] (67)

\[
I = \sum_{n, c_n \leq 1} c_n^{-2}E \left(X^2 I(|X| \leq c_n)\right)
\leq \sum_{n, c_n > 1} c_n^{-2}E \left(X^2 I(|X| \leq c_n)\right)
\leq I_1 + I_2.
\] (68)

Since \(N(1) = \text{Card}\{n : c_n \leq 1\} \leq 2R(1) < \infty,\) following from (65) and condition (ii), then we have \(I_1 < \infty.\)

\[
I_2 = \sum_{n, c_n > 1} c_n^{-2}E \left(X^2 I(|X| \leq c_n)\right)
= \sum_{k=2}^{\infty} \sum_{(k-1) \leq c_n \leq k} c_n^{-2}E \left(X^2 I(|X| \leq c_n)\right)
\leq \sum_{k=2}^{\infty} \left(\text{C}(k) - \text{C}(k-1)\right)(k-1)^{-2}E \left(X^2 I(|X| \leq k)\right)
\leq \sum_{k=2}^{\infty} \left(\text{C}(k) - \text{C}(k-1)\right)(k-1)^{-2}E \left(X^2 I(|X| \leq 1)\right)
\leq \sum_{k=2}^{\infty} \left(\text{C}(k) - \text{C}(k-1)\right)(k-1)^{-2}E \left(X^2 I(|X| \leq 1)\right)
\times (k-1)^{-1}E \left(X^2 I(1 < |X| \leq k)\right)
\leq I_{21} + I_{22}.
\] (69)
To prove $I_2 < \infty$, we need to prove that $I_{21} < \infty$ and $I_{22} < \infty$:

$$I_{21} \leq C \sum_{k=2}^{\infty} \left( N(k) - N(k-1) \right) \sum_{j=k-1}^{\infty} j^{-3}$$

$$= C \sum_{j=1}^{\infty} j^{-3} \sum_{k=2}^{j+1} \left( N(k) - N(k-1) \right)$$

$$\leq C \sum_{j=1}^{\infty} (j+1)^{-3} N(j+1)$$

$$\leq C \int_{1}^{\infty} y^{-3} N(y) \, dy < \infty. \quad (70)$$

Since $N(x)$ is nondecreasing and $R(x)$ is nonincreasing, then

$$I_{22} = \sum_{k=2}^{\infty} \left( N(k) - N(k-1) \right) (k-1)^{-2}$$

$$\times \sum_{m=2}^{k} E \left( X^2 I(m-1 < |X| \leq m) \right)$$

$$= \sum_{m=2}^{\infty} E \left( X^2 I(m-1 < |X| \leq m) \right)$$

$$\times \sum_{k=m}^{\infty} \left( (k-1)^{-2} - k^{-2} \right)$$

$$\leq C \sum_{m=2}^{\infty} E \left( X^2 I(m-1 < |X| \leq m) \right)$$

$$\times \sum_{k=m}^{\infty} \int_{k}^{k+1} N(x) x^{-3} \, dx$$

$$= C \sum_{m=2}^{\infty} R(m) E \left( X^2 I(m-1 < |X| \leq m) \right)$$

$$\leq C \sum_{m=2}^{\infty} E \left( X^2 R(|X|) I(m-1 < |X| \leq m) \right)$$

$$\leq C E \left( X^2 R(|X|) \right) < \infty. \quad (71)$$

Therefore, it follows from Corollary 9 and Kronecker’s lemma that

$$b_n^{-1} \sum_{i=1}^{n} a_i (X_i^{(c)} - E X_i^{(c)}) \rightarrow 0 \quad a.s. \quad (73)$$

Taking $d_n = b_n^{-1} \sum_{i=1}^{n} a_i EX_i^{(c)}$, $n \geq 1$, we can get (64). The proof is complete. \( \square \)

Remark 24. Since NA implies AANA, Theorem 20 extends corresponding result for NA random variable (see Wang et al. [24]) to AANA random variables without adding any extra condition.

Similar to the proof of Corollary 21, we can get the following corollary.

**Corollary 25.** Let the conditions of Theorem 23 be satisfied and $EX_n = 0$ for each $n \geq 1$. If $\int_{1}^{\infty} EN(|X|/s) \, ds < \infty$, then $b_n^{-1} \sum_{i=1}^{n} a_i X_i \rightarrow 0 \quad a.s.$

**Corollary 26.** Let $(X_{mn}, n \geq 1)$ be a sequence of AANA random variables with $\sum_{n=1}^{\infty} q^2(n) < \infty$, which is stochastically dominated by a random variable $X$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be two sequences of positive numbers with $c_a = b_a/a_n$ and $b_n \uparrow \infty$. Let $f(x) = x^r h(x)$, where $h(x) > 0$ is a slowly varying function as $x \rightarrow \infty$, $1 < r < 2$. Define $N(x) = \text{Card} \{n : c_a \leq x\}$, $x > 0$. If

(i) $N(n) = O(f(n))$ for each $n \geq 1$;

(ii) $Ef(|X|) < \infty$.

Then there exist $d_n \in \mathbb{R}$, $n = 1, 2, \ldots$, such that

$$b_n^{-1} \sum_{i=1}^{n} a_i X_i - d_n \rightarrow 0 \quad a.s. \quad (74)$$

**Proof.** It is easy to verify that conditions (i)–(iii) of Theorem 23 hold under the conditions of Corollary 25. So Corollary 25 is true by Theorem 23. \( \square \)

**Corollary 27.** Suppose that the conditions of Corollary 26 are satisfied. If $EX_n = 0$ for each $n \geq 1$, then $b_n^{-1} \sum_{i=1}^{n} a_i X_i \rightarrow 0 \quad a.s.$

**Proof.** According to Corollary 26, we need only to prove (60). By $EX_n = 0$ for each $n \geq 1$ and Lemma 12, we have

$$\sum_{i=1}^{\infty} a_i \left| EX_i^{(c)} \right| / b_i \leq \sum_{i=1}^{\infty} \left( c_0 E \left( |X_i| > c_i \right) + |EX_i| I(|X_i| \leq c_i) \right)$$

$$\leq \sum_{i=1}^{\infty} c_i^{-1} \left[ E |X_i| I(|X_i| > c_i) + |EX_i| I(|X_i| > c_i) \right]$$
\[ \sum_{i=1}^{\infty} c_i^{-1} E[|X_i| I(|X_i| > c_i)] \leq C \sum_{i=1}^{\infty} c_i^{-1} E[|X| I(|X| > c_i)] \]

\[ = C \sum_{k=1}^{\infty} \sum_{(k-1) < c_i < k} c_i^{-1} E[|X| I(|X| > c_i)] \]

\[ \leq C \sum_{k=2}^{\infty} (k-1)^{-1} (N(k) - N(k-1)) \]

\[ \times \sum_{j=k}^{\infty} E[|X| I(j \leq |X| < j+1)] \]

\[ \leq C \sum_{j=2}^{\infty} \sum_{k=2}^{j} \sum_{(k-1) < c_i < k} c_i^{-1} E[|X| I(|X| > c_i)] \]

\[ \times \sum_{j=k}^{\infty} E[|X| I(j \leq |X| < j+1)] . \]

(75)

Since \( r > 1 \), we can take \( \delta > 0 \) such that \( r - \delta > 1 \). By Lemma 13 and differential mean value theorem, we can obtain

\[ \sum_{k=2}^{j} N(k) (k-1)^{-1} - k^{-1} \]

\[ \leq C \sum_{k=2}^{j} k' h(k) (k-1)^{-1} - k^{-1} \]

\[ = C \sum_{k=2}^{j} k'^{-\delta} (k^\delta h(k)) (k-1)^{-1} - k^{-1} \]

\[ \leq C f'(h) \sum_{k=2}^{j} k'^{-\delta-2} \]

\[ \leq C f'(h) \int_{2}^{j} x'^{-\delta-2} dx \leq C f'^{-1} (h) . \]

(76)

It is easily seen that \( x'^{-1} h(x) \) is a quasimonotonically increasing function by Lemma 13. Hence, we have by (75) and (76) that

\[ \sum_{j=1}^{\infty} a_j \frac{E X_j^{(c_j)}}{b_j} \]

\[ \leq C \sum_{j=2}^{\infty} f'^{-1} (h) E[|X| I(j \leq |X| < j+1)] \]

\[ \leq C \sum_{j=2}^{\infty} E \left( |X|^{-1} h(|X|) |X| I(j \leq |X| < j+1) \right) \]

\[ = C \sum_{j=2}^{\infty} E \left( |X|^{-1} h(|X|) I(j \leq |X| < j+1) \right) \]

(77)

By Kronecker's lemma, we can get (60) immediately. The proof is complete.

Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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