Research Article

(1, 1)-Coherent Pairs on the Unit Circle

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A pair (U, V) of Hermitian regular linear functionals on the unit circle is said to be a (1, 1)-coherent pair if their corresponding sequences of monic orthogonal polynomials (φn(x))n≥0 and (ψn(x))n≥0 satisfy

\[ \frac{P_{n+1}'(x)}{n+1} + a_n \frac{P_n'(x)}{n} = Q_n(x) + b_n Q_{n-1}(x), \quad a_n \neq 0, \ n \geq 1. \]  

(1)

This concept is a generalization of the notion of coherent pair, for us (1, 0)-coherent pair, introduced by Iserles et al. in [1], where b_n = 0, for every n ≥ 1.

In the work by Delgado and Marcellán [2], the notion of a generalized coherent pair of measures, in short, (1, 1)-coherent pair of measures, arose as a necessary and sufficient condition for the existence of an algebraic relation between the SMOP \{S_n(x; \lambda)\}_{n≥0} associated with the Sobolev inner product

\[ \langle p(x), r(x) \rangle_\lambda = \int_R p(x) r(x) \, d\mu_0 \]

(2)

and the SMOP \{P_n(x)\}_{n≥0} associated with the positive Borel measure \mu_0 in the real line as follows:

\[ S_{n+1}(x; \lambda) + c_n(\lambda) S_n(x; \lambda) = P_{n+1}(x) + \frac{n+1}{n} a_n P_n(x), \quad n \geq 1, \]

(3)

where \{c_n(\lambda)\}_{n≥1} are rational functions in \lambda > 0. Besides, they obtained the classification of all (1, 1)-coherent pairs of regular functionals (U, V) and proved that at least one of them must be semiclassical of class at most 1, and U and V are related by a rational type expression. This is a generalization of the results of Meijer [3] for the (1, 0)-coherence case (when b_n = 0, n ≥ 1), where either U or V must be a classical linear functional.

The most general case of the notion of coherent pair was studied by de Jesus et al. in [4] (see also [5]), the so-called (M, N)-coherent pairs of order (m, k), where the derivatives of order m and k of two SMOP \{P_n(x)\}_{n≥0} and \{Q_n(x)\}_{n≥0} with respect to the regular linear functionals U and V are related by

\[ \sum_{j=0}^{M} a_{n,i,m} P_{n+m-j}^{(m)}(x) = \sum_{j=0}^{N} b_{n,i,k} Q_{n+k-j}^{(k)}(x), \quad n \geq 0, \]

(4)
where $M, N, m, k \in \mathbb{Z}^+ \cup \{0\}$ and the real numbers $a_{n-j,m,n}, b_{n-j,m,k}$ satisfy some natural conditions. They showed that the regular linear functionals $\mathcal{U}$ and $\mathcal{V}$ are related by a rational factor, and, when $m \neq k$, those linear functionals are semiclassical. Besides, they proved that if $(\mu_0, \mu_1)$ is a $(M, N)$-coherent pair of order $(m, 0)$ of positive Borel measures on the real line, then

$$
\max\{M, N\} \sum_{j=0}^{\min\{M, N\}} c_{n-j,m,n}(\lambda) S_{n-j+m,n}(x; \lambda) = \sum_{j=0}^{M} a_{n-j,m,n} P_{n-j,m}(x), \quad n \geq 0,
$$

holds, where $c_{n-j,m,n}(\lambda), 0 < j \leq \max\{M, N\}, n \geq 0$, are rational functions in $\lambda$ such that $c_{n-j,m,n}(\lambda) = 0$ for $n < j \leq \max\{M, N\}$, and $\{S_{n,m,n}(x; \lambda)\}_{n \geq 0}$ is the Sobolev SMOP with respect to the inner product

$$
\langle p(x), r(x) \rangle_{\lambda,m} = \int_{\mathbb{R}} p(x) r(x) d\mu_0 + \lambda \int_{\mathbb{R}} p^{(m)}(x) r^{(m)}(x) d\mu_1, \quad \lambda > 0, m \in \mathbb{Z}^+,
$$

$p, r \in \mathbb{P}$. Also, they showed that $(M, \max\{M, N\})$-coherence of order $(m, 0)$ is a necessary condition for the algebraic relation (5). For a historical summary about coherent pairs on the real line, see, for example, the introductory sections in the recent papers of de Jesus et al. [6] and of Marcellán and Pinzón-Cortés [7].

On the other hand, the notion of coherent pair was extended to the theory of orthogonal polynomials in a discrete variable by Area et al. in [8–10]. They used the difference operator $D_\omega$ as well as the $q$-derivative operator $D_q$ defined by

$$
(D_\omega p)(x) = \frac{p(x + \omega) - p(x)}{\omega}, \quad \omega \in \mathbb{C} \setminus \{0\},
$$

$$
(D_q p)(x) = \frac{p(qx) - p(x)}{(q-1)x} \quad \text{for } x \neq 0,
$$

$$
(D_q p)(0) = p'(0), \quad q \in \mathbb{C} \setminus \{0, 1\},
$$

instead of the usual derivative operator $D$. In this way, they obtained similar results to those by Meijer and similar classification as a limit case when $\omega \to 0$ or $q \to 1$, respectively. Likewise, Marcellán and Pinzón-Cortés in [11, 12] studied the analogue of the generalized coherent pairs introduced by Delgado and Marcellán, that is, $(1,1)-D_\omega$-coherent pairs and $(1,1)-D_q$-coherent pairs. Finally, Álvarez-Nodarse et al. [13] analyzed the more general case, $(M,N)$-$D_\omega$-coherent pairs of order $(m,k)$ and $(M,N)$-$D_q$-coherent pairs of order $(m,k)$, proving the analogue results to those in [4].

Furthermore, Branquinho et al. in [14] extended the concept of coherent pair to Hermitian linear functionals associated with nontrivial probability measures supported on the unit circle. They studied (3) in the framework of orthogonal polynomials on the unit circle (OPUC). Also, they concluded that if $(\mathcal{U}, \mathcal{V})$ is a $(1,0)$-coherent pair of Hermitian regular linear functionals, then $\{P_n(z)\}_{n \geq 0}$ is semiclassical and $\{Q_n(z)\}_{n \geq 0}$ is quasiorthogonal of order at most 6 with respect to the functional $[zA(z) + (1/2)A(1/2)]\mathcal{U}$, $A \in \mathbb{P}$. Besides, they analyzed the cases when either $\mathcal{U}$ or $\mathcal{V}$ is the Bernstein-Szegő measure.

Later on, Branquinho and Rebocho in [15] obtained that if the sequences $\{P_n(z)\}_{n \geq 0}$ and $\{Q_n(z)\}_{n \geq 0}$ satisfy, for $n \geq 0$,

$$
\sum_{j=0}^{M_1} a_{n+j+M_1,n}(z) P_{n+j+M_1,n}(z) = \sum_{j=0}^{N_1} b_{n+j,N_1,n}(z) Q_{n+j,N_1,n}(z) = \delta_n,
$$

with $N_1 = M_1$, $\max\{M_2, N_2\} < N_1$, and some extra conditions, then $\{P_n(z)\}_{n \geq 0}$ and $\{Q_n(z)\}_{n \geq 0}$ are semiclassical sequences of OPUC. Moreover, when $P_n(z) = Q_n(z)$ for all $n$ and under some extra conditions, (8) is a necessary condition for the semiclassical character of $\{P_n(z)\}_{n \geq 0}$. Finally, they analyzed the $(0,1)$-coherence case $\{P_{n,M}(z)\}/(n + 1) = Q_{n,N}(z) + b_{n,N}Q_{n,N}(z), b_n \neq 0, n \geq 1$, when $\mathcal{U}$ is the linear functional associated with either the Lebesgue measure or the Bernstein-Szegő measure.

The aim of our contribution is to describe the $(1,1)$-coherence pair $(\mathcal{U}, \mathcal{V})$ when $\mathcal{U}$ and $\mathcal{V}$ are regular linear functionals, focusing our attention on the cases when $\mathcal{U}$ is either the Lebesgue or the Bernstein-Szegő linear functional. The structure of this work is as follows. In Section 2, we state some definitions and basic results which will be useful in the forthcoming sections. In Section 3, we introduce the concept of $(1,1)$-coherent pair of Hermitian regular linear functionals, and we obtain some results that will be applied in the sequel. In Section 4, we analyze $(1,1)$-coherent pairs when $\mathcal{U}$ is the linear functional associated with the Lebesgue measure on the unit circle. We determine the cases when the linear functional $\mathcal{V}$ is associated with a positive measure on the unit circle, or a rational spectral transformation of it. Finally, in Section 5, we deal with a similar analysis for the case when $\mathcal{U}$ is the linear functional associated with the Bernstein-Szegő measure.

2. Preliminaries

Let us consider the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the linear space of Laurent polynomials with complex coefficients $\Lambda = \text{span}\{z^n : n \in \mathbb{Z}\}$, and a linear functional $\mathcal{U} : \Lambda \to \mathbb{C}$. We can associate with $\mathcal{U}$ a sequence of moments $\{c_n\}_{n \in \mathbb{Z}}$ defined by

$$
\langle p(z), (z^n) \rangle = \langle \mathcal{U}, p(z) z^n \rangle = \left( \frac{1}{z} \right)^n, \quad n \in \mathbb{Z},
$$

where $p, q \in \mathbb{P}$, the linear space of polynomials with complex coefficients. Its Gram matrix with respect to $\{z^n\}_{n \geq 0}$
is an infinite Toeplitz matrix \((c_{j-k})_{j,k \geq 0}\) with leading principal minors given by \(\Delta_n = \det(c_{j-k})_{j,k < n}, n \in \mathbb{Z}^* \cup \{0\}\).

The linear functional \(\mathcal{U}\) is said to be Hermitian if \(c_{n-m} = \overline{c}_n\), quasidefinite or regular if \(\Delta_n \neq 0\) for all \(n \in \mathbb{Z}^* \cup \{0\}\), and positive definite if \(\Delta_n > 0\) for all \(n \in \mathbb{Z}^* \cup \{0\}\). We will denote by \(\mathcal{H}\) the set of Hermitian linear functionals defined on \(A\).

\(\mathcal{U} \in \mathcal{H}\) is regular if and only if there exists a (unique) sequence of monic orthogonal polynomials \(\{\phi_n(z)\}_{n \geq 0}\), which satisfies (11) and (12), as well as the integral representation (see [19]):

\[\phi_n(z) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_{-1} & c_{-2} & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{-(n-1)} & c_{-(n-2)} & \cdots & c_1 \end{vmatrix}, \quad n \geq 1, \quad \phi_0(z) = 1.\]

Besides, they satisfy the forward and backward Szegő recurrence relations

\[\phi_n(z) = z\phi_{n-1}(z) + \alpha_n\phi_n(z), \quad \phi_n(z) = (1 - \alpha_n z^2)\phi_{n-1}(z) + \alpha_n \phi_n(z), \quad n \geq 1, \quad \phi_0(z) = 1,\]

where \(\alpha_n = \phi_0(0), n \geq 1\), are said to be the Verblunsky (reflection, Schur, Szegő, or Geronomic) coefficients and \(\phi_n(z) = z^n\phi_n(1/z), n \in \mathbb{Z}^* \cup \{0\}\), is called the reversed polynomial of \(\phi_n(z)\). Conversely, if \(\{\phi_n(z)\}_{n \geq 0}\) is a sequence of monic polynomials which satisfies (11) and \(|\alpha_n| \neq 1\) for \(n \geq 1\), then \(\{\phi_n(z)\}_{n \geq 0}\) is the sequence of monic OPUC with respect to some Hermitian regular linear functional.

If \(\mathcal{U}\) is a Hermitian regular (resp., positive definite) linear functional, then (see [16–18]) \(|\alpha_n| \neq 1\) (resp., \(|\alpha_n| < 1\)), for \(n \geq 1\).

A positive definite Hermitian linear functional \(\mathcal{U}\) has an integral representation (see [19])

\[\langle p(z), q(z) \rangle = \left\langle \mathcal{U}, \frac{p(z) \overline{q}(1/z)}{z} \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} p(z) \overline{q}(1/z) d\mu(\theta), \quad z = e^{i\theta}, \quad p, q \in P,\]

where \(\mu\) is a nontrivial probability measure supported on an infinite subset of \(T\). A measure \(\mu\) belongs to the Nevai class (see [20, 21]) if \(\lim_{n \to \infty} \phi_n(0) = 0\).

On the other hand (see [19]), an analytic function \(F(z)\), defined on \(\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}\), is said to be a Carathéodory function if and only if \(F(0) = 1\) and \(\text{Re } F(z) > 0\) on \(\mathbb{D}\). If \(\mu\) is a probability measure on \(\mathbb{T}\), then

\[F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)\]

is a Carathéodory function. Conversely, the Herglotz representation theorem claims that every Carathéodory function \(F(z)\) has a representation given by (13) for a unique probability measure \(\mu\) on \(T\).

Besides (see [22]), a Carathéodory function (13) admits the expansions

\[F(z) = \phi_0 + \sum_{n=1}^{\infty} c_n z^n, \quad |z| < 1,\]

and

\[F(z) = -\phi_0 - \sum_{n=1}^{\infty} c_n z^{-n}, \quad |z| > 1,\]

where \(\{c_n\}_{n \geq 0}\) are the moments of the measure associated with \(F(z)\).

To complete this section, we state the following definitions. Let \(\{\phi_n(z)\}_{n \geq 0}\) be a sequence of monic OPUC with corresponding Verblunsky coefficients \(\{\alpha_n\}_{n \geq 1}\), and let \(N \in \mathbb{Z}^* \cup \{0\}\). The polynomials defined by

\[\phi_n^{(N)}(z) = z\phi_{n-1}(z) + \alpha_n + N\phi_n(z), \quad n \geq 1, \quad \phi_0^{(N)}(z) = 1,\]

are called the associated polynomials of \(\{\phi_n(z)\}_{n \geq 0}\) of order \(N\). Similarly, given a finite set of complex numbers \(\{\gamma_n\}_{n=1}^{N}\), with \(|\gamma_n| \neq 1\), \(n = 1, 2, \ldots, N\), let us define the new Verblunsky coefficients \(\{\alpha_n\}_{n \geq 1} = \{\gamma_1, \gamma_2, \ldots, \gamma_N, \alpha_1, \alpha_2, \ldots\}\). Then the monic OPUC defined by the forward Szegő relation associated with \(\{\alpha_n\}_{n \geq 1}\) are said to be the antiasociated polynomials of \(\{\phi_n(z)\}_{n \geq 0}\) of order \(N\).

### 3. (1, 1)–Coherent Pairs on the Unit Circle

A pair of Hermitian regular linear functionals \((\mathcal{U}, \mathcal{V})\) defined on the linear space of Laurent polynomials is said to be a (1, 1)–coherent pair if their corresponding sequences of monic OPUC, \(\{\phi_n(z)\}_{n \geq 0}\) and \(\{\psi_n(z)\}_{n \geq 0}\), are related by

\[\phi_n^{[1]}(z) + a_n\phi_{n-1}^{[1]}(z) = \psi_{n-1}(z), \quad a_n \neq 0, n \geq 1,\]

where \(\phi_n^{[1]}(z) = (\phi_{n+1}(z))/(n+1)\), for \(n \in \mathbb{N}\). In such a case, the pair \(\{\phi_n(z)\}_{n \geq 0}\) and \(\{\psi_n(z)\}_{n \geq 0}\) is also said to be a (1, 1)–coherent pair. If \(b_n = 0\) for every \(n \geq 1\), then \((\mathcal{U}, \mathcal{V})\) is called a (1, 0)–coherent pair.

**Lemma 1.** If \((\mathcal{U}, \mathcal{V})\) satisfies (16), then, one has the following.

(i) \(a_1 \neq b_1\) if and only if \(\phi_n^{[1]}(z) \neq \psi_n(z)\), for every \(n \geq 1\).
(ii) For \( n \geq 1 \), one has
\[
\phi_n^{[1]}(z) = \psi_n(z) + (b_n - a_n) \psi_{n-1}(z) + \sum_{k=0}^{n-2} (-1)^{n-k+1} a_n a_{n-1} \cdots a_{k+2} (b_{k+1} - a_{k+1}) \psi_k(z),
\]
where \( a_0 = \phi_n(0) = 0 \) for \( n \geq 1 \). Furthermore, its moments are
\( c_n = \delta_{n,0}, \) for \( n \in \mathbb{Z}^+ \cup \{0\} \), and its Carathéodory function is \( F(z) = 1 \).

The Bernstein-Szegő linear functional is associated with the measure \( d\mu(\theta) = (1 - |z|)^n/|1 + C e^{i\theta}|^2 (d\theta/2\pi) \), where \( C \in \mathbb{C} \) and \( |C| < 1 \). Its corresponding monic OPUC are \( \phi_n(z) = z^n + C \) for \( n \geq 1 \) and \( \phi_0(z) = 1 \). Its reversed polynomials are \( \phi_n^{*}(z) = 1 + Cz \), for \( n \geq 1 \), and its Verblunsky coefficients are \( a_n = \phi_n(0) = 0 \) for \( n \geq 2 \); \( a_0 = \alpha_1 = C \). Furthermore, its moments are \( c_n = (-C)^n, \) for \( n \in \mathbb{Z}^+ \cup \{0\} \), and its Carathéodory function is \( F(z) = (1 - zC)/(1 + zC) \).

We begin by analyzing the first one.

4. The Lebesgue Linear Functional

Theorem 3. Let \((\mathcal{H}, \mathcal{Y})\) be a \((1,1)\)-coherent pair on the unit circle such that their corresponding monic OPUC satisfy (16), and let \( \mathcal{U} \) be the Lebesgue linear functional.

(i) If \( a_1 = b_1 \), then \( \mathcal{Y} \) is also the linear functional associated with the Lebesgue measure, and \( a_n = b_n \) for \( n \geq 1 \).

(ii) If \( a_1 \neq b_1 \) and \( |\psi_n(0)| = |\beta_n| \neq 1, n \geq 1 \), then
\[
\begin{align*}
\nu_n &= (-a_2)^{n-1} (b_1 - a_1) v_0, \quad n \geq 1, \\
\alpha_2 &= b_1 (1 - |\beta_1|^2) + \beta_1, \quad a_n = a_2, \\
b_n &= \frac{b_{n-1}}{(1 - |\beta_{n-1}|)^2} = \frac{b_2}{\prod_{k=2}^{n-1} (1 - |\beta_k|^2)}, \quad n \geq 3, \\
\beta_1 &= a_1 - b_1, \\
\beta_n &= \frac{(-1)^{n-1} b_n \cdots b_2 \beta_2}{b_n \cdots b_2} = -b_n \beta_{n-1}, \quad n \geq 2, \\
\psi_1(z) &= z + a_1 - b_1, \quad \text{and for } n \geq 2, \\
\nu_n(z) &= z^n + \alpha_2 (z^n - a_2), \quad \text{with } \{v_n\}_{n \geq 0} \text{ is the sequence of moments associated with } \mathcal{Y}. 
\end{align*}
\]

Proof. From (16) it is easy to check that \( a_1 = b_1 \) if and only if there exists \( N \in \mathbb{N}, N \geq 1 \), such that \( \phi_n^{[1]}(z) = \psi_n(z) \). Also, from (16) and using induction on \( n \), it is immediate to prove (17) and (18).

Corollary 2. If \((\mathcal{H}, \mathcal{Y})\) is a \((1,1)\)-coherent pair given by (16), then
\[
\begin{align*}
\langle \mathcal{Y}, \phi_n^{[1]}(z) \rangle &= (-1)^n \alpha_2 (b_1 - a_1) \prod_{j=1}^{n} a_j \langle \mathcal{Y}, 1 \rangle, \quad n \geq 1, \\
\end{align*}
\]
where \( \prod_{j=k_1}^{k_2} a_j = 1 \) whenever \( k_2 < k_1 \).

We will study the \((1,1)\)-coherence relations when \( \mathcal{H} \) is the linear functional associated with basic positive measures on the unit circle, namely, the Lebesgue and Bernstein-Szegő measures.
Since \( a_n \neq 0, n \geq 1 \), and \( a_1 \neq b_1 \), (25) yields \( v_n \neq 0 \) for \( n \geq 1 \). Thus, from (28), we conclude that \( a_{n+1} = a_n \) for \( n \geq 2 \) or, equivalently, \( a_{n+1} = a_2 \) for \( n \geq 2 \). Therefore, (25) becomes (20).

On the other hand, from (26) we obtain (22) and (23). Besides, from the forward Szegő relation and (26), we can obtain another expression for \( \psi_{n+1}(z) \), \( n \geq 0 \). By comparing the coefficients of \( z^n \), we get \( d_{m+1} - b_{m+1} = a_n - b_n - b_{m+1} |\beta_m|^2 \), for \( n \geq 1 \). Hence, since \( a_{n+1} = a_n \) and \( |\beta_{n-1}| \neq 1 \), for \( n \geq 2 \), (21) follows.

We are interested in the cases where \( \mathcal{V}' \) is also a positive definite linear functional. Notice that, aside from the trivial case when \( a_1 = b_1 \), all of the coherence coefficients are determined from the values of \( a_1, b_1, a_2, b_2 \), and \( a_3 \) (or, equivalently, \( a_1, b_1, a_2, b_2 \) and \( a_3 \)). Not every choice of these parameters will yield a positive definite linear functional \( \mathcal{V}' \). For instance, if \( |b_1| = 1 \) and \( |a_1 - b_1| = |\beta_1| = \sqrt{2} \), then we can see from (22) that \( |b_n| = 1, n \geq 3 \), and \( |\beta_n| = \sqrt{2}, n \geq 2 \). However, it is possible to choose the values of \( a_1, b_1, a_2 \), and \( b_2 \) in order to get a positive definite linear functional \( \mathcal{V}' \), or at least its rational spectral transformation. We have the following cases.

**Proposition 4.** Let \( (\mathcal{U}, \mathcal{V}) \) be a \((1,1)\)-coherent pair on the unit circle such that their corresponding monic OPUC satisfy (16), and let \( \mathcal{U} \) be the linear functional associated with the Lebesgue measure. Assume that \( \mathcal{V}' \) is normalized (i.e., \( v_0 = 1 \)). Then, one has the following.

(i) Let \( |b_1 - a_1| < 1 \). If \( a_2 = a_1 - b_1 \) (i.e., \( b_2 = 0 \)), then \( b_n = 0 \) for every \( n \geq 2 \). Besides, \( \mathcal{V}' \) is the linear functional associated with the Bernstein-Szegő measure with parameter \( b_1 - a_1 \). Furthermore, if \( \mathcal{V}_N = 0 \) for some \( N \geq 2 \), then \( b_n = 0 \).

(ii) If \( a_1, b_1, a_2 \in \mathbb{R} \) and either \( 0 < a_1 < b_1 < a_2 < 1 \) or \(-1 < a_1 < a_2 < b_1 < 0 \) holds, then the Carathéodory function associated with \( \mathcal{V}' \) is

\[
F_{\mathcal{V}}(z) = \frac{b_1 - a_1}{a_2} F_B(z) + \frac{b_1 - a_1 + a_2}{a_2},
\]

where \( F_B(z) \) is the Carathéodory function associated with the Bernstein-Szegő measure with parameter \(-a_2\). As a consequence, the orthogonality measure associated with \( \mathcal{V}' \) is

\[
d\mu_2 = -\frac{b_1 - a_1}{a_2} \frac{1 - |a_2|^2}{1 + a_2 e^{i\theta}} \frac{d\theta}{2\pi} + \frac{b_1 - a_1 + a_2}{a_2} \frac{d\theta}{2\pi}.
\]

(iii) For any values of \( a_1, b_1 \), the value of \( b_2 \) can be chosen in such a way that \( \mathcal{V}' \) is the linear functional associated with a rational spectral transformation of a Nevan class measure.

**Proof.** (i) Notice that \( a_1 \neq b_1 \) because \( a_2 \neq 0 \). We first prove that if \( b_N = 0 \) for some \( N \geq 2 \), then \( b_n = 0 \) for \( n \geq 2 \). Assume that for some \( N \geq 2 \), \( b_N = 0 \). From (20), (22), and (23) it follows that \( b_n = 0 = \beta_n \). Hence, since \( \mathcal{V}_N = 0 \), we conclude that \( b_n = 0 \) for \( n = 2, \ldots, N - 1 \) and \( a_2 = b_2 = 0 \). Therefore, \( b_n = 0 = \beta_n \) for \( n \geq 2 \). Besides, from the forward Szegő relation and (26), we can obtain another expression for \( \psi_{n+1}(z) \). Therefore, (29) holds.

(ii) From (20), the Carathéodory function associated with \( \mathcal{V}' \) is

\[
F_{\mathcal{V}}(z) = 1 + 2 \sum_{k=1}^{N-1} (\beta_{k-1} a_k - a_k b_{k-1} z^k).
\]

(iii) From (21), given \( \beta_1 = a_1 - b_1 \), we have \( b_n = b_n / (1 - |\beta_n|^2) = b_n / (1 - |\beta_n|^2) \), so we can choose \( \beta_n \) small enough so that \( \beta_n \) is sufficiently close to 0. Thus, \( b_2 \) will also be close to 0, and since

\[
\beta_n = \frac{b_n b_{n-1}}{1 - |\beta_{n-1}|^2}, \quad n \geq 2,
\]

\[
b_n = \frac{b_n}{1 - |\beta_{n-1}|^2}, \quad n \geq 3,
\]

\(|\beta_n|_{n=2}^\infty \) will be an increasing sequence and, as a consequence, \(|\beta_n|_{n=2}^\infty \) will be a decreasing sequence. Besides, \( b_n \) can be chosen so that \( |b_n| \) converges to a constant \( b \), \( 0 < b < 1 \), and therefore the product \( \prod_{k=1}^{N-1} (1 - |\beta_k|^2) \) will also converge to \( |b|^2 b^N \). This shows that \( \beta_n \to 0 \), and thus \( |\beta_n|_{n=2}^\infty \) defines a Nevan measure \( \mu \). As a consequence, since \( \mathcal{V}' \) has \(|\beta_n|_{n=2}^\infty \) as Verblunsky coefficients, \( \mathcal{V}' \) can be expressed as an antipodal perturbation of order 1 (see [24]) applied to the measure \( \mu \).

5. The Bernstein-Szegő Linear Functional

Now, we proceed to analyze the companion measure \( \mathcal{V}' \) when \( \mathcal{U} \) is the Bernstein-Szegő linear functional defined as above.
Theorem 5. Let $\mathcal{U}$ be the Bernstein-Szegő linear functional, and let $(\mathcal{U}, \mathcal{V})$ be a $(1, 1)$-coherent pair on the unit circle given by (16). Then, the moments of $\mathcal{V}$ are

$$v_n = (-1)^n \sum_{j=0}^{n-k} \frac{(a_j - b_j)}{n+1} + \frac{1}{n+1} \sum_{j=2}^{n-k} a_j \sum_{k=0}^{n-1} \prod_{j=2}^{n-k} a_j v_0,$$

where $\prod_{j=2}^{n-k} a_j = 1$ whenever $k_2 < k_1$, and the sequence of monic OPUC $(\psi_n(z))_{n \geq 0}$ is given by $\psi_0(z) = 1$, $\psi_1(z) = z + (a_1 - b_1) + (1/2) C$, and, for $n \geq 2$,

$$\psi_n(z) = z^n \sum_{k=0}^{n-1} \left( a_n - b_n + \frac{n}{n+1} C \right)^z a_j \sum_{k=2}^{n-3} a_j \left( a_n - b_n - \frac{n-1}{n} C \right)^z + \sum_{k=0}^{n-3} (-1)^{n-k+1} b_k b_{n-k+1} \cdots b_{n-3} \times \left[ \frac{k+1}{n+1} C (a_k - b_k) + \frac{k+2}{n+1} C (a_{k+2} - b_{k+2}) \right] \psi_{n-1}.$$

Furthermore, $|\beta_n| = |\psi_n(0)| \neq 1$, $n \geq 1$, and

$$\beta_1 = (a_1 - b_1) + \frac{1}{2} C, \quad \beta_2 = \frac{1}{2} C (a_2 - b_2),$$

and for $n \geq 3$,

$$\beta_n = (-1)^{n-1} b_n \sum_{k=0}^{n-3} \left( b_k (a_k - b_k - \frac{1}{2} C) (a_k - b_k) \right)$$

$$= \left( -b_n b_{n-1} \right),$$

$$a_n + b_n \left( \frac{1}{2} C \right)^2 - 1$$

$$= -\frac{1}{n+1} C + \frac{1}{2} C \sum_{k=2}^{n-k} a_j \left( b_k (a_k - b_k) \right)^2,$$

for $n \geq 2$.

Proof. Since $\psi_n^{(1)}(z) = z^n + n/(n+1) C z^{n-1}$, for $n \geq 0$, then, from (19), we get

$$v_n = -\frac{n}{n+1} C v_{n-1} + (-1)^n (a_1 - b_1) \prod_{j=2}^{n-k} a_j v_0,$$

where $\prod_{j=2}^{n-k} a_j = 1$ whenever $k_2 < k_1$. From (36) and using induction on $n$, it is easy to verify that the moments of $\mathcal{V}$ are given by (32). Besides, from (18) and (33), (34) holds. Furthermore, since $(\psi_n(z))_{n \geq 0}$ is a sequence of monic OPUC, then it follows that $|\beta_n| \neq 1$, $n \geq 1$.

On the other hand, from the forward Szegő relation and (33), we can get another expression of $\psi_n(z)$, for $n \geq 2$. Hence, comparing the coefficients of $z$ and using (34), (35) follows.

As in the previous section, we are interested in the situations where $\mathcal{V}$ is also a positive definite linear functional. Notice now that the values of $a_1, b_1, a_2, b_2, a_3$ determine all other coherence coefficients. We have the following cases.

Proposition 6. Let $\mathcal{U}$ be the Bernstein-Szegő linear functional, and let $(\mathcal{U}, \mathcal{V})$ be a $(1, 1)$-coherent pair on the unit circle given by (16). Then, one has the following.

(i) If $a_1 = b_1$, then $C = 0$ and, therefore, $\mathcal{U}$ and $\mathcal{V}$ are Lebesgue linear functionals, and $a_n = b_n$ for $n \geq 1$.

(ii) Let $a_1 \neq b_1$.

(1) If $\mathcal{V}$ is normalized (i.e., $v_0 = 1$) and $b_N = 0$ for some $N \geq 3$, then $C = 0$; this is, $\mathcal{U}$ is the Lebesgue linear functional. As a consequence, $b_{n+1} = 0$, $a_{n+1} = a_n - b_n$, $\psi_n(z) = z^n + a_n - b_n$, and $v_n = (a_n - b_n)^{n+1}$ for every $n \geq 1$. In other words, for $|b_n - a_n| < 1$, $\mathcal{V}$ is the linear functional associated with the Bernstein-Szegő measure, with parameter $b_1 - a_1$.

(2) If $(1/2)C a_2 = b_1 b_3$, then $\psi_n(z) = z^{n-1}(z + a_n - b_n)$, and $v_n = (a_n - b_n)^{n+1}$ for every $n \geq 1$. In other words, for $|b_n - a_n| < 1$, $\mathcal{V}$ is the linear functional associated with the Bernstein-Szegő measure, with parameter $b_1 - a_1 - (1/2)C$.

(3) If $(1/2)C a_2 \neq b_1 b_3$ and $b_1 \neq 0$, for $n \geq 3$, then

$$b_n = \frac{b_n-1}{1 - |\beta_n|^{-2}},$$

$$= \frac{b_n}{\prod_{k=2}^{n-1} (1 - |\beta_k|^{-2})}, \quad n \geq 4,$

and $b_n$ can be chosen so that $\mathcal{V}$ is the linear functional associated with an antiaassociated perturbation of order 2 applied to a Nevai measure.

Proof. (i) If we multiply (33) by $z^{-1}$ and apply $\mathcal{V}$, then we get, for $n \geq 2$,

$$0 = v_{n-2} + \sum_{j=2}^{n-1} \left( a_j - b_j \right) \sum_{k=0}^{n-1} \prod_{j=2}^{n-k} a_j v_0,$$

$$= \left[ b_1 (a_1 - b_1 - \frac{1}{n-1} C (a_n - b_n)) \right] v_{n-3} + \sum_{k=0}^{n-k-1} (-1)^{n-k} b_k b_{n-k} \cdots b_{k+3} \times \left[ b_{k+2} (a_{k+1} - b_{k+1} - \frac{k+1}{k+2} C (a_{k+2} - b_{k+2})) \right] v_{k-1}.$$
If we multiply this equation by $b_{n+1}$ and we add it to the previous equation for $n+1$, then we obtain

$$0 = v_n + \left( a_{n+1} + \frac{n+1}{n+2}C \right) v_{n-1} + \frac{n}{n+1}Ca_{n-1}v_{n-2}, \quad n \geq 2. \quad (39)$$

Hence, from (39) and (36), it follows that

$$0 = (-1)^{n+1}(a_1 - b_1) \times \prod_{j=2}^{n+1} a_j \left[ a_{n+3} - a_{n+2} + \frac{1}{(n+3)(n+4)}C \right]$$

$$+ \left[ \frac{1}{(n+2)(n+3)}a_{n+3} + \frac{n+1}{(n+2)(n+3)(n+4)}C \right] C v_n, \quad n \geq 0. \quad (40)$$

On the other hand, if we apply the linear functional $\mathcal{V}$ to both sides of the $(1,1)$-coherence relation (16), we get $v_1 + \left[ a_1 + (C/2) \right] v_0 = b_1 v_0$ and

$$v_n + \left[ a_n + \frac{Cn}{n+1} \right] v_{n-1} + a_n \frac{(n-1)}{n} v_{n-2} = 0, \quad n \geq 2. \quad (41)$$

Thus, from (39) and (41), we obtain, for $n \geq 2$,

$$0 = \left[ a_{n+1} - a_n + \frac{C}{(n+1)(n+2)} \right] v_{n-1}$$

$$+ \left[ \frac{na_{n+1}}{n+1} - \frac{(n-1)a_n}{n} \right] C v_{n-2}. \quad (42)$$

Therefore, if $a_1 = b_1$, then from (32), the moments of $\mathcal{V}$ are $v_n = (1/(n+1))(-C)^n v_0$ for $n \geq 0$, and, as a consequence, (40) becomes

$$0 = (-1)^n \frac{1}{(n+1)(n+2)(n+3)} C^{n+1} \left[ a_{n+3} + \frac{n+1}{n+4} \right] v_0, \quad n \geq 0, \quad (43)$$

and (42) is, for $n \geq 2$,

$$0 = (-1)^{n+1} \frac{1}{n(n+1)} C^{n+1}$$

$$\times \left[ \frac{1}{n+2} C - \frac{1}{n-1} a_{n+1} \right] v_0, \quad n \geq 2. \quad (44)$$

Then, if $C \neq 0$, from (43) and (44) it follows that $a_n = -((n-\frac{1}{2})/(n+1))C$, for $n \geq 3$, and $a_n = ((n-2)/(n+1))C$, for $n \geq 3$, respectively, which is a contradiction. Thus, if $a_1 = b_1$, then $C = 0$; that is, $\mathcal{V}$ is the Lebesgue linear functional, and in case the part i of Theorem 3 holds.

Now, let us assume $a_1 \neq b_1$.

(ii) From part (i) of Proposition 4, it suffices to show that $\mathcal{V}$ is the Lebesgue linear functional. Thus, let us prove that if $b_N = 0$ for some $N \geq 3$ (and therefore $\beta_N = 0$), then $C = 0$. Indeed, if $b_N = 0$ for some $N \geq 3$, then from (33) for $n = N + 1, N \geq 2$, it follows that $\beta_{N+1} = 0$, for $N \geq 3$. Furthermore, from the forward Szegő relation and (33) for $n = N$, we obtain an expression of $\psi_{N+1}(z)$, for $N \geq 3$. Hence, comparing the coefficients of this expression and (33) for $n = N + 1$, we obtain, for $N \geq 3$,

$$-b_{N+1}a_N + \frac{N+1}{N+2} \frac{C}{(N+1)C a_N} = \frac{N-1}{N} C a_N, \quad (46)$$

$$b_{N+1} \frac{N-1}{N} C a_N = 0. \quad (47)$$

Since $a_N \neq 0$, then from (47) it follows that either $C = 0$ or $b_{N+1} = 0$. If $C = 0$, then from (46) we get $b_{N+1} = 0$, and, as a consequence, from (45) we have $a_{N+1} = a_N$. If $b_{N+1} = 0$, then from (46) it follows that either $C = 0$ (and thus, from (45), $a_{N+1} = a_N$) or $a_{N+1} = ((N^2-1)/N^2)a_N$. If $b_{N+1} = 0$ and $a_{N+1} = ((N^2-1)/N^2)a_N$, from (45) it follows that $C = ((N+2)(N+3)/N^3)\beta_{N+1}a_N$, and since $a_{N+1} = ((N^2-1)/N^2)a_N$, then we also have $C = ((N+2)(N+3)(N-1)/(N+1)^2)\beta_{N+1}a_N$, which yields a contradiction. Therefore, $C = 0$.

(iii) From the forward Szegő relation and (33) we obtain an expression of $\psi_{n}(z)$, for $n \geq 3$. If we compare the coefficients of $z$ of this expression and (33), we get $b_2[b_4 - b_3] = b_4b_2\sum_{k=3}^{N} |b_k| b_{k-1}|^2$ and

$$b_{n-1} \cdots b_{k+1} \frac{n-1}{n} \sum_{k=3}^{N} |b_k| b_{k-1}|^2, \quad n \geq 5. \quad (48)$$

Thus, if $(1/2)Ca_2 \neq b_2$, then from (34) it follows that $\beta_2 \neq 0$, and, as a consequence, if $b_4, \ldots, b_{n-1}, n \geq 5$, are nonzero, then from (48) we get

$$b_n = \frac{b_1}{1 - \sum_{k=3}^{N} |b_k| b_{k-1}|^2}, \quad n \geq 4. \quad (49)$$

Besides, from (34), $|\beta_2| = |b_4b_{n-1}|$ for $n \geq 3$, and if $b_2 \neq 0$ then by induction on $n$ we can prove that $b_n = b_{n-1}/(1 - |\beta_{n-1}|^2)$, for $n \geq 4$, which is (37). Therefore, proceeding as in the proof of Proposition 4, we can choose $|b_1|$ small enough so that $\beta_3$ is sufficiently close to 0. As a consequence, $|\beta_3|_{n \geq 3}$ will be an increasing sequence, and hence $|\beta_3|_{n \geq 3}$ will be a decreasing sequence. Also, we can choose $b_1$ such that $|b_1|$ converges to a constant $b$, with $0 < b < 1$. The infinite product $\prod_{k=3}^{N} (1 - |\beta_k|^2)$ will then converge to $|b_3|^2/b$. Therefore, since $|\beta_3|_{n \geq 3}$ are the Verblunsky coefficients of $\mathcal{V}$, this linear functional $\mathcal{V}$ is an antiasociated perturbation of order 2 (see [24]) applied to a Neva measure $\mu$. \qed
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