Research Article

On Abstract Economies and Their Applications

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We establish a new equilibrium existence theorem of generalized abstract economies with general preference correspondences. As an application, we derive an existence theorem of generalized quasi-variational inequalities in the general setting of l.c.-spaces without any linear structure.

1. Introduction and Preliminary

Let \( I \) be any (finite or infinite) set of agents. A generalized abstract economy is defined as a family of order quintuples \( \Omega = (X, \Gamma, A, B, \Gamma_a, P_a)_{a \in I} \) with \( X := \prod_{a \in I} X_a \) such that for each \( a \in I \), \( X_a \) is a topological space, \( A_a, B_a : X \to 2^{X_a} \) are constraint correspondences, \( F_a : X \to 2^{X_a} \) is a fuzzy constraint correspondence, and \( P_a : X \times X \to 2^{X_a} \) is a preference correspondence. In a real market, any preference of a real agent would be unstable by the fuzziness of consumers’ behavior or market situations. Thus, it is reasonable to introduce fuzzy constraint correspondences in defining an abstract economy. An equilibrium point of \( \Omega \) is a point \( (\tilde{x}, \tilde{y}) \in X \times X \) such that for each \( a \in I \), \( \tilde{x}_a \in \text{cl} B_a(\tilde{x}), \tilde{y}_a \in F_a(\tilde{x}), \text{ and } A_a(\tilde{x}) \cap P_a(\tilde{x}, \tilde{y}) = \emptyset \), where \( \tilde{x}_a \) and \( \tilde{y}_a \) denote the projections of \( \tilde{x} \) and \( \tilde{y} \) from \( X \) to \( X_a \), respectively.

In case \( F_a(x) = X_a \) for each \( x \in X \) and \( P_a \) is independent of the second variable, that is, \( P_a : X \to 2^{X_a} \), the above generalized abstract economy reduces to the standard abstract economy \( \Omega_s := (X_a, A_a, B_a, \Gamma_a)_{a \in I} \). In which an equilibrium point of \( \Omega_s \) is a point \( \tilde{x} \in X \) such that for each \( a \in I \), \( \text{cl} B_a(\tilde{x}), \text{ and } A_a(\tilde{x}) = \emptyset \). When \( A_a = B_a \) and each \( X_a \) is a topological vector space, the standard abstract economy \( \Omega_s \) coincides with the classical definition of Shapley and Sonnenschein [1]. For more details on abstract economies, see, for example, [2–14] and the references therein.

Throughout this paper, all topological spaces are assumed to be Hausdorff. In order to establish our main results, we first give some basic notations. For a nonempty set \( C \) of a topological space \( X \), we denote the set of all subsets of \( C \) by \( 2^C \), the set of all nonempty finite subsets of \( C \) by \( \langle X \rangle \), the interior of \( C \) by \( \text{int} C \), and the closure of \( C \) by \( \text{cl} C \).

Let \( \{\Gamma_D\} \) be a family of some nonempty contractible subsets of a topological space \( X \) indexed by \( D \in \langle X \rangle \) such that \( \Gamma_D \subset \Gamma_D' \) whenever \( D \subset D' \). The pair \( (X, \{\Gamma_D\}) \) is called an H-space. Given an H-space \( (X, \{\Gamma_D\}) \), a nonempty subset \( C \) of \( X \) is said to be \( H \)-convex if \( \Gamma_D \subset C \) for all \( D \in \langle C \rangle \). For a nonempty subset \( C \) of \( X \), we define the \( H \)-convex hull of \( C \) as

\[
H \text{-co} C := \bigcap \{K \mid K \text{ is } H \text{-convex in } X \text{ and } C \subseteq K\}.
\] (1)

It is known that if \( x \in H \text{-co} C \), then there exists a finite subset \( D \) of \( C \) such that \( x \in H \text{-co} D \). Moreover, for any \( D \in \langle X \rangle \), \( H \text{-co} D \) is called a polytope. We will say that \( X \) is an \( H \)-space with precompact polytopes if every polytope of \( X \) is precompact. For example, a locally convex topological vector space \( X \) is an \( H \)-space with precompact polytopes, by setting \( \Gamma_D = \text{co} D \) for all \( D \in \langle X \rangle \).

An \( H \)-space \( (X, \{\Gamma_D\}) \) is called an l.c.-space if \( X \) is a uniform space whose topology is induced by its uniformity \( \mathcal{U} \), and there is a base \( \mathcal{B} \) consisting of symmetric entourages in \( \mathcal{U} \) such that for each \( V, V' \in \mathcal{B} \), the set \( V(E) := \{y \in X \mid (x, y) \in V \text{ for some } x \in E\} \) is \( H \)-convex whenever \( E \) is \( H \)-convex.

We will use the notation \( (X, \mathcal{U}, \mathcal{B}) \) to stand for an l.c.-space. For details of uniform spaces, we refer to [15]. In a recent
paper [16], we introduce a new measure of precompactness of a subset $A$ in an l.c.-space $(X, \mathcal{U}, \mathcal{B})$ by

$$Q(A) := \{ V \in \mathcal{B} \mid A \subseteq cl(V(K)) \}$$

for some precompact set $K$ of $X$.

Let $(X_{\alpha}, \mathcal{U}_{\alpha}, \mathcal{B}_{\alpha})_{\alpha \in \mathcal{I}}$ be a family of l.c.-spaces with precompact polytopes, where $I$ is a finite or infinite index set and $X = \prod_{\alpha \in I} X_{\alpha}$. For each $\alpha \in I$, let $\pi_{\alpha}$ be the projection of $X$ onto $X_{\alpha}$ and $Q_{\alpha}$ a measure of precompactness in $X_{\alpha}$. We say that a set-valued mapping $T_{\alpha} : X \to 2^{X_{\alpha}}$ is $Q_{\alpha}$-condensing if $Q_{\alpha}(\pi_{\alpha}(C)) \subseteq Q_{\alpha}(T_{\alpha}(C))$ for every $C$ satisfying $\pi_{\alpha}(C)$ is a nonprecompact subset of $X_{\alpha}$. It is clear that for any set-valued mapping $T : X \to 2^{Y}$ and any measure $Q$ in $Y$, $T$ is $Q$-condensing whenever $Y$ is compact.

Let $X$ be a topological space, let $Y$ be an $H$-space, and let $S, T : X \to 2^{Y}$ be two set-valued mappings.

1. $T$ is said to be upper semicontinuous (u.s.c.) if for each $x \in X$ and each open subset $V$ of $Y$ with $T(x) \subseteq V$, there exists a neighborhood $N_x$ of $x$ such that $T(z) \subseteq V$ for all $z \in N_x$.

2. $T$ is said to be transfer open valued on $X$ if for each $x \in X$, for each $y \in T(x)$, there exists some $x' \in X$ such that $y \in \text{int}T(x')$.

3. $T$ is said to be transfer open inverse valued in $Y$ if $T^{-1}$ is transfer open valued on $Y$, where $T^{-1} : Y \to 2^{X}$ is defined by

$$T^{-1}(y) := \{ x \in X \mid y \in T(x) \} \quad \forall y \in Y.$$  

4. The set-valued mappings $S \cap T : X \to 2^{Y}$ and $\text{cl}T : X \to 2^{Y}$ are defined by

$$(S \cap T)(x) := S(x) \cap T(x),$$

$$(\text{cl}T)(x) := \text{cl}(T(x)), \quad \forall x \in X.$$

Further, we denote by $\mathfrak{H}(X, Y)$ the class of all u.s.c. set-valued mappings $T : X \to 2^{Y}$ with nonempty closed $H$-convex values.

### 2. Main Results

The following fundamental theorems will play an important role in proving our main theorem.

**Theorem A** (see [16]). Let $(X_{\alpha}, \mathcal{U}_{\alpha}, \mathcal{B}_{\alpha})_{\alpha \in \mathcal{I}}$ be a family of l.c.-spaces with precompact polytopes, $X := \prod_{\alpha \in I} X_{\alpha}$, and let $T_{\alpha} : X \to 2^{X_{\alpha}}$ be $Q_{\alpha}$-condensing. Then there exists a nonempty compact $H$-convex subset $K := \prod_{\alpha \in \mathcal{I}} K_{\alpha}$ of $X$ such that $T_{\alpha}(K) \subseteq K_{\alpha}$.

**Theorem B** (see [16]). Let $(X_{\alpha}, \mathcal{U}_{\alpha}, \mathcal{B}_{\alpha})_{\alpha \in \mathcal{I}}$ be a family of l.c.-spaces with precompact polytopes and $X := \prod_{\alpha \in I} X_{\alpha}$. If $T_{\alpha} : X \to 2^{X_{\alpha}}$ is an u.s.c. $Q_{\alpha}$-condensing mapping with closed $H$-convex values for each $\alpha \in I$, then $T := \prod_{\alpha \in \mathcal{I}} T_{\alpha}$ has a fixed point.

Next, we list and establish some essential lemmas as follows.

**Lemma 1** (see [12]). If $X$ is an l.c.-space and $E$ is an $H$-convex subset of $X$, then $\text{cl}E$ is also $H$-convex.

**Lemma 2** (see [12]). Let $X$ be a topological space and let $(Y, \Gamma_{\alpha})$ be a compact l.c.-space. If $T : X \to 2^{Y}$ is an u.s.c. set-valued mapping, then the mapping $x \mapsto \text{cl}[H-coT(x)]$ is also u.s.c. with compact $H$-convex values.

**Lemma 3** (see [7]). Let $X$ and $Y$ be topological spaces and let $S : X \to 2^{Y}$ be a transfer open valued mapping. Then $\bigcup_{x \in X} S(x) = Y \setminus \bigcap_{x \in X} \text{cl}(Y \setminus S(x))$ and hence $\bigcup_{x \in X} S(x)$ is open in $Y$.

**Lemma 4.** Let $X$ be paracompact, $(Y, \Gamma_{\alpha})$ an $H$-space, and $S, T : X \to 2^{Y}$ be two set-valued mappings such that

1. $S(x) \neq \emptyset$ and $H-coS(x) \subseteq T(x)$ for each $x \in X$,
2. $S$ is transfer open inverse valued in $Y$.

Then $T$ has a continuous selection; that is, there exists a continuous function $f : X \to Y$ such that $f(x) \in T(x)$ for each $x \in X$.

**Proof.** Since for each $x \in X$, $S(x) \neq \emptyset$, it follows that $x \in S^{-1}(y)$ for some $y \in Y$. Since $S$ is transfer open inverse valued in $Y$, there exists some $y_{x} \in Y$ such that $x \in \text{int} S^{-1}(y_{x})$. This yields that $\text{int}S^{-1}(y_{x}) \subseteq \text{int}S^{-1}(y)$ forms an open cover of $X$. Since $X$ is paracompact, there exists a locally finite open cover $(U_y \mid y \in Y)$ such that $U_y \subseteq \text{int}S^{-1}(y)$ for each $y \in Y$. By [17, Theorem 3.1], there exists a continuous function $f : X \to Y$ such that $f(x) \in \Gamma_{\{ y \mid x \in \text{int}S^{-1}(y) \}}$ for all $x \in X$. Note that for any $x \in X$, there exist finitely many $y \in Y$ such that $x \in \bigcup_{y \in Y} \text{int}S^{-1}(y) \subseteq S^{-1}(y)$. This implies $y \in S(x) \subseteq H-coS(x)$, and hence $\{ y \mid x \in U_y \} \in (H-coS(x))$. It follows that for each $x \in X$, we get

$$f(x) \in \bigcap_{y \in \text{int}S^{-1}(y)} \bigcap_{x \in U_y} H-coS(x) \subseteq H-coS(x) \subseteq T(x).$$

Thus, the proof is complete.

We remark that Lemma 4 extends [7, Theorem 2] from topological vector spaces to general $H$-spaces. When $S = T$ and $S$ has open lower sections, Lemma 4 reduces to [18, Theorem 3.1].

**Lemma 5.** Let $X$ be a compact $H$-space, and let $P : X \times X \to 2^{X}$ be a set-valued mapping such that for each $x \in X$, $P^{-1}(x)$ is open: then so is $(H-coP)^{-1}(x)$.

**Proof.** For each $z_0 \in X$, we fix an $(x_0, y_0) \in (H-coP)^{-1}(z_0)$. Since $z_0 \in H-coP(x_0, y_0)$, there is a finite set $\{ z_1, z_2, \ldots, z_n \}$ in $P(x_0, y_0)$ such that $z_0 \in H-co\{ z_1, z_2, \ldots, z_n \}$. Since each $P^{-1}(z)$ is open, it follows that the set $U := \bigcap_{i=1}^{n} P^{-1}(z_i)$ is also open and $(x_0, y_0) \in U$. To complete the proof, we will show that $U \subseteq (H-coP)^{-1}(z_0)$. For any $(x, y) \in U$, we have...
\((x, y) \in P^{-1}(z_i)\) for all \(i = 1, 2, \ldots, n\). Accordingly, \(z_i \in P(x, y)\) for all \(i = 1, 2, \ldots, n\). Hence,

\[
z_0 \in H \text{-co} \{z_1, z_2, \ldots, z_n\} \subseteq H \text{-co} P(x, y).
\] (6)

That is, \((x, y) \in (H \text{-co} P)^{-1}(z_0)\). Consequently, \(U \subseteq (H \text{-co} P)^{-1}(z_0)\).

**Theorem 6.** Let \(\Omega = (X_\alpha, A_\alpha, B_\alpha, F_\alpha, P_\alpha)_{\alpha \in I}\) be a generalized abstract economy, where \(I\) is a set of agents and \(X = \prod_{\alpha \in I} X_\alpha\) such that for each \(\alpha \in I\),

1. \(X_\alpha\) is an l.c.-space with precompact polytopes,
2. \(A_\alpha(x) \subseteq cl B_\alpha(x)\) for each \(x \in X\),
3. both \(cl B_\alpha\) and \(F_\alpha\) are \(Q_\alpha\)-condensing mappings in \(H(X, X_\alpha)\),
4. \(x_\alpha \notin cl(H \text{-co} P_\alpha(x, y))\) for each \(x, y \in X\),
5. \(A_\alpha \cap (H \text{-co} P_\alpha)\) is transfer open inverse valued in \(X_\alpha\),
6. \(W_\alpha := \{(x, y) \in X \times X \mid A_\alpha(x) \cap (H \text{-co} P_\alpha(x, y)) \neq \emptyset\}\) is paracompact.

Then \(\Omega\) has an equilibrium point \((\tilde{x}, \tilde{y}) \in X \times X\).

**Proof.** For each \(\alpha \in I\), we define \(\phi_\alpha : X \times X \to 2^{X_\alpha}\) by

\[
\phi_\alpha(x, y) := A_\alpha(x) \cap (H \text{-co} P_\alpha(x, y)), \quad \forall (x, y) \in X \times X.
\] (7)

Assume that \(W_\alpha \neq \emptyset\). Then for each \((x, y) \in W_\alpha\), we have some \(y_\alpha \in \phi_\alpha(x, y)\). Equivalently, \((x, y) \in \phi_\alpha^{-1}(y_\alpha)\). It follows that \(W_\alpha = \bigcup_{y \in \phi_\alpha^{-1}(y_\alpha)} W^\alpha\). Since each \(\phi_\alpha\) is transfer open inverse valued in \(X_\alpha\) by (5), it follows from Lemma 3 that \(W_\alpha\) is open in \(X \times X\).

For \(z_\alpha \in X_\alpha\), if \((x, y) \in \phi_\alpha^{-1}(z_\alpha)\), by using (5), we have some \(z_\alpha' \in X_\alpha\) such that \((x, y) \in \inf \phi_\alpha^{-1}(z_\alpha') \subseteq W_\alpha\). Thus, the restriction \(\phi_\alpha|_{W_\alpha} : W_\alpha \to 2^{X_\alpha}\) is transfer open inverse valued in \(X_\alpha\). Moreover, by (3), each \(\phi_\alpha|_{W_\alpha}(x, y)\) is nonempty and \(H\)-convex. Therefore, by Lemma 4, there exists a continuous function \(f_\alpha : W_\alpha \to X_\alpha\) such that \(f_\alpha(x, y) \in \phi_\alpha|_{W_\alpha}(x, y)\) for each \((x, y) \in W_\alpha\).

Since \(cl B_\alpha\) and \(F_\alpha\) are \(Q_\alpha\)-condensing, applying Theorem A, we have two nonempty compact \(H\)-convex subsets \(K := \prod_{\alpha \in I} X_\alpha\) and \(K' := \prod_{\alpha \in I} X'_\alpha\) of \(X\) such that \(cl B_\alpha(K) \subseteq K_\alpha\) and \(F_\alpha(K') \subseteq K'_\alpha\). Using these notations, we define a set-valued mapping \(S_\alpha : K \times K' \to 2^{K_\alpha \times K'_\alpha}\) by

\[
S_\alpha(x, y) = \begin{cases} cl(H \text{-co} f_\alpha(x, y)) \times F_\alpha(x), & \text{if } (x, y) \in (K \times K') \cap W_\alpha, \\ cl B_\alpha(x) \times F_\alpha(x), & \text{if } (x, y) \in (K \times K') \setminus W_\alpha. \end{cases}
\] (8)

We will show that \(S_\alpha \in H(K \times K', K_\alpha \times K'_\alpha)\). Let \(U_\alpha\) be an open subset of \(K_\alpha \times K'_\alpha\). Since \(cl[H \text{-co} f_\alpha(x, y)] \subseteq cl[H \text{-co} f_\alpha(x, y)] \subseteq cl B_\alpha(x) \times F_\alpha(x)\) for each \((x, y) \in W_\alpha\), we have

\[
U_\alpha = \{(x, y) \in K \times K' \mid S_\alpha(x, y) \subseteq U_\alpha\} = \{(x, y) \in K \times K' \mid S_\alpha(x, y) \subseteq U_\alpha\} 
\times F_\alpha(x) \subseteq V_\alpha \\
\cup \{(x, y) \in K \times K' \setminus W_\alpha \mid cl B_\alpha(x) \times F_\alpha(x) \subseteq U_\alpha\} = \{(x, y) \in K \times K' \setminus W_\alpha \mid cl B_\alpha(x) \times F_\alpha(x) \subseteq U_\alpha\} \\
\times F_\alpha(x) \subseteq V_\alpha \cup \{(x, y) \in K \times K' \mid cl B_\alpha(x) \times F_\alpha(x) \subseteq U_\alpha\}.
\] (9)

It follows from Lemma 2 and the upper semicontinuity of \(cl B_\alpha \times F_\alpha\) that \(U_\alpha\) is open in \(K \times K'\). Hence, \(S_\alpha\) is u.s.c. Further, by (3) and Lemma 1, each \(S_\alpha\) is nonempty, closed, and \(H\)-convex. Therefore, \(S_\alpha \in H(K \times K', K_\alpha \times K'_\alpha)\).

Next, we define a set-valued mapping \(T_\alpha : K \times K' \to 2^{K_\alpha \times K'_\alpha}\) by

\[
T_\alpha(x, y) = \begin{cases} S_\alpha(x, y), & \text{if } (x, y) \in (K \times K') \cap W_\alpha, \\ cl B_\alpha(x) \times F_\alpha(x), & \text{if } (x, y) \in (K \times K') \setminus W_\alpha. \end{cases}
\] (10)

Since \(K_\alpha \times K'_\alpha\) is compact, each \(T_\alpha\) is \(Q_\alpha\)-condensing in \(H(K \times K', K_\alpha \times K'_\alpha)\). Hence, by Theorem B, the set-valued mapping \(\prod_{\alpha \in I} T_\alpha\) has a fixed point \((\tilde{x}, \tilde{y}) \in K \times K'\); that is, \((\tilde{x}_\alpha, \tilde{y}_\alpha) \in T_\alpha(\tilde{x}, \tilde{y})\) for each \(\alpha \in I\). If \((\tilde{x}, \tilde{y}) \in W_\alpha\), then

\[
(\tilde{x}_\alpha, \tilde{y}_\alpha) \in cl(S_\alpha(\tilde{x}, \tilde{y})) \\
\subseteq cl(B_\alpha(x) \times (H \text{-co} P_\alpha(x, y))) \times F_\alpha(x).
\] (11)

Thus, \(\tilde{x}_\alpha \in \phi(\tilde{S}_\alpha(\tilde{x}, \tilde{y}), \tilde{y})\), which contradicts with (4). Therefore, \((\tilde{x}, \tilde{y}) \notin W_\alpha\) and hence \(\tilde{x}_\alpha \in cl B_\alpha(\tilde{x}), \tilde{y}_\alpha \in F_\alpha(\tilde{x}),\) and \(A_\alpha(\tilde{x}) \cap P_\alpha(\tilde{y}) = \emptyset\) for each \(\alpha \in I\). That is, \((\tilde{x}, \tilde{y})\) is an equilibrium of \(\Omega\).

Remark that condition (4) of Theorem 6 can be replaced by a milder condition \(x_\alpha \notin cl(B_\alpha(x) \times (H \text{-co} P_\alpha(x, y)))\) for each \((x, y) \in W_\alpha\). Further, when each l.c.-space \((X_\alpha, \Gamma_\alpha)\) satisfies \(\Gamma_\alpha(x) = \{x_\alpha\}\) for each \(\alpha \in I\), condition (4) can be modified by \(x_\alpha \notin H \text{-co} P_\alpha(x, y)\) without affecting the conclusion.

**Corollary 7.** Let \(\Omega = (X_\alpha, A_\alpha, B_\alpha, F_\alpha, P_\alpha)_{\alpha \in I}\) be a generalized abstract economy, where \(I\) is a set of agents and \(X = \prod_{\alpha \in I} X_\alpha\) such that for each \(\alpha \in I\),

1. \((X_\alpha, \Gamma_\alpha)\) is an l.c.-space with precompact polytopes, and
2. \(\Gamma_\alpha(x) = \{x_\alpha\}\) for each \(x_\alpha \in X_\alpha\).
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(2) \( A_\alpha(x) \subseteq \text{cl}B_\alpha(x) \) for each \( x \in X \),
(3) both \( \text{cl}B_\alpha \) and \( F_\alpha \) are \( \mathcal{Q}_\alpha \)-condensing mappings in \( H(X, X_\alpha) \),
(4) \( x_\alpha \notin H\cdot\text{co}P_\alpha(x, y) \) for each \( x, y \in X \),
(5) \( A_\alpha \cap (H\cdot\text{co}P_\alpha) \) is transfer open inverse valued in \( X_\alpha \).

Then \( \Omega \) has an equilibrium point \((\bar{x}, \bar{y}) \in X \times X \).

Proof. According to the proof of Theorem 6 and by virtue of the condition \( \Gamma_{\{x_\alpha\}} := \{x_\alpha\} \) for each \( x_\alpha \in X_\alpha \), we obtain \( \text{cl}(H\cdot\text{co}P_\alpha(x, y)) = f_\alpha(x, y) \). It follows that the set-valued mapping \( S_\alpha \) can be defined by

\[
S_\alpha(x, y) = \begin{cases} f_\alpha(x, y) \times F_\alpha(x), & \text{if } (x, y) \in (K \times K') \setminus W_\alpha, \\ \text{cl}B_\alpha(x) \times F_\alpha(x), & \text{if } (x, y) \in (K \times K') \setminus W_\alpha. \end{cases}
\]

Thus, by an analogue proof to Theorem 6, we may conclude that \( \Omega \) has an equilibrium point.

Following the proof of Theorem 6 by taking \( \phi_\alpha(x, y) := \text{cl}B_\alpha(x) \cap (H\cdot\text{co}P_\alpha(x, y)) \), we may obtain a new version of equilibrium existence theorem as follows.

Corollary 8. Let \( \Omega = (X_\alpha, A_\alpha, B_\alpha, F_\alpha, P_\alpha)_{\alpha \in I} \) be a generalized abstract economy, where \( I \) is a set of agents and \( X = \prod_{\alpha \in I} X_\alpha \) such that for each \( \alpha \in I \),

(1) \( X_\alpha \) is an \( l.c. \)-space with precompact polytopes,
(2) \( A_\alpha(x) \subseteq \text{cl}B_\alpha(x) \) for each \( x \in X \),
(3) both \( \text{cl}B_\alpha \) and \( F_\alpha \) are \( \mathcal{Q}_\alpha \)-condensing mappings in \( H(X, X_\alpha) \),
(4) \( x_\alpha \notin \text{cl}(H\cdot\text{co}P_\alpha(x, y)) \) for each \( x, y \in X \),
(5) \( \text{cl}B_\alpha \cap (H\cdot\text{co}P_\alpha) \) is transfer open inverse valued in \( X_\alpha \),
(6) \( W_\alpha := \{(x, y) \in X \times X \mid \text{cl}B_\alpha(x) \cap (H\cdot\text{co}P_\alpha(x, y)) \neq \emptyset\} \) is paracompact.

Then \( \Omega \) has an equilibrium point \((\bar{x}, \bar{y}) \in X \times X \).

Notice that Theorem 6 generalizes [7, Kim-Tan, Theorem 2], in which they deal with the case of locally convex topological vector spaces under some compactness conditions, and it also improves [19, Wu-Yuan, Theorem 3] in the setting of locally \( H \)-convex spaces. We also note that if \( X \) is metrizable, the set \( W_\alpha \) is also metrizable and hence is paracompact. Therefore, the assumption (6) of Theorem 6 is automatically satisfied. Furthermore, if each \( X_\alpha \) is compact, then both \( \text{cl}B_\alpha \) and \( F_\alpha \) are obviously \( \mathcal{Q}_\alpha \)-condensing. Thus, we have an immediate consequence, which is a generalization of [7, Kim-Tan, Corollary 1] to \( H \)-spaces.

Corollary 9. Let \( \Omega = (X_\alpha, A_\alpha, B_\alpha, F_\alpha, P_\alpha)_{\alpha \in I} \) be a generalized abstract economy, where \( I \) is a set of agents such that for each \( \alpha \in I \),

(1) \( (X_\alpha, \Gamma_\alpha^\alpha) \) is a metrizable compact \( l.c. \)-space, and \( \Gamma_\alpha^\alpha_{\{x_\alpha\}} = \{x_\alpha\} \) for each \( x_\alpha \in X_\alpha \),
(2) \( A_\alpha(x) \subseteq \text{cl}B_\alpha(x) \) for each \( x \in X \),
(3) \( \text{cl}B_\alpha \in H(X, X_\alpha) \), and \( F_\alpha \in \mathcal{H}(X, X_\alpha) \),
(4) \( x_\alpha \notin H\cdot\text{co}P_\alpha(x, y) \) for each \( x, y \in X \),
(5) \( A_\alpha \cap (H\cdot\text{co}P_\alpha) \) is transfer open inverse valued in \( X_\alpha \).

Then \( \Omega \) has an equilibrium point \((\bar{x}, \bar{y}) \in X \times X \).

We note that our main results focus on the setting of general \( l.c. \)-spaces without any linear structure; further, the correspondences are not necessarily lower semicontinuous and do not require the usual open lower section assumption, such as the earlier works [3, Theorem 4], [13, Theorem 3 and its Corollary], [19, Theorem 1 and 3], and [18, Theorem 6.1]. In fact, we can give a simple example applicable for Corollary 9, while previous results do not.

Example 10. Consider the set \( I \) of agents is singleton. Let \( X = [0, 1] \) and the correspondences \( A, B, F : X \to 2^X \) be defined by \( A(x) = B(x) = \{0, 1\} \), and \( F(x) = \{x\} \) for each \( x \in X \). The preference correspondence \( P : X \times X \to 2^X \) is defined as follows:

\[
P(x_1, x_2) = \begin{cases} \left\{ \frac{x_1 + x_2}{2}, 1 \right\}, & \text{if } x_1 < x_2, x_1, x_2 \in Q, \\ \left\{ \frac{x_1 + 2x_2}{3}, 1 \right\}, & \text{if } x_1 < x_2, x_1 \notin Q, \\ 0, & \text{if } x_1 = x_2, \\ \left\{ 0 \right\}, & \text{if } x_1 > x_2. \end{cases}
\]

Then \( A \cap (H\cdot\text{co}P) = A \cap P \) is transfer open inverse valued in \( X \). Indeed, \( (A \cap P)^{-1}(0) = P^{-1}(0) = \{(x_1, x_2) \mid x_1 > x_2\} \) is open in \( X \times X \), and for any \( t \in (0, 1] \) and \( (a_1, b_1) \notin (A \cap P)^{-1}(t) \), we always have \( (x_1, x_2) \in \text{int}(P^{-1}(1)) = \text{int}(A \cap P)^{-1}(1) \). However, the lower section \( (A \cap P)^{-1}(1/2) \) is not open. Indeed, let \( a_1 = (1/2) - (1/\sqrt{2}) \) and let \( b_1 = 9/10 \); then \( (a_1, b_1) \in X \times X \\setminus (A \cap P)^{-1}(1/2) \) and \( (a_1, b_1) \) converges to \((1/2, 9/10)\), which does not belong to \( X \times X \\setminus (A \cap P)^{-1}(1/2) \). This means that the set \( X \times X \\setminus (A \cap P)^{-1}(1/2) \) is not closed, and hence \( (A \cap P)^{-1}(1/2) \) is not open. Further, for each \( x_1, x_2 \in X \), \( x_1 \notin P(x_1, x_2) = H\cdot\text{co}P(x_1, x_2) \). Thus, all hypotheses of Corollary 9 are satisfied so that the generalized abstract economy \( \Omega \) has an equilibrium point in \( X \times X \). In fact, all the equilibria of \( \Omega \) are the points \((a, a)\), where \( a \in [0, 1] \).

Let \( X \) and \( Y \) be two topological spaces. Given three set-valued mappings \( T : X \to 2^Y, F : X \to 2^X, A : X \to 2^X, \)
and a function $\phi: X \times X \times Y \to \mathbb{R}$, a generalized quasi-variational inequality is defined as follows:

\[(\text{GQVI}) \quad \left\{ \begin{array}{l}
\text{Find } (\tilde{x}, \tilde{w}, \tilde{y}) \in X \times X \times Y \\
\text{such that } \tilde{x} \in \text{cl} A(\tilde{x}), \quad \tilde{w} \in F(\tilde{x}), \quad \\
\tilde{y} \in T(\tilde{x}), \\
\phi(z, \tilde{x}, \tilde{y}) \geq 0, \quad \forall z \in A(\tilde{x}) \cap (F^{-1}(\tilde{w}))^C.
\end{array} \right. \tag{14}
\]

In particular, if $F(x) = \{x\}$ for each $x \in X$, then $(F^{-1}(\tilde{w}))^C = X \setminus \{w\}$. Therefore, the (GQVI) reduces to the usual quasi-variational inequality as follows:

\[(\text{QVI}) \quad \left\{ \begin{array}{l}
\text{Find } (\tilde{x}, \tilde{y}) \in X \times Y \\
\text{such that } \tilde{x} \in \text{cl} A(\tilde{x}), \quad \tilde{y} \in T(\tilde{x}), \\
\phi(z, \tilde{x}, \tilde{y}) > 0, \quad \forall z \in A(\tilde{x}) \setminus \{\tilde{x}\}.
\end{array} \right. \tag{15}
\]

**Theorem 11.** Let $(X, \Gamma)$ be an l.c.-space with precompact polytopes, $\Gamma_{\{x\}} = \{x\}$ for each $x \in X$, and let $Y$ be a topological space. The set-valued mappings $T: X \to 2^Y$ and $A: X \to 2^X$ satisfy $T \in \mathcal{H}(X,Y)$, $F \in \mathcal{H}(X,X)$, and $\text{cl} A \in \mathcal{H}(X,X)$, and $A^{-1}(x)$ is open for each $x \in X$. Suppose that $\phi: X \times X \times Y \to \mathbb{R}$ is a function such that

1. $\phi(x, x, y) \geq 0$ for all $x \in X$ and $y \in T(x)$,
2. for each fixed $z \in X$, the mapping $(x, y) \mapsto \phi(z, x, y)$ is lower semicontinuous,
3. for each fixed $x \in X \times Y$, the mapping $z \mapsto \phi(z, x, y)$ is $H$-quasiconvex in the following sense that for any finite set $D$ in $X$,

$$\phi(z, x, y) \leq \max_{u \in D} \phi(u, x, y), \quad \forall z \in H^{-co}D. \tag{16}$$

Then there is a solution to (GQVI).

**Proof.** Define a set-valued mapping $P: X \times X \to 2^X$ by

$$P(x, w) := \left\{ z \in X \mid \inf_{y \in T(z)} \phi(z, x, y) < 0 \right\} \cap (F^{-1}(w))^C, \quad \forall (x, w) \in X \times X. \tag{17}$$

By [20, Proposition 23, page 121], for each fixed $z \in X$, the mapping $x \mapsto \inf_{y \in T(z)} \phi(z, x, y)$ is lower semicontinuous. Thus, the set $\{x \in X \mid \inf_{y \in T(z)} \phi(z, x, y) > 0\}$ is open for each $z \in X$. It follows that

$$P^{-1}(z) = \left( \left\{ x \in X \mid \inf_{y \in T(z)} \phi(z, x, y) > 0 \right\} \times X \right) \cap \left( X \times (T(z))^C \right)^C \tag{18}$$

is open. By Lemma 5, $(H^{-co}P)^{-1}(z)$ is also open. Next, we show that $x \notin H^{-co}P(x, w)$ for all $x, w \in X$. Assume that there are $x_0$ and $w_0$ satisfying $x_0 \in H^{-co}P(x_0, w_0)$. Then there is a finite subset $D$ of $P(x_0, w_0)$ such that $x_0 \in H^{-co}D$. For each fixed $y \in T(x_0)$, since the mapping $z \mapsto \phi(z, x_0, y)$ is $H$-quasiconvex, it follows that

$$0 \leq \inf_{y \in T(x_0)} \phi(x_0, x_0, y) \leq \inf_{y \in T(x_0)} \max_{z \in D} \phi(z, x_0, y). \tag{19}$$

By Kneser’s minimax theorem [21], together with $z \in P(x_0, w_0)$ for all $z \in D$, we have

$$\inf_{y \in T(x_0)} \max_{z \in D} \phi(z, x_0, y) = \max_{z \in D} \inf_{y \in T(x_0)} \phi(z, x_0, y) < 0. \tag{20}$$

This is a contradiction. Thus, all hypotheses of Corollary 7 are satisfied. Therefore, there exist $\tilde{x}, \tilde{w} \in X$ such that $\tilde{x} \in \text{cl} A(\tilde{x}), \tilde{w} \in F(\tilde{x})$, and $A(\tilde{x}) \cap P(\tilde{x}, \tilde{w}) = \emptyset$. It follows that

$$\inf_{y \in T(\tilde{x})} \phi(z, \tilde{x}, y) \geq 0, \quad \forall z \in A(\tilde{x}) \cap (F^{-1}(\tilde{w}))^C. \tag{21}$$

Since $T(\tilde{x})$ is compact, there is $\tilde{y} \in T(\tilde{x})$ such that $\phi(\tilde{x}, \tilde{x}, \tilde{y}) \geq 0$ for all $z \in A(\tilde{x}) \cap (F^{-1}(\tilde{w}))^C$. That is, $(\tilde{x}, \tilde{x}, \tilde{y})$ is a solution to (GQVI). \(\square\)

**References**


