Research Article

On the Mean Values of Certain Character Sums

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Let $q \geq 5$ be an odd number. In this paper, we study the fourth power mean of certain character sums $\sum_{\chi \bmod q, (\chi-1)=-1}^* |\sum_{1 \leq a \leq q/4} a \chi(a)|^4$ and $\sum_{\chi \bmod q, (\chi-1)=1}^* |\sum_{1 \leq a \leq q/4} a \chi(a)|^4$, where $\sum^*$ denotes the summation over primitive characters modulo q , and give some asymptotic formulae.

1. Introduction

The sum

$$S_\chi(n) = \frac{1}{q^n} \sum_{a=1}^q a^n \chi(a) \quad (1)$$

appears frequently in number theory, where χ is a nonprincipal primitive character modulo q , and has been studied by several experts. For example, for $q \equiv 3 \pmod{4}$ being a prime p and χ being the Legendre symbol, Ayoub et al. [1] have proved that $S_\chi(n) < 0$ for $n = 1, 2$ and for $n \geq p - 2$. Fine [2] has showed that for $n > 2$, there exist infinitely many primes $p \equiv 3 \pmod{4}$ with $S_\chi(n) > 0$ and infinitely many with $S_\chi(n) < 0$.

Williams [3] proved that

$$S_\chi(n) = O(p^{1/2} \log p) \quad (2)$$

for χ being the Legendre symbol modulo p . For primitive character χ modulo q , Toyozumi [4] used the generalized Bernoulli numbers to express $S_\chi(n)$ in terms of Gauss sums and Dirichlet L -functions as follows:

$$\sum_{a=1}^q a^n \chi(a) = \begin{cases} 2q^n \tau(\chi) \times \sum_{1 \leq m \leq n/2} \frac{\binom{n}{2m-1} (2m-1)! L(2m, \bar{\chi})}{(-1)^{m+1} (2\pi)^{2m}}, & \text{if } \chi(-1) = 1, \\ 2q^n \tau(\chi) \times \sum_{0 \leq m \leq (n-1)/2} \frac{\binom{n}{2m} (2m)! L(2m+1, \bar{\chi})}{(-1)^{m+1} (2\pi)^{2m+1} i}, & \text{if } \chi(-1) = -1, \end{cases} \quad (3)$$

where $\tau(\chi) = \sum_{a=1}^q \chi(a) e(a/q)$ is the Gauss sum, $e(y) = e^{2\pi i y}$, $L(s, \chi)$ is the Dirichlet L -function corresponding to χ , and $\binom{n}{m}$ denotes the binomial coefficient.

Toyozumi [4] also gave explicit bounds for $S_\chi(n)$.

Proposition 1. (a) Assume that $\chi(-1) = 1$ and $n \geq 2$. Then for any primitive character $\chi \bmod q$, one has

$$|S_\chi(n)| < C_1(n) q^{1/2}, \quad (4)$$

where

$$C_1(n) = \frac{2\zeta(2)n!}{(2\pi)^{n+1}} \sum_{1 \leq m \leq n/2} \frac{(2\pi)^{n+1-2m}}{(n+1-2m)!}, \quad (5)$$

and $\zeta(s)$ is the Riemann zeta function.

(b) Assume that $\chi(-1) = -1$ and $n \geq 3$. Then for any primitive character $\chi \pmod q$, one has

$$|S_\chi(n)| < \left(C_2(n) + \frac{|L(1, \chi)|}{\pi} \right) q^{1/2}, \quad (6)$$

where

$$C_2(n) = \frac{2\zeta(3)n!}{(2\pi)^{n+1}} \sum_{1 \leq m \leq (n-1)/2} \frac{(2\pi)^{n-2m}}{(n-2m)!}. \quad (7)$$

In [5], Peral used the Gauss sums and adequate Fourier expansion to greatly improve the result in Proposition 1.

Proposition 2. (a) Assume that $\chi(-1) = 1$ is a primitive nonprincipal character modulo q , and then

$$|S_\chi(n)| \leq q^{1/2} \left(\frac{n-1}{2(n+1)} \right). \quad (8)$$

(b) Assume that $\chi(-1) = -1$ is a primitive character modulo q ; then,

$$\left| S_\chi(n) + \frac{\tau(\chi)L(1, \bar{\chi})}{\pi i} \right| \leq q^{1/2} \left(\frac{n}{\pi} \int_0^1 \ln \frac{1}{2 \sin(\pi t)} t^{n-1} dt \right). \quad (9)$$

Furthermore, Liu and Zhang [6] gave an upper bound for $S_\chi(n)$ when χ is a nonprincipal character modulo q .

It may be interesting to consider the mean value of certain character sums. For example, Burgess [7] proved that

$$\sum_{\chi \pmod q}^* \sum_{n=1}^q \left| \sum_{m=1}^h \chi(n+m) \right|^4 \leq 8d^2(q) q^2 h^2, \quad (10)$$

where $\sum^*$ denotes the summation over primitive characters modulo q , and $d(q)$ is the Dirichlet divisor function. Xu and Zhang studied the power mean

$$\sum_{\substack{\chi \pmod q \\ \chi(-1)=\pm 1}}^* \left| \sum_{1 \leq a \leq q/4} \chi(a) \right|^4 \quad (11)$$

in [8, 9] and obtained some sharper results.

In this paper, we study the fourth power mean of certain character sums

$$\begin{aligned} & \sum_{\substack{\chi \pmod q \\ \chi(-1)=-1}}^* \left| \sum_{1 \leq a \leq q/4} a\chi(a) \right|^4, \\ & \sum_{\substack{\chi \pmod q \\ \chi(-1)=1}}^* \left| \sum_{1 \leq a \leq q/4} a\chi(a) \right|^4 \end{aligned} \quad (12)$$

and give a few asymptotic formulae.

Theorem 3. Let $q \geq 5$ be an odd number. Then one has

$$\begin{aligned} & \sum_{\substack{\chi \pmod q \\ \chi(-1)=-1}}^* \left| \sum_{1 \leq a \leq q/4} a\chi(a) \right|^4 \\ &= \frac{7}{2^{17} \cdot 3^2} q^6 J(q) \prod_{p|q} \frac{(1-1/p^2)^3}{1+1/p^2} \\ &+ \frac{35}{2^{16} \cdot 3^2 \cdot 17} q^6 J(q) \prod_{p|q} \frac{(1-1/p^4)^3}{1+1/p^4} \\ &- \frac{L^2(3, \chi_4)}{2^{14} \cdot 3^2 \cdot \pi \cdot L(5, \chi_4)} \\ &\times q^6 J(q) \prod_{p|q} \frac{(1-1/p^2)^2 (1-\chi_4(p)/p^3)^2}{1-\chi_4(p)/p^5} \\ &+ \frac{L(3, \chi_4)}{2^8 \cdot 5 \cdot 3^2 \cdot \pi^2} q^6 J(q) \\ &\times \prod_{p|q} (1-\chi_4(p)/p^3) (1-1/p^4) \\ &\times \prod_{p \nmid q} \left(1 + \frac{1}{p^2(1-\chi_4(p)/p)} \right) + O(q^{6+\epsilon}), \end{aligned} \quad (13)$$

where $J(q)$ is the number of primitive characters modulo q , χ_4 is the nonprincipal character modulo 4, and ϵ is any fixed positive real number.

Theorem 4. Let $q \geq 5$ be an odd number. Then one has

$$\begin{aligned} & \sum_{\substack{\chi \pmod q \\ \chi(-1)=1}}^* \left| \sum_{1 \leq a \leq q/4} a\chi(a) \right|^4 \\ &= \frac{385}{2^{20} \cdot 51} \cdot q^6 J(q) \prod_{p|q} \frac{(1-1/p^4)^3}{1+1/p^4} \\ &+ \frac{15L^4(3, \chi_4)}{\pi^{12}} \cdot q^6 J(q) \prod_{p|q} \frac{(1-\chi_4(p)/p^3)^3}{1+\chi_4(p)/p^3} \\ &- \frac{1423\pi L^2(3, \chi_4)}{2^{16} \cdot 5^2 \cdot 3^4 \cdot L(7, \chi_4)} \\ &\cdot q^6 J(q) \prod_{p|q} \frac{(1-\chi_4(p)/p^3)^2 (1-1/p^4)^2}{1-\chi_4(p)/p^7} \\ &+ \frac{3}{2^{16}} \cdot q^6 J(q) \prod_{p|q} \frac{(1-1/p^2)^3}{1+1/p^2} \end{aligned}$$

$$\begin{aligned}
 & - \frac{L^2(3, \chi_4)}{2^{12} \cdot \pi \cdot L(5, \chi_4)} \\
 & \cdot q^6 J(q) \prod_{p|q} \frac{(1 - 1/p^2)^2 (1 - \chi_4(p)/p^3)^2}{1 - \chi_4(p)/p^5} \\
 & + \frac{L(3, \chi_4)}{2^{10} \cdot 3 \cdot \pi^2} \cdot q^6 J(q) \\
 & \times \prod_{p|q} (1 - \chi_4(p)/p^3) (1 - 1/p^4) \\
 & \times \prod_{p \nmid q} \left(1 + \frac{1}{p^2 (1 - \chi_4(p)/p)} \right) + O(q^{6+\epsilon}).
 \end{aligned} \tag{14}$$

From Theorems 3 and 4, we immediately get the following corollaries.

Corollary 5. Let $p \geq 5$ be a prime. Then one has

$$\begin{aligned}
 & \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left| \sum_{1 \leq a \leq p/4} a \chi(a) \right|^4 \\
 & = \frac{21}{2^{17} \cdot 17} p^7 - \frac{L^2(3, \chi_4)}{2^{14} \cdot 3^2 \cdot \pi \cdot L(5, \chi_4)} p^7 \\
 & + \frac{L(3, \chi_4)}{2^8 \cdot 5 \cdot 3^2 \cdot \pi^2} p^7 \prod_{p_1} \left(1 + \frac{1}{p_1^2 (1 - \chi_4(p_1)/p_1)} \right) \\
 & + O(p^{6+\epsilon}),
 \end{aligned} \tag{15}$$

where $\prod_{p_1}$ denotes the product over all primes.

Corollary 6. Let $p \geq 5$ be a prime. Then

$$\begin{aligned}
 & \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \left| \sum_{1 \leq a \leq p/4} a \chi(a) \right|^4 \\
 & = \frac{2833}{2^{20} \cdot 51} p^7 + \frac{15L^4(3, \chi_4)}{\pi^{12}} p^7 \\
 & - \frac{1423\pi L^2(3, \chi_4)}{2^{16} \cdot 5^2 \cdot 3^4 \cdot L(7, \chi_4)} p^7 \\
 & - \frac{L^2(3, \chi_4)}{2^{12} \cdot \pi \cdot L(5, \chi_4)} p^7 \\
 & + \frac{L(3, \chi_4)}{2^{10} \cdot 3 \cdot \pi^2} p^7 \prod_{p_1} \left(1 + \frac{1}{p_1^2 (1 - \chi_4(p_1)/p_1)} \right) \\
 & + O(p^{6+\epsilon}).
 \end{aligned} \tag{16}$$

Remark 7. It seems that the contributions of odd and even primitive characters to the fourth power moment of character sums over $[1, q/4]$ are very different.

2. Express the Character Sum in terms of Gauss Sums and L -Functions (I)

Let χ be an odd primitive character modulo q . In this section, we will express $\sum_{1 \leq a \leq q/4} a \chi(a)$ in terms of Gauss sums and Dirichlet L -functions. We need the following lemmas.

Lemma 8. Suppose that $q \geq 5$ is an odd number, and χ is an odd character modulo q .

(i) For $q \equiv 1 \pmod{4}$, one has

$$\begin{aligned}
 & \sum_{1 \leq a \leq (3q-3)/4} a \chi(a) \\
 & = \sum_{1 \leq a \leq q} a \chi(a) + q \sum_{1 \leq a \leq (q-1)/4} \chi(a) \\
 & - \sum_{1 \leq a \leq (q-1)/4} a \chi(a), \\
 & \sum_{1 \leq a \leq (3q-3)/4} \chi(a) = \sum_{1 \leq a \leq (q-1)/4} \chi(a).
 \end{aligned} \tag{17}$$

(ii) For $q \equiv 3 \pmod{4}$, one has

$$\begin{aligned}
 & \sum_{1 \leq a \leq (3q-1)/4} a \chi(a) \\
 & = \sum_{1 \leq a \leq q} a \chi(a) + q \sum_{1 \leq a \leq (q-3)/4} \chi(a) \\
 & - \sum_{1 \leq a \leq (q-3)/4} a \chi(a), \\
 & \sum_{1 \leq a \leq (3q-1)/4} \chi(a) = \sum_{1 \leq a \leq (q-3)/4} \chi(a).
 \end{aligned} \tag{18}$$

Proof. It is easy to show that

$$\begin{aligned}
 & \sum_{1 \leq a \leq (3q-3)/4} a \chi(a) \\
 & = \sum_{1 \leq a \leq q} a \chi(a) - \sum_{(3q+1)/4 \leq a \leq q} a \chi(a) \\
 & = \sum_{1 \leq a \leq q} a \chi(a) - \sum_{1 \leq b \leq (q-1)/4} (q-b) \chi(q-b) \\
 & = \sum_{1 \leq a \leq q} a \chi(a) + q \sum_{1 \leq a \leq (q-1)/4} \chi(a) \\
 & - \sum_{1 \leq a \leq (q-1)/4} a \chi(a),
 \end{aligned}$$

$$\begin{aligned}
& \sum_{1 \leq a \leq (3q-3)/4} \chi(a) \\
&= \sum_{1 \leq a \leq q} \chi(a) - \sum_{(3q+1)/4 \leq a \leq q} \chi(a) = \sum_{1 \leq a \leq (q-1)/4} \chi(a).
\end{aligned} \tag{19}$$

This proves (i). Similarly, we can deduce (ii). $\square$

Lemma 9. Suppose that $q \geq 5$ is an odd number, and χ is an odd character modulo q . Let χ_4 be the nonprincipal character modulo 4. For $q \equiv 1 \pmod{4}$, one has

$$\begin{aligned}
& \sum_{a=1}^{4q} a^2 \chi(a) \chi_4(a) \\
&= 16\chi(4)q \sum_{a=1}^q a\chi(a) + 16\chi(4)q^2 \sum_{1 \leq a \leq (q-1)/4} \chi(a) \\
&\quad - 64\chi(4)q \sum_{1 \leq a \leq (q-1)/4} a\chi(a).
\end{aligned} \tag{20}$$

Proof. Note that $\chi_4(1) = 1$ and $\chi_4(3) = -1$, and we get

$$\begin{aligned}
\sum_{a=1}^{4q} a^2 \chi(a) \chi_4(a) &= \sum_{a=0}^{q-1} (4a+1)^2 \chi(4a+1) \\
&\quad - \sum_{a=0}^{q-1} (4a+3)^2 \chi(4a+3).
\end{aligned} \tag{21}$$

First we have

$$\begin{aligned}
& \sum_{a=0}^{q-1} (4a+1)^2 \chi(4a+1) \\
&= 16 \sum_{a=0}^{q-1} a^2 \chi(4a+1) + 8 \sum_{a=0}^{q-1} a\chi(4a+1) \\
&\quad + \sum_{a=0}^{q-1} \chi(4a+1) \\
&= 16\chi(4) \sum_{a=0}^{q-1} a^2 \chi(a + \bar{4}) + 8\chi(4) \sum_{a=0}^{q-1} a\chi(a + \bar{4}) \\
&= 16\chi(4) \sum_{a=0}^{q-1} (a + \bar{4})^2 \chi(a + \bar{4}) \\
&\quad + 8\chi(4) (1 - 4 \cdot \bar{4}) \sum_{a=0}^{q-1} (a + \bar{4}) \chi(a + \bar{4}),
\end{aligned} \tag{22}$$

where $\bar{4}$ is the inverse of 4 modulo q with $4 \cdot \bar{4} \equiv 1 \pmod{q}$ and $1 \leq \bar{4} \leq q$. Since $q \equiv 1 \pmod{4}$, we get $\bar{4} = (3q+1)/4$. Then from Lemma 8, we have

$$\begin{aligned}
& \sum_{a=0}^{q-1} (a + \bar{4})^2 \chi(a + \bar{4}) \\
&= \sum_{0 \leq a \leq (q-1)/4} (a + \bar{4})^2 \chi(a + \bar{4}) \\
&\quad + \sum_{(q-1)/4 < a \leq q-1} (a + \bar{4})^2 \chi(a + \bar{4}) \\
&= \sum_{0 \leq a \leq (q-1)/4} (a + \bar{4})^2 \chi(a + \bar{4}) \\
&\quad + \sum_{(q-1)/4 < a \leq q-1} (a + \bar{4} - q)^2 \chi(a + \bar{4} - q) \\
&\quad + 2q \sum_{(q-1)/4 < a \leq q-1} (a + \bar{4} - q) \chi(a + \bar{4} - q) \\
&\quad + q^2 \sum_{(q-1)/4 < a \leq q-1} \chi(a + \bar{4} - q) \\
&= \sum_{a=1}^q a^2 \chi(a) + 2q \sum_{1 \leq a \leq (3q-3)/4} a\chi(a) \\
&\quad + q^2 \sum_{1 \leq a \leq (3q-3)/4} \chi(a) \\
&= \sum_{a=1}^q a^2 \chi(a) + 2q \sum_{a=1}^q a\chi(a) \\
&\quad + 3q^2 \sum_{1 \leq a \leq (q-1)/4} \chi(a) \\
&\quad - 2q \sum_{1 \leq a \leq (q-1)/4} a\chi(a),
\end{aligned}$$

$$\begin{aligned}
& \sum_{a=0}^{q-1} (a + \bar{4}) \chi(a + \bar{4}) \\
&= \sum_{0 \leq a \leq (q-1)/4} (a + \bar{4}) \chi(a + \bar{4}) \\
&\quad + \sum_{(q-1)/4 < a \leq q-1} (a + \bar{4}) \chi(a + \bar{4}) \\
&= \sum_{0 \leq a \leq (q-1)/4} (a + \bar{4}) \chi(a + \bar{4}) \\
&\quad + \sum_{(q-1)/4 < a \leq q-1} (a + \bar{4} - q) \chi(a + \bar{4} - q) \\
&\quad + q \sum_{(q-1)/4 < a \leq q-1} \chi(a + \bar{4} - q)
\end{aligned}$$

$$\begin{aligned}
 &= \sum_{a=1}^q a\chi(a) + q \sum_{(q-1)/4 \leq a \leq q-1} \chi(a + \bar{4} - q) \\
 &= \sum_{a=1}^q a\chi(a) + q \sum_{1 \leq a \leq (3q-3)/4} \chi(a) \\
 &= \sum_{a=1}^q a\chi(a) + q \sum_{1 \leq a \leq (q-1)/4} \chi(a).
 \end{aligned} \tag{23}$$

Therefore

$$\begin{aligned}
 &\sum_{a=0}^{q-1} (4a+1)^2 \chi(4a+1) \\
 &= 16\chi(4) \sum_{a=0}^{q-1} (a+\bar{4})^2 \chi(a+\bar{4}) \\
 &\quad + 8\chi(4) \left(1 - 4 \cdot \bar{4}\right) \sum_{a=0}^{q-1} (a+\bar{4}) \chi(a+\bar{4}) \\
 &= 16\chi(4) \sum_{a=1}^q a^2 \chi(a) + 8\chi(4) q \sum_{a=1}^q a\chi(a) \\
 &\quad + 24\chi(4) q^2 \sum_{1 \leq a \leq (q-1)/4} \chi(a) \\
 &\quad - 32\chi(4) q \sum_{1 \leq a \leq (q-1)/4} a\chi(a).
 \end{aligned} \tag{24}$$

On the other hand, we get

$$\begin{aligned}
 &\sum_{a=0}^{q-1} (4a+3)^2 \chi(4a+3) \\
 &= 16 \sum_{a=0}^{q-1} a^2 \chi(4a+3) \\
 &\quad + 24 \sum_{a=0}^{q-1} a\chi(4a+3) + 9 \sum_{a=0}^{q-1} \chi(4a+3) \\
 &= 16\chi(4) \sum_{a=0}^{q-1} a^2 \chi(a+3 \cdot \bar{4}) \\
 &\quad + 24\chi(4) \sum_{a=0}^{q-1} a\chi(a+3 \cdot \bar{4}).
 \end{aligned} \tag{25}$$

Since $\bar{4} = (3q+1)/4$ and $3 \cdot \bar{4} = 2q + (q+3)/4$, we have

$$\begin{aligned}
 &\sum_{a=0}^{q-1} (4a+3)^2 \chi(4a+3) \\
 &= 16\chi(4) \sum_{a=0}^{q-1} a^2 \chi\left(a + \frac{q+3}{4}\right)
 \end{aligned}$$

$$\begin{aligned}
 &\quad + 24\chi(4) \sum_{a=0}^{q-1} a\chi\left(a + \frac{q+3}{4}\right) \\
 &= 16\chi(4) \sum_{a=0}^{q-1} \left(a + \frac{q+3}{4}\right)^2 \chi\left(a + \frac{q+3}{4}\right) \\
 &\quad + 24\left(1 - \frac{q+3}{3}\right) \chi(4) \\
 &\quad \times \sum_{a=0}^{q-1} \left(a + \frac{q+3}{4}\right) \chi\left(a + \frac{q+3}{4}\right).
 \end{aligned}$$

(26)

Note that

$$\begin{aligned}
 &\sum_{a=0}^{q-1} \left(a + \frac{q+3}{4}\right)^2 \chi\left(a + \frac{q+3}{4}\right) \\
 &= \sum_{0 \leq a \leq (3q-3)/4} \left(a + \frac{q+3}{4}\right)^2 \chi\left(a + \frac{q+3}{4}\right) \\
 &\quad + \sum_{(3q+1)/4 \leq a \leq q-1} \left(a + \frac{q+3}{4}\right)^2 \chi\left(a + \frac{q+3}{4}\right) \\
 &= \sum_{0 \leq a \leq (3q-3)/4} \left(a + \frac{q+3}{4}\right)^2 \chi\left(a + \frac{q+3}{4}\right) \\
 &\quad + \sum_{(3q+1)/4 \leq a \leq q-1} \left(a + \frac{q+3}{4} - q\right)^2 \chi\left(a + \frac{q+3}{4} - q\right) \\
 &\quad + 2q \sum_{(3q+1)/4 \leq a \leq q-1} \left(a + \frac{q+3}{4} - q\right) \chi\left(a + \frac{q+3}{4} - q\right) \\
 &\quad + q^2 \sum_{(3q+1)/4 \leq a \leq q-1} \chi\left(a + \frac{q+3}{4} - q\right) \\
 &= \sum_{a=1}^q a^2 \chi(a) + q^2 \sum_{1 \leq a \leq (q-1)/4} \chi(a) \\
 &\quad + 2q \sum_{1 \leq a \leq (q-1)/4} a\chi(a), \\
 &\sum_{a=0}^{q-1} \left(a + \frac{q+3}{4}\right) \chi\left(a + \frac{q+3}{4}\right) \\
 &= \sum_{0 \leq a \leq (3q-3)/4} \left(a + \frac{q+3}{4}\right) \chi\left(a + \frac{q+3}{4}\right) \\
 &\quad + \sum_{(3q+1)/4 \leq a \leq q-1} \left(a + \frac{q+3}{4}\right) \chi\left(a + \frac{q+3}{4}\right) \\
 &= \sum_{0 \leq a \leq (3q-3)/4} \left(a + \frac{q+3}{4}\right) \chi\left(a + \frac{q+3}{4}\right)
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{(3q+1)/4 \leq a \leq q-1} \left(a + \frac{q+3}{4} - q \right) \chi \left(a + \frac{q+3}{4} - q \right) \\
& + q \sum_{(3q+1)/4 \leq a \leq q-1} \chi \left(a + \frac{q+3}{4} - q \right) \\
& = \sum_{a=1}^q a \chi(a) + q \sum_{1 \leq a \leq (q-1)/4} \chi(a), \tag{27}
\end{aligned}$$

so we get

$$\begin{aligned}
& \sum_{a=0}^{q-1} (4a+3)^2 \chi(4a+3) \\
& = 16\chi(4) \sum_{a=0}^{q-1} \left(a + \frac{q+3}{4} \right)^2 \chi \left(a + \frac{q+3}{4} \right) \\
& \quad + 24 \left(1 - \frac{q+3}{3} \right) \chi(4) \\
& \quad \times \sum_{a=0}^{q-1} \left(a + \frac{q+3}{4} \right) \chi \left(a + \frac{q+3}{4} \right) \\
& = 16\chi(4) \sum_{a=1}^q a^2 \chi(a) - 8\chi(4) q \sum_{a=1}^q a \chi(a) \\
& \quad + 8\chi(4) q^2 \sum_{1 \leq a \leq (q-1)/4} \chi(a) + 32\chi(4) q \sum_{1 \leq a \leq (q-1)/4} a \chi(a). \tag{28}
\end{aligned}$$

Now combine (21)–(28); we have

$$\begin{aligned}
& \sum_{a=1}^{4q} a^2 \chi(a) \chi_4(a) \\
& = \sum_{a=0}^{q-1} (4a+1)^2 \chi(4a+1) \\
& \quad - \sum_{a=0}^{q-1} (4a+3)^2 \chi(4a+3) \\
& = 16\chi(4) q \sum_{a=1}^q a \chi(a) \\
& \quad + 16\chi(4) q^2 \sum_{1 \leq a \leq (q-1)/4} \chi(a) \\
& \quad - 64\chi(4) q \sum_{1 \leq a \leq (q-1)/4} a \chi(a). \tag{29}
\end{aligned}$$

□

Lemma 10. Suppose that $q \geq 5$ is an odd number, and χ is an odd character modulo q . Let χ_4 be the nonprincipal character modulo 4. For $q \equiv 3 \pmod{4}$, one has

$$\begin{aligned}
& \sum_{a=1}^{4q} a^2 \chi(a) \chi_4(a) = -16\chi(4) q \sum_{a=1}^q a \chi(a) \\
& \quad - 16\chi(4) q^2 \sum_{1 \leq a \leq (q-1)/4} \chi(a) \tag{30} \\
& \quad + 64\chi(4) q \sum_{1 \leq a \leq (q-1)/4} a \chi(a).
\end{aligned}$$

Proof. For $q \equiv 3 \pmod{4}$, we get $\bar{4} = (q+1)/4$ and $3 \cdot \bar{4} = (3q+3)/4$. Using the methods of proving Lemma 9, we have

$$\sum_{a=1}^{4q} a^2 \chi(a) \chi_4(a) \tag{31}$$

$$= \sum_{a=0}^{q-1} (4a+1)^2 \chi(4a+1) - \sum_{a=0}^{q-1} (4a+3)^2 \chi(4a+3),$$

$$\begin{aligned}
& \sum_{a=0}^{q-1} (4a+1)^2 \chi(4a+1) \\
& = 16\chi(4) \sum_{a=0}^{q-1} (a+\bar{4})^2 \chi(a+\bar{4}) \tag{32} \\
& \quad + 8\chi(4) (1-4 \cdot \bar{4}) \sum_{a=0}^{q-1} (a+\bar{4}) \chi(a+\bar{4}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{a=0}^{q-1} (4a+3)^2 \chi(4a+3) \\
& = 16 \sum_{a=0}^{q-1} a^2 \chi(4a+3) + 24 \sum_{a=0}^{q-1} a \chi(4a+3) \\
& \quad + 9 \sum_{a=0}^{q-1} \chi(4a+3) \\
& = 16\chi(4) \sum_{a=0}^{q-1} a^2 \chi(a+3 \cdot \bar{4}) \\
& \quad + 24\chi(4) \sum_{a=0}^{q-1} a \chi(a+3 \cdot \bar{4}) \\
& = 16\chi(4) \sum_{a=0}^{q-1} \left(a + \frac{3q+3}{4} \right)^2 \chi \left(a + \frac{3q+3}{4} \right) \\
& \quad - 24\chi(4) q \sum_{a=0}^{q-1} \left(a + \frac{3q+3}{4} \right) \chi \left(a + \frac{3q+3}{4} \right). \tag{33}
\end{aligned}$$

(33)

It is not hard to show that

$$\begin{aligned}
 & \sum_{a=0}^{q-1} (a + \bar{4})^2 \chi(a + \bar{4}) \\
 &= \sum_{0 \leq a \leq (3q-1)/4} (a + \bar{4})^2 \chi(a + \bar{4}) \\
 &+ \sum_{(3q-1)/4 < a \leq q-1} (a + \bar{4})^2 \chi(a + \bar{4}) \\
 &= \sum_{0 \leq a \leq (3q-1)/4} (a + \bar{4})^2 \chi(a + \bar{4}) \\
 &+ \sum_{(3q-1)/4 < a \leq q-1} (a + \bar{4} - q)^2 \chi(a + \bar{4} - q) \\
 &+ 2q \sum_{(3q-1)/4 < a \leq q-1} (a + \bar{4} - q) \chi(a + \bar{4} - q) \\
 &+ q^2 \sum_{(3q-1)/4 < a \leq q-1} \chi(a + \bar{4} - q) \\
 &= \sum_{a=1}^q a^2 \chi(a) + q^2 \sum_{1 \leq a \leq (q-3)/4} \chi(a) \\
 &+ 2q \sum_{1 \leq a \leq (q-3)/4} a \chi(a), \\
 &\sum_{a=0}^{q-1} (a + \bar{4}) \chi(a + \bar{4}) \\
 &= \sum_{0 \leq a \leq (3q-1)/4} (a + \bar{4}) \chi(a + \bar{4}) \\
 &+ \sum_{(3q-1)/4 < a \leq q-1} (a + \bar{4}) \chi(a + \bar{4}) \\
 &= \sum_{0 \leq a \leq (3q-1)/4} (a + \bar{4}) \chi(a + \bar{4}) \\
 &+ \sum_{(3q-1)/4 < a \leq q-1} (a + \bar{4} - q) \chi(a + \bar{4} - q) \\
 &+ q \sum_{(3q-1)/4 < a \leq q-1} \chi(a + \bar{4} - q) \\
 &= \sum_{a=1}^q a \chi(a) + q \sum_{1 \leq a \leq (q-3)/4} \chi(a).
 \end{aligned} \tag{34}$$

Then by (32), we have

$$\begin{aligned}
 & \sum_{a=0}^{q-1} (4a + 1)^2 \chi(4a + 1) \\
 &= 16 \chi(4) \sum_{a=0}^{q-1} (a + \bar{4})^2 \chi(a + \bar{4})
 \end{aligned}$$

$$\begin{aligned}
 & + 8 \chi(4) (1 - 4 \times \bar{4}) \sum_{a=0}^{q-1} (a + \bar{4}) \chi(a + \bar{4}) \\
 &= 16 \chi(4) \sum_{a=1}^q a^2 \chi(a) \\
 &- 8 \chi(4) q \sum_{a=1}^q a \chi(a) + 8 \chi(4) q^2 \sum_{1 \leq a \leq (q-3)/4} \chi(a) \\
 &+ 32 \chi(4) q \sum_{1 \leq a \leq (q-3)/4} a \chi(a).
 \end{aligned} \tag{35}$$

On the other hand, by Lemma 8, we get

$$\begin{aligned}
 & \sum_{a=0}^{q-1} \left(a + \frac{3q+3}{4}\right)^2 \chi\left(a + \frac{3q+3}{4}\right) \\
 &= \sum_{0 \leq a \leq (q-3)/4} \left(a + \frac{3q+3}{4}\right)^2 \chi\left(a + \frac{3q+3}{4}\right) \\
 &+ \sum_{(q+1)/4 \leq a \leq q-1} \left(a + \frac{3q+3}{4}\right)^2 \chi\left(a + \frac{3q+3}{4}\right) \\
 &= \sum_{0 \leq a \leq (q-3)/4} \left(a + \frac{3q+3}{4}\right)^2 \chi\left(a + \frac{3q+3}{4}\right) \\
 &+ \sum_{(q+1)/4 \leq a \leq q-1} \left(a + \frac{3q+3}{4} - q\right)^2 \chi\left(a + \frac{3q+3}{4} - q\right) \\
 &+ 2q \sum_{(q+1)/4 \leq a \leq q-1} \left(a + \frac{3q+3}{4} - q\right) \chi\left(a + \frac{3q+3}{4} - q\right) \\
 &+ q^2 \sum_{(q+1)/4 \leq a \leq q-1} \chi\left(a + \frac{3q+3}{4} - q\right) \\
 &= \sum_{a=1}^q a^2 \chi(a) + 2q \sum_{1 \leq a \leq (3q-1)/4} a \chi(a) \\
 &+ q^2 \sum_{1 \leq a \leq (3q-1)/4} \chi(a) \\
 &= \sum_{a=1}^q a^2 \chi(a) + 2q \sum_{a=1}^q a \chi(a) \\
 &+ 3q^2 \sum_{1 \leq a \leq (q-3)/4} \chi(a) - 2q \sum_{1 \leq a \leq (q-3)/4} a \chi(a),
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{a=0}^{q-1} \left(a + \frac{3q+3}{4}\right) \chi\left(a + \frac{3q+3}{4}\right) \\
 &= \sum_{0 \leq a \leq (q-3)/4} \left(a + \frac{3q+3}{4}\right) \chi\left(a + \frac{3q+3}{4}\right)
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{(q+1)/4 \leq a \leq q-1} \left(a + \frac{3q+3}{4}\right) \chi\left(a + \frac{3q+3}{4}\right) \\
& = \sum_{0 \leq a \leq (q-3)/4} \left(a + \frac{3q+3}{4}\right) \chi\left(a + \frac{3q+3}{4}\right) \\
& + \sum_{(q+1)/4 \leq a \leq q-1} \left(a + \frac{3q+3}{4} - q\right) \chi\left(a + \frac{3q+3}{4} - q\right) \\
& + q \sum_{(q+1)/4 \leq a \leq q-1} \chi\left(a + \frac{3q+3}{4} - q\right) \\
& = \sum_{a=1}^q a \chi(a) + q \sum_{1 \leq a \leq (3q-1)/4} \chi(a) \\
& = \sum_{a=1}^q a \chi(a) + q \sum_{1 \leq a \leq (q-3)/4} \chi(a).
\end{aligned} \tag{36}$$

Then from (33), we have

$$\begin{aligned}
& \sum_{a=0}^{q-1} (4a+3)^2 \chi(4a+3) \\
& = 16\chi(4) \sum_{a=0}^{q-1} \left(a + \frac{3q+3}{4}\right)^2 \chi\left(a + \frac{3q+3}{4}\right) \\
& - 24\chi(4) q \sum_{a=0}^{q-1} \left(a + \frac{3q+3}{4}\right) \chi\left(a + \frac{3q+3}{4}\right) \\
& = 16\chi(4) \sum_{a=1}^q a^2 \chi(a) + 8\chi(4) q \sum_{a=1}^q a \chi(a) \\
& + 24\chi(4) q^2 \sum_{1 \leq a \leq (q-3)/4} \chi(a) \\
& - 32\chi(4) q \sum_{1 \leq a \leq (q-3)/4} a \chi(a).
\end{aligned} \tag{37}$$

Combining (31), (35), and (37), we have

$$\begin{aligned}
& \sum_{a=1}^{4q} a^2 \chi(a) \chi_4(a) \\
& = \sum_{a=0}^{q-1} (4a+1)^2 \chi(4a+1) \\
& - \sum_{a=0}^{q-1} (4a+3)^2 \chi(4a+3) \\
& = -16\chi(4) q \sum_{a=1}^q a \chi(a) \\
& - 16\chi(4) q^2 \sum_{1 \leq a \leq (q-3)/4} \chi(a) + 64\chi(4) q \sum_{1 \leq a \leq (q-3)/4} a \chi(a).
\end{aligned} \tag{38}$$

Now we can express $\sum_{1 \leq a \leq q/4} a \chi(a)$ in terms of Gauss sums and Dirichlet L -functions.

Theorem 11. Let χ be an odd primitive character modulo odd integer $q \geq 5$, and let χ_4 be the nonprincipal character modulo 4. Then one has

$$\begin{aligned}
& \sum_{1 \leq a \leq q/4} a \chi(a) \\
& = \frac{q}{8\pi i} \tau(\chi) \left(\bar{\chi}(2) L(1, \bar{\chi}) - \bar{\chi}(4) L(1, \bar{\chi}) \right. \\
& \quad \left. + \frac{4}{\pi} L(2, \bar{\chi} \chi_4) \right).
\end{aligned} \tag{39}$$

Proof. By Lemmas 9 and 10, we get

$$\begin{aligned}
& 16\chi(4) q \sum_{a=1}^q a \chi(a) + 16\chi(4) q^2 \sum_{1 \leq a \leq q/4} \chi(a) \\
& - 64\chi(4) q \sum_{1 \leq a \leq q/4} a \chi(a) \\
& = \begin{cases} \sum_{a=1}^{4q} a^2 \chi(a) \chi_4(a), & \text{if } q \equiv 1 \pmod{4} \\ -\sum_{a=1}^{4q} a^2 \chi(a) \chi_4(a), & \text{if } q \equiv 3 \pmod{4} \end{cases} \\
& = \chi_4(q) \sum_{a=1}^{4q} a^2 \chi \chi_4(a).
\end{aligned} \tag{40}$$

From the Fourier expansion for primitive character sums (see [10] or [11])

$$\sum_{a < \lambda q} \chi(a) = \begin{cases} \frac{\tau(\chi) \sum_{n=1}^{+\infty} \bar{\chi}(n) \sin(2\pi n \lambda)}{\pi \tau(\chi) L(1, \bar{\chi})}, & \text{if } \chi(-1) = 1, \\ -\frac{\pi i \tau(\chi)}{\tau(\chi)} \times \sum_{n=1}^{+\infty} \frac{\bar{\chi}(n) \cos(2\pi n \lambda)}{n}, & \text{if } \chi(-1) = -1, \end{cases} \tag{41}$$

we easily have

$$\sum_{1 \leq a \leq q/4} \chi(a) = \frac{2 + \bar{\chi}(2) - \bar{\chi}(4)}{2\pi i} \tau(\chi) L(1, \bar{\chi}). \tag{42}$$

Note that $\chi \chi_4$ is a primitive character modulo $4q$ satisfying $\chi \chi_4(-1) = 1$, and

$$\begin{aligned}
& \tau(\chi \chi_4) = \sum_{a=1}^{4q} \chi \chi_4(a) e\left(\frac{a}{4q}\right) \\
& = \sum_{a=1}^4 \sum_{b=1}^q \chi(4b+qa) \chi_4(4b+qa) e\left(\frac{4b+qa}{4q}\right) \\
& = \sum_{a=1}^4 \sum_{b=1}^q \chi(4b) \chi_4(qa) e\left(\frac{b}{q} + \frac{a}{4}\right)
\end{aligned}$$

$$\begin{aligned}
 &= \chi(4) \chi_4(q) \left(\sum_{a=1}^4 \chi_4(a) e\left(\frac{a}{4}\right) \right) \\
 &\quad \times \left(\sum_{b=1}^q \chi(b) e\left(\frac{b}{q}\right) \right) \\
 &= 2i\chi(4) \chi_4(q) \tau(\chi),
 \end{aligned} \tag{43}$$

then from (3) we have

$$\begin{aligned}
 \sum_{a=1}^q a\chi(a) &= -\frac{q}{\pi i} \tau(\chi) L(1, \bar{\chi}), \\
 \sum_{a=1}^{4q} a^2 \chi \chi_4(a) &= \frac{16q^2}{\pi^2} \tau(\chi \chi_4) L(2, \bar{\chi} \chi_4) \\
 &= \frac{32i\chi(4) \chi_4(q)}{\pi^2} q^2 \tau(\chi) L(2, \bar{\chi} \chi_4).
 \end{aligned} \tag{44}$$

Therefore

$$\begin{aligned}
 &-\frac{16\chi(4)}{\pi i} q^2 \tau(\chi) L(1, \bar{\chi}) \\
 &+ \frac{8\chi(4)(2 + \bar{\chi}(2) - \bar{\chi}(4))}{\pi i} q^2 \tau(\chi) L(1, \bar{\chi}) \\
 &- 64\chi(4) q \sum_{1 \leq a \leq q/4} a\chi(a) \\
 &= \frac{32i\chi(4)}{\pi^2} q^2 \tau(\chi) L(2, \bar{\chi} \chi_4).
 \end{aligned} \tag{45}$$

Then we have

$$\begin{aligned}
 &\sum_{1 \leq a \leq q/4} a\chi(a) \\
 &= \frac{q}{8\pi i} \tau(\chi) \left(\bar{\chi}(2) L(1, \bar{\chi}) - \bar{\chi}(4) L(1, \bar{\chi}) \right. \\
 &\quad \left. + \frac{4}{\pi} L(2, \bar{\chi} \chi_4) \right).
 \end{aligned} \tag{46}$$

□

3. Express the Character Sum in terms of Gauss Sums and L -Functions (II)

Let χ be an even primitive character modulo q . In this section, we express $\sum_{1 \leq a \leq q/4} a\chi(a)$ in terms of Gauss sums and Dirichlet L -functions.

Lemma 12. *Let $q > 2$ be an odd number, and let χ be a nonprincipal character modulo q . Then*

$$\begin{aligned}
 &4\chi(2) q \sum_{a=1}^{(q-1)/2} a\chi(a) \\
 &= (2\chi(2) + 1) q \sum_{a=1}^q a\chi(a) - (4\chi(2) - 1) \sum_{a=1}^q a^2 \chi(a),
 \end{aligned}$$

$$\begin{aligned}
 &4\chi(2) q \sum_{a=(q+1)/2}^q a\chi(a) \\
 &= (2\chi(2) - 1) q \sum_{a=1}^q a\chi(a) + (4\chi(2) - 1) \sum_{a=1}^q a^2 \chi(a).
 \end{aligned} \tag{47}$$

Proof. We have

$$\begin{aligned}
 &\sum_{a=1}^q (2a)^2 \chi(2a) \\
 &= \sum_{a=1}^{(q-1)/2} (2a)^2 \chi(2a) + \sum_{a=(q+1)/2}^q (2a)^2 \chi(2a) \\
 &= \sum_{a=1}^{(q-1)/2} (2a)^2 \chi(2a) \\
 &\quad + \sum_{b=1}^{(q+1)/2} (2b + q - 1)^2 \chi(2b + q - 1) \\
 &= \sum_{a=1}^{(q-1)/2} (2a)^2 \chi(2a) \\
 &\quad + \sum_{b=1}^{(q+1)/2} (2b - 1)^2 \chi(2b - 1) \\
 &\quad + 2q \sum_{b=1}^{(q+1)/2} (2b - 1) \chi(2b - 1) \\
 &\quad + q^2 \sum_{b=1}^{(q+1)/2} \chi(2b - 1) \\
 &= \sum_{a=1}^q a^2 \chi(a) + 2q \sum_{a=1}^{(q+1)/2} (2a - 1) \chi(2a - 1) \\
 &\quad + q^2 \sum_{a=1}^{(q+1)/2} \chi(2a - 1).
 \end{aligned} \tag{48}$$

Since

$$\begin{aligned}
 &\sum_{a=1}^{(q+1)/2} \chi(2a - 1) + \sum_{a=1}^{(q-1)/2} \chi(2a) = \sum_{a=1}^q \chi(a) = 0, \\
 &\sum_{a=1}^{(q+1)/2} (2a - 1) \chi(2a - 1) + \sum_{a=1}^{(q-1)/2} 2a\chi(2a) = \sum_{a=1}^q a\chi(a),
 \end{aligned} \tag{49}$$

we have

$$\sum_{a=1}^q (2a)^2 \chi(2a) = \sum_{a=1}^q a^2 \chi(a)$$

$$\begin{aligned}
& + 2q \sum_{a=1}^q a\chi(a) - 4\chi(2)q \sum_{a=1}^{(q-1)/2} a\chi(a) \\
& - \chi(2)q^2 \sum_{a=1}^{(q-1)/2} \chi(a).
\end{aligned} \tag{50}$$

It is not hard to show that

$$\begin{aligned}
\chi(2)q \sum_{a=1}^{(q-1)/2} \chi(a) &= (1 - 2\chi(2)) \sum_{a=1}^q a\chi(a), \\
\chi(2)q \sum_{a=(q+1)/2}^q \chi(a) &= (2\chi(2) - 1) \sum_{a=1}^q a\chi(a).
\end{aligned} \tag{51}$$

Therefore

$$\begin{aligned}
4\chi(2)q \sum_{a=1}^{(q-1)/2} a\chi(a) \\
&= (2\chi(2) + 1)q \\
&\quad \times \sum_{a=1}^q a\chi(a) - (4\chi(2) - 1) \sum_{a=1}^q a^2\chi(a).
\end{aligned} \tag{52}$$

Note that

$$\sum_{a=1}^{(q-1)/2} a\chi(a) + \sum_{a=(q+1)/2}^q a\chi(a) = \sum_{a=1}^q a\chi(a), \tag{53}$$

we have

$$\begin{aligned}
4\chi(2)q \sum_{a=(q+1)/2}^q a\chi(a) \\
&= (2\chi(2) - 1)q \\
&\quad \times \sum_{a=1}^q a\chi(a) + (4\chi(2) - 1) \sum_{a=1}^q a^2\chi(a).
\end{aligned} \tag{54}$$

□

Lemma 13. Let $q \geq 5$ be an odd number, and let χ be a nonprincipal even character modulo q . If $q \equiv 1 \pmod{4}$, then

$$\begin{aligned}
16\chi(4) \sum_{a=1}^{q-1} a^2\chi(a) \\
&= (8\chi(4) - 2\chi(2) + 1) \\
&\quad \times \sum_{a=1}^q a^2\chi(a) - 16q\chi(4) \\
&\quad \times \sum_{a=1}^{(q-1)/4} a\chi(a) + 4q^2\chi(4) \sum_{a=1}^{(q-1)/4} \chi(a).
\end{aligned} \tag{55}$$

While if $q \equiv 3 \pmod{4}$, we have

$$\begin{aligned}
16\chi(4) \sum_{a=1}^{q-1} a^2\chi(a) \\
&= (8\chi(4) - 2\chi(2) + 1) \sum_{a=1}^q a^2\chi(a) - 16q\chi(4) \\
&\quad \times \sum_{a=1}^{(q-3)/4} a\chi(a) + 4q^2\chi(4) \sum_{a=1}^{(q-3)/4} \chi(a).
\end{aligned} \tag{56}$$

Proof. First suppose that $q \equiv 1 \pmod{4}$. Then $\bar{4} = (3q+1)/4$. We have

$$\begin{aligned}
16\chi(4) \sum_{a=1}^{q-1} a^2\chi(a) \\
&= \sum_{a=1}^{q-1} (4a)^2\chi(4a) = \sum_{a=1}^{(q-1)/4} (4a)^2\chi(4a) \\
&\quad + \sum_{a=(q+3)/4}^{(2q-2)/4} (4a)^2\chi(4a) + \sum_{a=(2q+2)/4}^{(3q-3)/4} (4a)^2\chi(4a) \\
&\quad + \sum_{a=(3q+1)/4}^{q-1} (4a)^2\chi(4a) \\
&= \sum_{a=1}^{(q-1)/4} (4a)^2\chi(4a) \\
&\quad + \sum_{a=1}^{(q-1)/4} (4a+q-1)^2\chi(4a-1) \\
&\quad + \sum_{a=1}^{(q-1)/4} (4a+2q-2)^2\chi(4a-2) \\
&\quad + \sum_{a=1}^{(q-1)/4} (4a+3q-3)^2\chi(4a-3) \\
&= \sum_{a=1}^{(q-1)/4} (4a)^2\chi(4a) \\
&\quad + \sum_{a=1}^{(q-1)/4} (4a-1)^2\chi(4a-1) \\
&\quad + \sum_{a=1}^{(q-1)/4} (4a-2)^2\chi(4a-2) \\
&\quad + \sum_{a=1}^{(q-1)/4} (4a-3)^2\chi(4a-3)
\end{aligned}$$

$$\begin{aligned}
 & + 2q \sum_{a=1}^{(q-1)/4} (4a-1) \chi(4a-1) \\
 & + 4q \sum_{a=1}^{(q-1)/4} (4a-2) \chi(4a-2) \\
 & + 6q \sum_{a=1}^{(q-1)/4} (4a-3) \chi(4a-3) \\
 & + q^2 \sum_{a=1}^{(q-1)/4} \chi(4a-1) + 4q^2 \sum_{a=1}^{(q-1)/4} \chi(4a-2) \\
 & + 9q^2 \sum_{a=1}^{(q-1)/4} \chi(4a-3) \\
 & = \sum_{a=1}^{q-1} a^2 \chi(a) + 8q\chi(4) \sum_{a=1}^{(q-1)/4} \left(a + \frac{q-1}{4}\right) \chi\left(a + \frac{q-1}{4}\right) \\
 & - 2q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi\left(a + \frac{q-1}{4}\right) \\
 & + 16q\chi(4) \sum_{a=1}^{(q-1)/4} \left(a + \frac{q-1}{2}\right) \chi\left(a + \frac{q-1}{2}\right) \\
 & - 8q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi\left(a + \frac{q-1}{2}\right) \\
 & + 24q\chi(4) \sum_{a=1}^{(q-1)/4} \left(a + \frac{3q-3}{4}\right) \chi\left(a + \frac{3q-3}{4}\right) \\
 & - 18q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi\left(a + \frac{3q-3}{4}\right) \\
 & + q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi\left(a + \frac{q-1}{4}\right) \\
 & + 4q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi\left(a + \frac{q-1}{2}\right) \\
 & + 9q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi\left(a + \frac{3q-3}{4}\right) \\
 & = \sum_{a=1}^{q-1} a^2 \chi(a) + 8q\chi(4) \sum_{a=(q+3)/4}^{(q-1)/2} a\chi(a) \\
 & + 16q\chi(4) \sum_{a=(q+1)/2}^{(3q-3)/4} a\chi(a) \\
 & + 24q\chi(4) \sum_{a=(3q+1)/4}^{q-1} a\chi(a)
 \end{aligned}$$

$$\begin{aligned}
 & - q^2 \chi(4) \sum_{a=(q+3)/4}^{(q-1)/2} \chi(a) - 4q^2 \chi(4) \sum_{a=(q+1)/2}^{(3q-3)/4} \chi(a) \\
 & - 9q^2 \chi(4) \sum_{a=(3q+1)/4}^{q-1} \chi(a).
 \end{aligned} \tag{57}$$

Note that $\sum_{1 \leq a \leq (q-1)/2} \chi(a) = 0$ and $\sum_{a=1}^q a\chi(a) = 0$ for even character χ . By Lemma 12 we have

$$\begin{aligned}
 & 16\chi(4) \sum_{a=1}^{q-1} a^2 \chi(a) \\
 & = \sum_{a=1}^q a^2 \chi(a) - 16q\chi(4) \\
 & \quad \times \sum_{a=1}^{(q-1)/4} a\chi(a) - 8q\chi(4) \sum_{a=1}^{(q-1)/2} a\chi(a) \\
 & \quad + 4q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi(a) \\
 & = (8\chi(4) - 2\chi(2) + 1) \\
 & \quad \times \sum_{a=1}^q a^2 \chi(a) - 16q\chi(4) \sum_{a=1}^{(q-1)/4} a\chi(a) \\
 & \quad + 4q^2 \chi(4) \sum_{a=1}^{(q-1)/4} \chi(a).
 \end{aligned} \tag{58}$$

Now assume that $q \equiv 3 \pmod{4}$. Then $\bar{4} = (q+1)/4$. We have

$$\begin{aligned}
 & 16\chi(4) \sum_{a=1}^{q-1} a^2 \chi(a) \\
 & = \sum_{a=1}^{q-1} (4a)^2 \chi(4a) \\
 & = \sum_{a=1}^{(q-3)/4} (4a)^2 \chi(4a) + \sum_{a=(q+1)/4}^{(2q-2)/4} (4a)^2 \chi(4a) \\
 & \quad + \sum_{a=(2q+2)/4}^{(3q-1)/4} (4a)^2 \chi(4a) + \sum_{a=(3q+3)/4}^{q-1} (4a)^2 \chi(4a) \\
 & = \sum_{a=1}^{(q-3)/4} (4a)^2 \chi(4a) \\
 & \quad + \sum_{a=1}^{(q+1)/4} (4a+q-3)^2 \chi(4a-3) \\
 & \quad + \sum_{a=1}^{(q+1)/4} (4a+2q-2)^2 \chi(4a-2)
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{a=1}^{(q-3)/4} (4a+3q-1)^2 \chi(4a-1) \\
& = \sum_{a=1}^{(q-3)/4} (4a)^2 \chi(4a) + \sum_{a=1}^{(q-3)/4} (4a-1)^2 \chi(4a-1) \\
& + \sum_{a=1}^{(q+1)/4} (4a-2)^2 \chi(4a-2) \\
& + \sum_{a=1}^{(q+1)/4} (4a-3)^2 \chi(4a-3) \\
& + 6q \sum_{a=1}^{(q-3)/4} (4a-1) \chi(4a-1) \\
& + 4q \sum_{a=1}^{(q+1)/4} (4a-2) \chi(4a-2) \\
& + 2q \sum_{a=1}^{(q+1)/4} (4a-3) \chi(4a-3) \\
& + 9q^2 \sum_{a=1}^{(q-3)/4} \chi(4a-1) + 4q^2 \sum_{a=1}^{(q+1)/4} \chi(4a-2) \\
& + q^2 \sum_{a=1}^{(q+1)/4} \chi(4a-3) \\
& = \sum_{a=1}^{q-1} a^2 \chi(a) \\
& + 24q\chi(4) \sum_{a=1}^{(q-3)/4} \left(a + \frac{3q-1}{4}\right) \chi\left(a + \frac{3q-1}{4}\right) \\
& - 18q^2 \chi(4) \sum_{a=1}^{(q-3)/4} \chi\left(a + \frac{3q-1}{4}\right) \\
& + 16q\chi(4) \sum_{a=1}^{(q+1)/4} \left(a + \frac{q-1}{2}\right) \chi\left(a + \frac{q-1}{2}\right) \\
& - 8q^2 \chi(4) \sum_{a=1}^{(q+1)/4} \chi\left(a + \frac{q-1}{2}\right) \\
& + 8q\chi(4) \sum_{a=1}^{(q+1)/4} \left(a + \frac{q-3}{4}\right) \chi\left(a + \frac{q-3}{4}\right) \\
& - 2q^2 \chi(4) \sum_{a=1}^{(q+1)/4} \chi\left(a + \frac{q-3}{4}\right) \\
& + 9q^2 \chi(4) \sum_{a=1}^{(q-3)/4} \chi\left(a + \frac{3q-1}{4}\right) \\
& + 4q^2 \chi(4) \sum_{a=1}^{(q+1)/4} \chi\left(a + \frac{q-1}{2}\right)
\end{aligned}$$

$$\begin{aligned}
& + q^2 \chi(4) \sum_{a=1}^{(q+1)/4} \chi\left(a + \frac{q-3}{4}\right) \\
& = \sum_{a=1}^{q-1} a^2 \chi(a) + 24q\chi(4) \sum_{a=(3q+3)/4}^{q-1} a\chi(a) \\
& + 16q\chi(4) \sum_{a=(q+1)/2}^{(3q-1)/4} a\chi(a) \\
& + 8q\chi(4) \sum_{a=(q+1)/4}^{(q-1)/2} a\chi(a) \\
& - 9q^2 \chi(4) \sum_{a=(3q+3)/4}^{q-1} \chi(a) \\
& - 4q^2 \chi(4) \sum_{a=(q+1)/2}^{(3q-1)/4} \chi(a) \\
& - q^2 \chi(4) \sum_{a=(q+1)/4}^{(q-1)/2} \chi(a).
\end{aligned} \tag{59}$$

Note that $\sum_{1 \leq a \leq (q-1)/2} \chi(a) = 0$ and $\sum_{a=1}^q a\chi(a) = 0$ for even character χ . By Lemma 12 we have

$$\begin{aligned}
& 16\chi(4) \sum_{a=1}^{q-1} a^2 \chi(a) \\
& = \sum_{a=1}^q a^2 \chi(a) - 16q\chi(4) \sum_{a=1}^{(q-3)/4} a\chi(a) \\
& - 8q\chi(4) \sum_{a=1}^{(q-1)/2} a\chi(a) + 4q^2 \chi(4) \sum_{a=1}^{(q-3)/4} \chi(a) \\
& = (8\chi(4) - 2\chi(2) + 1) \sum_{a=1}^q a^2 \chi(a) \\
& - 16q\chi(4) \sum_{a=1}^{(q-3)/4} a\chi(a) \\
& + 4q^2 \chi(4) \sum_{a=1}^{(q-3)/4} \chi(a).
\end{aligned} \tag{60}$$

□

Now we express $\sum_{1 \leq a \leq q/4} a\chi(a)$ in terms of Gauss sums and Dirichlet L -functions.

Theorem 14. *Let χ be an even primitive character modulo odd integer $q \geq 5$. Then one has*

$$\begin{aligned}
\sum_{1 \leq a \leq q/4} a\chi(a) & = \frac{q}{16\pi^2} (\bar{\chi}(4) - 2\bar{\chi}(2) - 8) \tau(\chi) L(2, \bar{\chi}) \\
& + \frac{q}{4\pi} \tau(\chi) L(1, \bar{\chi}\chi_4).
\end{aligned} \tag{61}$$

Proof. By Lemma 13, (3), and (37), we have

$$\begin{aligned} & 16q\chi(4) \sum_{1 \leq a \leq q/4} a\chi(a) \\ &= (1 - 2\chi(2) - 8\chi(4)) \sum_{a=1}^q a^2\chi(a) \\ &\quad + 4q^2\chi(4) \sum_{1 \leq a \leq q/4} \chi(a) \\ &= \frac{q^2}{\pi^2} (1 - 2\chi(2) - 8\chi(4)) \tau(\chi) L(2, \bar{\chi}) \\ &\quad + \frac{4q^2}{\pi} \chi(4) \tau(\chi) L(1, \bar{\chi}\chi_4). \end{aligned} \quad (62)$$

Therefore

$$\begin{aligned} \sum_{1 \leq a \leq q/4} a\chi(a) &= \frac{q}{16\pi^2} (\bar{\chi}(4) - 2\bar{\chi}(2) - 8) \\ &\quad \times \tau(\chi) L(2, \bar{\chi}) + \frac{q}{4\pi} \tau(\chi) L(1, \bar{\chi}\chi_4). \quad \square \end{aligned} \quad (63)$$

4. Mean Values of Dirichlet L -Functions

In this section, we will study the mean values of Dirichlet L -functions, which will be used to prove Theorems 3 and 4.

Lemma 15. Let q and r be integers with $q \geq 2$ and $(r, q) = 1$. Then one has the identities

$$\begin{aligned} \sum_{\chi \bmod q}^* \chi(r) &= \sum_{d|(q, r-1)} \mu\left(\frac{q}{d}\right) \phi(d), \\ J(q) &= \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d), \end{aligned} \quad (64)$$

where $\sum_{\chi \bmod q}^*$ denotes the summation over all primitive characters modulo q , and $J(q)$ is the number of primitive characters modulo q .

Proof. This is Lemma 3 of [12]. $\square$

Lemma 16. Let $q \geq 2$ be an odd number, and let $k \geq 0$ be an integer. Then one has

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{d(n) d(2^k n)}{n^2} &= \frac{5+3k}{5} \cdot \frac{\zeta^4(2)}{\zeta(4)} \prod_{p|q} \frac{(1-1/p^2)^3}{1+1/p^2}, \\ \sum_{n=1}^{\infty} \frac{d(n) d(2^k n)}{n^4} &= \frac{15k+17}{17} \cdot \frac{\zeta^4(4)}{\zeta(8)} \prod_{p|q} \frac{(1-1/p^4)^3}{1+1/p^4}, \end{aligned}$$

$$\begin{aligned} \sum_{\substack{n=1 \\ (n, 2q)=1}}^{\infty} \frac{d^2(n)}{n^2} &= \frac{27}{80} \cdot \frac{\zeta^4(2)}{\zeta(4)} \prod_{p|q} \frac{(1-1/p^2)^3}{1+1/p^2}, \\ \sum_{\substack{n=1 \\ (n, 2q)=1}}^{\infty} \frac{d^2(n)}{n^4} &= \frac{3375}{4352} \cdot \frac{\zeta^4(4)}{\zeta(8)} \prod_{p|q} \frac{(1-1/p^4)^3}{1+1/p^4}, \\ \sum_{\substack{n=1 \\ (n, 2q)=1}}^{\infty} \frac{d^2(n) \chi_4(n)}{n^3} \\ &= \frac{1}{(1-1/2^6)} \cdot \frac{L^4(3, \chi_4)}{\zeta(6)} \prod_{p|q} \frac{(1-\chi_4(p)/p^3)^3}{1+\chi_4(p)/p^3}, \\ \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{d(n) \tau_1(2^k n)}{n^2} \\ &= \frac{\zeta^2(2) L^2(3, \chi_4)}{L(5, \chi_4)} \\ &\quad \times \prod_{p|q} \frac{(1-1/p^2)^2 (1-\chi_4(p)/p^3)^2}{1-\chi_4(p)/p^5}, \\ \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{\tau_1(n) \tau_1(2^k n)}{n^2} \\ &= \zeta(4) L(3, \chi_4) \prod_{p|q} \left(1 - \frac{\chi_4(p)}{p^3}\right) \left(1 - \frac{1}{p^4}\right) \\ &\quad \times \prod_{p \nmid q} \left(1 + \frac{1}{p^2(1-\chi_4(p)/p)}\right), \\ \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{d(2^k n) \tau_2(n)}{n^3} \\ &= \left(\frac{15}{16}k + 1\right) \cdot \frac{\zeta^2(4) L^2(3, \chi_4)}{L(7, \chi_4)} \\ &\quad \times \prod_{p|q} \frac{(1-\chi_4(p)/p^3)^2 (1-1/p^4)^2}{1-\chi_4(p)/p^7}, \\ \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{d(n) \tau_2(2^k n)}{n^3} \\ &= \frac{1}{2^k} \cdot \frac{\zeta^2(4) L^2(3, \chi_4)}{L(7, \chi_4)} \\ &\quad \times \prod_{p|q} \frac{(1-\chi_4(p)/p^3)^2 (1-1/p^4)^2}{1-\chi_4(p)/p^7}, \end{aligned}$$

$$\begin{aligned}
& \sum_{\substack{n=1 \\ (n,2q)=1}}^{\infty} \frac{\chi_4(n) d(n) \tau_2(n)}{n^2} \\
&= \frac{9}{16} \cdot \frac{\zeta^2(2) L^2(3, \chi_4)}{L(5, \chi_4)} \\
&\quad \times \prod_{p|q} \frac{(1-1/p^2)^2 (1-\chi_4(p)/p^3)^2}{1-\chi_4(p)/p^5}, \\
& \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\tau_2(n) \tau_2(2^k n)}{n^2} \\
&= \frac{4}{2^{k5}} \cdot \zeta(4) L(3, \chi_4) \\
&\quad \times \prod_{p|q} \left(1 - \frac{\chi_4(p)}{p^3}\right) \left(1 - \frac{1}{p^4}\right) \\
&\quad \times \prod_{p \nmid q} \left(1 + \frac{1}{p^2(1-\chi_4(p)/p)}\right), \tag{65}
\end{aligned}$$

where $d(n) = \sum_{d|n} 1$, $\tau_1(n) = \sum_{d|n} (\chi_4(d)/d)$, and $\tau_2(n) = \sum_{d|n} (\chi_4(n/d)/d)$.

Proof. By the Euler product, we have

$$\begin{aligned}
& \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{d(n) d(2^k n)}{n^2} \\
&= \sum_{i=0}^{\infty} \sum_{\substack{m=1 \\ (m,2q)=1}}^{\infty} \frac{d(2^i m) d(2^{k+i} m)}{(2^i m)^2} \\
&= \sum_{i=0}^{\infty} \frac{d(2^i) d(2^{k+i})}{2^{2i}} \sum_{\substack{m=1 \\ (m,2q)=1}}^{\infty} \frac{d^2(m)}{m^2} \\
&= \left(\sum_{i=0}^{\infty} \frac{(i+1)(k+i+1)}{2^{2i}} \right) \prod_{p \nmid 2q} \left(\sum_{j=0}^{\infty} \frac{(j+1)^2}{p^{2j}} \right) \\
&= \frac{5/4 + (3/4)k}{(1-1/2^2)^3} \prod_{p \nmid 2q} \frac{1+1/p^2}{(1-1/p^2)^3} \\
&= \frac{5+3k}{5} \prod_p \frac{1+1/p^2}{(1-1/p^2)^3} \prod_{p|q} \frac{(1-1/p^2)^3}{1+1/p^2} \\
&= \frac{5+3k}{5} \cdot \frac{\zeta^4(2)}{\zeta(4)} \prod_{p|q} \frac{(1-1/p^2)^3}{1+1/p^2}. \tag{66}
\end{aligned}$$

Similarly, we can deduce the other identities. $\square$

Lemma 17. Let $q \geq 2$ be an odd number. For integers $k \geq 0$ and $l \geq 1$, one has

$$\begin{aligned}
& \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \chi(2^k) |L(1, \chi)|^4 \\
&= \frac{5+3k}{2^{k+1} \cdot 5} \cdot \frac{\zeta^4(2)}{\zeta(4)} J(q) \prod_{p|q} \frac{(1-1/p^2)^3}{1+1/p^2} + O(q^\epsilon), \\
& \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* |L(2, \chi \chi_4)|^4 \\
&= \frac{3375}{8704} \cdot \frac{\zeta^4(4)}{\zeta(8)} J(q) \prod_{p|q} \frac{(1-1/p^4)^3}{1+1/p^4} + O(q^\epsilon), \\
& \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \bar{\chi}(2^k) L^2(1, \bar{\chi}) L(1, \chi) L(2, \chi \chi_4) \\
&= \frac{\zeta^2(2) L^2(3, \chi_4)}{2^{k+1} L(5, \chi_4)} J(q) \\
&\quad \times \prod_{p|q} \frac{(1-1/p^2)^2 (1-\chi_4(p)/p^3)^2}{1-\chi_4(p)/p^5} + O(q^\epsilon), \\
& \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \bar{\chi}(2^l) L^2(1, \bar{\chi}) L^2(2, \chi \chi_4) \ll q^\epsilon, \\
& \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \bar{\chi}(2^l) L(1, \bar{\chi}) L(2, \bar{\chi} \chi_4) L^2(2, \chi \chi_4) \ll q^\epsilon, \\
& \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \chi(2^k) |L(1, \chi)|^2 |L(2, \chi \chi_4)|^2 \\
&= \frac{\zeta(4) L(3, \chi_4)}{2^{k+1}} J(q) \prod_{p|q} \left(1 - \frac{\chi_4(p)}{p^3}\right) \left(1 - \frac{1}{p^4}\right) \\
&\quad \times \prod_{p \nmid q} \left(1 + \frac{1}{p^2(1-\chi_4(p)/p)}\right) + O(q^\epsilon). \tag{67}
\end{aligned}$$

Proof. We only prove the first formula since, similarly, we can get the others. Let $d(n) = \sum_{d|n} 1$ be the divisor function. For $N \geq q^2$, by Abel's identity, we get

$$\begin{aligned}
L^2(1, \chi) &= \left(\sum_{n=1}^{\infty} \frac{\chi(n)}{n} \right)^2 \\
&= \sum_{1 \leq n \leq N} \frac{\chi(n) d(n)}{n} + O\left(\frac{q^{1/2} \log q \log N}{N^{1/2}} \right). \tag{68}
\end{aligned}$$

For $(r, q) = 1$, from Lemma 15, we have

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \chi(r)$$

$$= \frac{1}{2} \sum_{\chi \bmod q}^* (1 - \chi(-1)) \chi(r) \quad (69)$$

$$\begin{aligned} &= \frac{1}{2} \sum_{\chi \bmod q}^* \chi(r) - \frac{1}{2} \sum_{\chi \bmod q}^* \chi(-r) \\ &= \frac{1}{2} \sum_{d|(q, r-1)} \mu\left(\frac{q}{d}\right) \phi(d) \\ &\quad - \frac{1}{2} \sum_{d|(q, r+1)} \mu\left(\frac{q}{d}\right) \phi(d). \end{aligned}$$

(70)

Then from Lemma 16 we get

$$\begin{aligned} &\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \chi(2^k) |L(1, \chi)|^4 = \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \chi(2^k) \left| \sum_{1 \leq n \leq N} \frac{\chi(n) d(n)}{n} + O\left(\frac{q^{1/2} \log q \log N}{N^{1/2}}\right) \right|^2 \\ &= \sum_{1 \leq n \leq N} \sum_{1 \leq m \leq N} \frac{d(n) d(m)}{nm} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \chi(2^k n) \bar{\chi}(m) + O\left(\frac{q^{3/2} \log q \log^3 N}{N^{1/2}}\right) \\ &= \frac{1}{2} \sum_{1 \leq n \leq N} \sum_{1 \leq m \leq N} \frac{d(n) d(m)}{nm} \sum_{\substack{d|(q, 2^k n-m) \\ (n, q)=1}} \mu\left(\frac{q}{d}\right) \phi(d) \\ &\quad - \frac{1}{2} \sum_{1 \leq n \leq N} \sum_{1 \leq m \leq N} \frac{d(n) d(m)}{nm} \sum_{\substack{d|(q, 2^k n+m) \\ (n, q)=1}} \mu\left(\frac{q}{d}\right) \phi(d) + O\left(\frac{q^{3/2} \log q \log^3 N}{N^{1/2}}\right) \\ &= \frac{J(q)}{2^{k+1}} \sum_{\substack{1 \leq n \leq N/2^k \\ (n, q)=1}} \frac{d(n) d(2^k n)}{n^2} + \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sum_{\substack{1 \leq n \leq N \\ (n, q)=1}} \sum_{\substack{1 \leq m \leq N \\ (m, q)=1 \\ 2^k n \equiv m \pmod{d} \\ 2^k n \not\equiv m}} \frac{d(n) d(m)}{nm} \\ &\quad - \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sum_{\substack{1 \leq n \leq N \\ (n, q)=1}} \sum_{\substack{1 \leq m \leq N \\ (m, q)=1 \\ 2^k n \equiv -m \pmod{d}}} \frac{d(n) d(m)}{nm} + O\left(\frac{q^{3/2} \log q \log^3 N}{N^{1/2}}\right) \\ &= \frac{J(q)}{2^{k+1}} \sum_{n=1}^{\infty} \frac{d(n) d(2^k n)}{n^2} + O\left(\sum_{d|q} \phi(d) \sum_{1 \leq n \leq N(1-2^k n)/d} \sum_{l \leq (N-2^k n)/d} \frac{d(n) d(ld+2^k n)}{n(ld+2^k n)}\right) \\ &\quad + O\left(\sum_{d|q} \phi(d) \sum_{1 \leq n \leq N(1+2^k n)/d} \sum_{l \leq (N+2^k n)/d} \frac{d(n) d(ld-2^k n)}{n(ld-2^k n)}\right) + O\left(\frac{q^{3/2} \log q \log^3 N}{N^{1/2}}\right) \\ &= \frac{J(q)}{2^{k+1}} \sum_{n=1}^{\infty} \frac{d(n) d(2^k n)}{n^2} + O(N^\epsilon) + O\left(\frac{q^{3/2} \log q \log^3 N}{N^{1/2}}\right) \\ &= \frac{5+3k}{2^{k+1} \cdot 5} \cdot \frac{\zeta^4(2)}{\zeta(4)} J(q) \prod_{p|q} \frac{(1-1/p^2)^3}{1+1/p^2} + O(N^\epsilon) + O\left(\frac{q^{3/2} \log q \log^3 N}{N^{1/2}}\right). \end{aligned}$$

(71)

Now taking $N = q^4$, we immediately get

$$\begin{aligned} & \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \chi(2^k) |L(1, \chi)|^4 \\ &= \frac{5+3k}{2^{k+1} \cdot 5} \cdot \frac{\zeta^4(2)}{\zeta(4)} J(q) \\ & \times \prod_{p|q} \frac{(1-1/p^2)^3}{1+1/p^2} + O(q^\epsilon). \end{aligned} \quad (72)$$

□

Lemma 18. Let $q \geq 2$ be an odd number. For integers $k \geq 0$ and $l \geq 1$, one has

$$\begin{aligned} & \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(2^k) |L(2, \bar{\chi})|^4 \\ &= \frac{15k+17}{2^{2k+1} \cdot 17} \cdot \frac{\zeta^4(4)}{\zeta(8)} \cdot J(q) \prod_{p|q} \frac{(1-1/p^4)^3}{1+1/p^4} \\ & \quad + O(q^\epsilon), \\ & \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* L^2(2, \bar{\chi}) L^2(1, \chi \chi_4) \\ &= \frac{1}{2(1-1/2^6)} \cdot \frac{L^4(3, \chi_4)}{\zeta(6)} \cdot J(q) \prod_{p|q} \frac{(1-\chi_4(p)/p^3)^3}{1+\chi_4(p)/p^3} \\ & \quad + O(q^\epsilon), \\ & \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(2^l) L^2(2, \bar{\chi}) L^2(1, \chi \chi_4) \ll q^\epsilon, \\ & \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \chi(2^k) L^2(2, \bar{\chi}) L(1, \chi \chi_4) L(2, \chi) \\ &= \frac{((15/16)k+1)}{2^{2k+1}} \cdot \frac{\zeta^2(4) L^2(3, \chi_4)}{L(7, \chi_4)} \\ & \quad \cdot J(q) \prod_{p|q} \frac{(1-\chi_4(p)/p^3)^2 (1-1/p^4)^2}{1-\chi_4(p)/p^7} \\ & \quad + O(q^\epsilon), \\ & \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(2^k) L^2(2, \bar{\chi}) L(1, \chi \chi_4) L(2, \chi) \\ &= \frac{1}{2^{k+1}} \cdot \frac{\zeta^2(4) L^2(3, \chi_4)}{L(7, \chi_4)} \\ & \quad \cdot J(q) \prod_{p|q} \frac{(1-\chi_4(p)/p^3)^2 (1-1/p^4)^2}{1-\chi_4(p)/p^7} \\ & \quad + O(q^\epsilon), \end{aligned}$$

$$\begin{aligned} & \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* |L(1, \bar{\chi} \chi_4)|^4 \\ &= \frac{27}{160} \cdot \frac{\zeta^4(2)}{\zeta(4)} \cdot J(q) \prod_{p|q} \frac{(1-1/p^2)^3}{1+1/p^2} + O(q^\epsilon), \\ & \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \chi(2^l) L^2(1, \bar{\chi} \chi_4) L(1, \chi \chi_4) L(2, \chi) \ll q^\epsilon, \\ & \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* L^2(1, \bar{\chi} \chi_4) L(1, \chi \chi_4) L(2, \chi) \\ &= \frac{9}{32} \cdot \frac{\zeta^2(2) L^2(3, \chi_4)}{L(5, \chi_4)} \\ & \quad \cdot J(q) \prod_{p|q} \frac{(1-1/p^2)^2 (1-\chi_4(p)/p^3)^2}{1-\chi_4(p)/p^5} \\ & \quad + O(q^\epsilon), \\ & \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(2^k) |L(1, \bar{\chi} \chi_4)|^2 |L(2, \bar{\chi})|^2 \\ &= \frac{2}{2^{2k} 5} \cdot \zeta(4) L(3, \chi_4) \\ & \quad \cdot J(q) \prod_{p|q} (1-\chi_4(p)/p^3) (1-1/p^4) \\ & \quad \times \prod_{p \nmid q} \left(1 + \frac{1}{p^2(1-\chi_4(p)/p)} \right) + O(q^\epsilon). \end{aligned} \quad (73)$$

Proof. By Lemma 17 and the methods proving Lemma 18, we can get this lemma. □

5. Proof of Theorems 3 and 4

First we prove Theorem 3. By Theorem 11 and Lemma 18, we have

$$\begin{aligned} & \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \left| \sum_{1 \leq a \leq q/4} a \chi(a) \right|^4 \\ &= \frac{q^6}{8^4 \pi^4} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \left| \bar{\chi}(2) L(1, \bar{\chi}) - \bar{\chi}(4) L(1, \bar{\chi}) \right. \\ & \quad \left. + \frac{4}{\pi} L(2, \bar{\chi} \chi_4) \right|^4 \end{aligned}$$

$$\begin{aligned}
 &= \frac{q^6}{8^4 \pi^4} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \left| \bar{\chi}(4) L^2(1, \bar{\chi}) + \bar{\chi}(16) L^2(1, \bar{\chi}) \right. \\
 &\quad + \frac{16}{\pi^2} L^2(2, \bar{\chi}\chi_4) - 2\bar{\chi}(8) L^2(1, \bar{\chi}) \\
 &\quad + \frac{8}{\pi} \bar{\chi}(2) L(1, \bar{\chi}) L(2, \bar{\chi}\chi_4) \\
 &\quad \left. - \frac{8}{\pi} \bar{\chi}(4) L(1, \bar{\chi}) L(2, \bar{\chi}\chi_4) \right|^2 \\
 &= \frac{3}{2048 \pi^4} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* |L(1, \chi)|^4 \\
 &\quad - \frac{1}{512 \pi^4} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \chi(2) |L(1, \chi)|^4 \\
 &\quad + \frac{1}{2048 \pi^4} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \chi(4) |L(1, \chi)|^4 \\
 &\quad + \frac{1}{16 \pi^8} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* |L(2, \chi\chi_4)|^4 \\
 &\quad - \frac{1}{256 \pi^5} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* L^2(1, \bar{\chi}) L(1, \chi) L(2, \chi\chi_4) \\
 &\quad + \frac{3}{256 \pi^5} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \bar{\chi}(2) L^2(1, \bar{\chi}) L(1, \chi) L(2, \chi\chi_4) \\
 &\quad - \frac{3}{256 \pi^5} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \bar{\chi}(4) L^2(1, \bar{\chi}) L(1, \chi) L(2, \chi\chi_4) \\
 &\quad + \frac{1}{256 \pi^5} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \bar{\chi}(8) L^2(1, \bar{\chi}) L(1, \chi) L(2, \chi\chi_4) \\
 &\quad + \frac{1}{128 \pi^6} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \bar{\chi}(4) L^2(1, \bar{\chi}) L^2(2, \chi\chi_4) \\
 &\quad - \frac{1}{64 \pi^6} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \bar{\chi}(8) L^2(1, \bar{\chi}) L^2(2, \chi\chi_4) \\
 &\quad + \frac{1}{128 \pi^6} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \bar{\chi}(16) L^2(1, \bar{\chi}) L^2(2, \chi\chi_4)
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{1}{16 \pi^7} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \bar{\chi}(2) L(1, \bar{\chi}) L(2, \bar{\chi}\chi_4) L^2(2, \chi\chi_4) \\
 &- \frac{1}{16 \pi^7} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \bar{\chi}(4) L(1, \bar{\chi}) L(2, \bar{\chi}\chi_4) L^2(2, \chi\chi_4) \\
 &+ \frac{1}{32 \pi^6} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* |L(1, \chi)|^2 |L(2, \chi\chi_4)|^2 \\
 &- \frac{1}{32 \pi^6} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \chi(2) |L(1, \chi)|^2 |L(2, \chi\chi_4)|^2 \\
 &= \frac{7}{2^{17} \cdot 3^2} q^6 J(q) \prod_{p|q} \frac{(1 - 1/p^2)^3}{1 + 1/p^2} \\
 &\quad + \frac{35}{2^{16} \cdot 3^2 \cdot 17} q^6 J(q) \prod_{p|q} \frac{(1 - 1/p^4)^3}{1 + 1/p^4} \\
 &\quad - \frac{L^2(3, \chi_4)}{2^{14} \cdot 3^2 \cdot \pi \cdot L(5, \chi_4)} \\
 &\quad \times q^6 J(q) \prod_{p|q} \frac{(1 - 1/p^2)^2 (1 - \chi_4(p)/p^3)^2}{1 - \chi_4(p)/p^5} \\
 &\quad + \frac{L(3, \chi_4)}{2^8 \cdot 5 \cdot 3^2 \cdot \pi^2} q^6 J(q) \\
 &\quad \times \prod_{p|q} \left(1 - \frac{\chi_4(p)}{p^3} \right) \left(1 - \frac{1}{p^4} \right) \\
 &\quad \times \prod_{p \nmid q} \left(1 + \frac{1}{p^2 (1 - \chi_4(p)/p)} \right) + O(q^{6+\epsilon}).
 \end{aligned} \tag{74}$$

This proves Theorem 3.

On the other hand, by Theorem 14 and Lemma 18, we have

$$\begin{aligned}
 &\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \left| \sum_{1 \leq a \leq q/4} a \chi(a) \right|^4 \\
 &= \frac{q^6}{16^4 \pi^8} \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* |(\bar{\chi}(4) - 2\bar{\chi}(2) - 8) L(2, \bar{\chi}) \\
 &\quad + 4\pi L(1, \bar{\chi}\chi_4)|^4 \\
 &= \frac{q^6}{16^4 \pi^8} \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* |(\bar{\chi}(4) - 2\bar{\chi}(2) - 8)^2 L^2(2, \bar{\chi})
 \end{aligned}$$

$$\begin{aligned}
& + 16\pi^2 L^2(1, \bar{\chi}\chi_4) \\
& + 8\pi(\bar{\chi}(4) - 2\bar{\chi}(2) - 8) \\
& \times L(1, \bar{\chi}\chi_4) L(2, \bar{\chi}) \Big|^2 \\
= & \frac{5281}{65536\pi^8} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* |L(2, \bar{\chi})|^4 \\
& + \frac{427}{8192\pi^8} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(2) |L(2, \bar{\chi})|^4 \\
& - \frac{227}{8192\pi^8} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(4) |L(2, \bar{\chi})|^4 \\
& - \frac{7}{1024\pi^8} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(8) |L(2, \bar{\chi})|^4 \\
& + \frac{1}{512\pi^8} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(16) |L(2, \bar{\chi})|^4 \\
& + \frac{1}{32\pi^6} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* L^2(2, \bar{\chi}) L^2(1, \chi\chi_4) \\
& + \frac{1}{64\pi^6} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(2) L^2(2, \bar{\chi}) L^2(1, \chi\chi_4) \\
& - \frac{3}{512\pi^6} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(4) L^2(2, \bar{\chi}) L^2(1, \chi\chi_4) \\
& - \frac{1}{512\pi^6} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(8) L^2(2, \bar{\chi}) L^2(1, \chi\chi_4) \\
& + \frac{1}{2048\pi^6} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(16) L^2(2, \bar{\chi}) L^2(1, \chi\chi_4) \\
& + \frac{1}{64\pi^7} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \chi(4) L^2(2, \bar{\chi}) L(1, \chi\chi_4) L(2, \chi) \\
& - \frac{3}{128\pi^7} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \chi(2) L^2(2, \bar{\chi}) L(1, \chi\chi_4) L(2, \chi) \\
& - \frac{147}{1024\pi^7} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* L^2(2, \bar{\chi}) L(1, \chi\chi_4) L(2, \chi)
\end{aligned}$$

$$\begin{aligned}
& - \frac{59}{1024\pi^7} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(2) L^2(2, \bar{\chi}) L(1, \chi\chi_4) L(2, \chi) \\
& + \frac{105}{4096\pi^7} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(4) L^2(2, \bar{\chi}) L(1, \chi\chi_4) L(2, \chi) \\
& + \frac{15}{2048\pi^7} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(8) L^2(2, \bar{\chi}) L(1, \chi\chi_4) L(2, \chi) \\
& - \frac{1}{512\pi^7} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(16) L^2(2, \bar{\chi}) L(1, \chi\chi_4) L(2, \chi) \\
& + \frac{1}{256\pi^4} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* |L(1, \bar{\chi}\chi_4)|^4 \\
& + \frac{1}{256\pi^5} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \chi(4) L^2(1, \bar{\chi}\chi_4) L(1, \chi\chi_4) L(2, \chi) \\
& - \frac{1}{128\pi^5} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \chi(2) L^2(1, \bar{\chi}\chi_4) L(1, \chi\chi_4) L(2, \chi) \\
& - \frac{1}{32\pi^5} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* L^2(1, \bar{\chi}\chi_4) L(1, \chi\chi_4) L(2, \chi) \\
& + \frac{69}{1024\pi^6} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* |L(1, \bar{\chi}\chi_4)|^2 |L(2, \bar{\chi})|^2 \\
& + \frac{7}{256\pi^6} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(2) |L(1, \bar{\chi}\chi_4)|^2 |L(2, \bar{\chi})|^2 \\
& - \frac{1}{64\pi^6} q^6 \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(4) |L(1, \bar{\chi}\chi_4)|^2 |L(2, \bar{\chi})|^2 \\
= & \frac{385}{2^{20} \cdot 51} \cdot q^6 J(q) \prod_{p|q} \frac{(1 - 1/p^4)^3}{1 + 1/p^4} \\
& + \frac{15L^4(3, \chi_4)}{\pi^{12}} \cdot q^6 J(q) \prod_{p|q} \frac{(1 - \chi_4(p)/p^3)^3}{1 + \chi_4(p)/p^3} \\
& - \frac{1423\pi L^2(3, \chi_4)}{2^{16} \cdot 5^2 \cdot 3^4 \cdot L(7, \chi_4)} \\
& \cdot q^6 J(q) \prod_{p|q} \frac{(1 - \chi_4(p)/p^3)^2 (1 - 1/p^4)^2}{1 - \chi_4(p)/p^7}
\end{aligned}$$

$$\begin{aligned}
& + \frac{3}{2^{16}} \cdot q^6 J(q) \prod_{p|q} \frac{(1 - 1/p^2)^3}{1 + 1/p^2} \\
& - \frac{L^2(3, \chi_4)}{2^{12} \cdot \pi \cdot L(5, \chi_4)} \\
& \cdot q^6 J(q) \prod_{p|q} \frac{(1 - 1/p^2)^2 (1 - \chi_4(p)/p^3)^2}{1 - \chi_4(p)/p^5} \\
& + \frac{L(3, \chi_4)}{2^{10} \cdot 3 \cdot \pi^2} \cdot q^6 J(q) \prod_{p|q} (1 - \chi_4(p)/p^3) (1 - 1/p^4) \\
& \times \prod_{p \nmid q} \left(1 + \frac{1}{p^2 (1 - \chi_4(p)/p)} \right) \\
& + O(q^{6+\epsilon}).
\end{aligned} \tag{75}$$

This completes the proof of Theorem 4.

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