Research Article

A Best Proximity Point Result in Modular Spaces with the Fatou Property

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Consider a nonself-mapping $T : A \rightarrow B$, where (A, B) is a pair of nonempty subsets of a modular space X_ρ . A best proximity point of T is a point $z \in A$ satisfying the condition: $\rho(z - Tz) = \inf \{\rho(x - y) : (x, y) \in A \times B\}$. In this paper, we introduce the class of proximal quasicontraction nonself-mappings in modular spaces with the Fatou property. For such mappings, we provide sufficient conditions assuring the existence and uniqueness of best proximity points.

1. Introduction and Preliminaries

Through this paper, we denote by $\mathbb{N}$ the set of positive integers including zero. Let X be a vector space over $\mathbb{R}$. We denote by 0_X its zero vector. According to Orlicz [1], a functional $\rho : X \rightarrow [0, \infty]$ is said to be modular, if, for any pair $(x, y) \in X^2$, the following conditions are satisfied:

- (i) $\rho(x) = 0$ if and only if $x = 0_X$;
- (ii) $\rho(-x) = \rho(x)$;
- (iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ whenever $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

If ρ is a modular in X , then the set

$$X_\rho := \{x \in X : \rho(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0\}, \quad (1)$$

called a modular space, is a vector space.

As a classical example of modulars, we may give the Orlicz modular defined for every measurable real function f by

$$\rho_\varphi(f) = \int_{\mathbb{R}} \varphi(|f(t)|) d\lambda(t), \quad (2)$$

where λ is the Lebesgue measure in $\mathbb{R}$ and $\varphi : \mathbb{R} \rightarrow [0, \infty)$ is a function satisfying some conditions. The modular space induced by the Orlicz modular ρ_φ is called the Orlicz space. For more examples of modular spaces, we refer the reader to [2–4].

Definition 1. Let X_ρ be a modular space.

- (1) The sequence $\{x_n\} \subset X_\rho$ is said to be ρ -convergent to $x \in X_\rho$ if $\rho(x_n - x) \rightarrow 0$, as $n \rightarrow \infty$.
- (2) The sequence $\{x_n\} \subset X_\rho$ is said to be ρ -Cauchy if $\rho(x_n - x_m) \rightarrow 0$, as $n, m \rightarrow \infty$.
- (3) A subset C of X_ρ is called ρ -closed if the ρ -limit of a ρ -convergent sequence of C always belongs to C .
- (4) A subset C of X_ρ is called ρ -complete if any ρ -Cauchy sequence in C is ρ -convergent and its ρ -limit belongs to C .

Definition 2. The modular ρ has the Fatou property if $\rho(x) \leq \liminf_{n \rightarrow \infty} \rho(x_n)$ whenever $\{x_n\}$ ρ -converges to x .

Recently, the existence and uniqueness of best proximity points in metric spaces were investigated by many authors; see [2, 5–14] and references therein. In this paper, we introduce the family of proximal quasicontraction nonself-mappings on modular spaces with the Fatou property. Our main result is a best proximity point theorem providing sufficient conditions assuring the existence and uniqueness of best proximity points for such mappings.

Let (A, B) be a pair of nonempty closed subsets of a modular space X_ρ . Through this paper, we will use the following notations:

$$\gamma(A, B) := \inf \{ \rho(x - y) : (x, y) \in A \times B \},$$

$$A_0 := \{a \in A : \rho(a - b) = \gamma(A, B) \text{ for some } b \in B\}, \quad (3)$$

$$B_0 := \{b \in B : \rho(a - b) = \gamma(A, B) \text{ for some } a \in A\}.$$

Definition 3. Let $T : A \rightarrow B$ be a given nonself-mapping. We say that $z \in A_0$ is a best proximity point of T if

$$\rho(z - Tz) = \gamma(A, B). \quad (4)$$

Clearly, from condition (i), if $A = B$, a best proximity point of T will be a fixed point of T .

Definition 4. A nonself-mapping $T : A \rightarrow B$ is said to be a proximal quasicontraction if there exists a number $q \in (0, 1)$ such that

$$\begin{aligned} \left. \begin{aligned} \rho(u - Tx) &= \gamma(A, B) \\ \rho(v - Ty) &= \gamma(A, B) \end{aligned} \right\} \implies \rho(u - v) \\ \leq q \max \{ \rho(x - y), \rho(x - u), \\ \rho(y - v), \rho(x - v), \\ \rho(y - u) \}, \end{aligned} \quad (5)$$

where $x, y, u, v \in A$.

Lemma 5. Let $T : A \rightarrow B$ be a nonself-mapping. Suppose that

- (i) $A_0 \neq \emptyset$;
- (ii) $T(A_0) \subseteq B_0$.

Then, for any $a \in A_0$, there exists a sequence $\{x_n\} \subset A_0$ such that

$$\begin{aligned} x_0 &= a, \\ \rho(x_{n+1} - Tx_n) &= \gamma(A, B), \quad \forall n \in \mathbb{N}. \end{aligned} \quad (6)$$

Proof. Let $a \in A_0$. From (ii), we have $Ta \in B_0$. By definition of the set B_0 , there exists $x_1 \in A_0$ such that $\rho(x_1 - Ta) = \gamma(A, B)$. Again, we have $Tx_1 \in B_0$, which implies that there exists $x_2 \in A_0$ such that $\rho(x_2 - Tx_1) = \gamma(A, B)$. Continuing this process, by induction, we obtain a sequence $\{x_n\} \subset A_0$ satisfying (6). $\square$

Definition 6. Under the assumptions of Lemma 5, any sequence $\{x_n\} \subset A_0$ satisfying (6) is called a proximal Picard sequence associated to $a \in A_0$. We denote by $PP(a)$ the set of all proximal sequences associated to $a \in A_0$.

Definition 7. Under the assumptions of Lemma 5, we say that A_0 is proximal T -orbitally ρ -complete if every ρ -Cauchy sequence $\{x_n\} \in PP(a)$ for some $a \in A_0$ ρ -converges to an element in A_0 .

Let $a \in A_0$ and $\{x_n\} \in PP(a)$. For all $n \in \mathbb{N}$, We denote

$$\delta_p(x_n) := \sup \{ \rho(x_{n+s} - x_{n+r}) : r, s \in \mathbb{N} \}. \quad (7)$$

Since $x_0 = a$, we have

$$\delta_p(a) = \sup \{ \rho(x_s - x_r) : r, s \in \mathbb{N} \}. \quad (8)$$

2. A Best Proximity Point Theorem

The following lemmas will be useful later.

Lemma 8. Let X_ρ be a modular space. Suppose that a nonself-mapping $T : A \rightarrow B$, where (A, B) is a pair of subsets of X , satisfies the following conditions:

- (i) $\exists a \in A_0 \mid \delta_p(a) < \infty$;
- (ii) $T(A_0) \subseteq B_0$;
- (iii) T is proximal quasi-contraction.

Then, for any $\{x_n\} \in PP(a)$, one has

$$\rho(x_n - x_{n+m}) \leq \delta_p(x_n) \leq q^n \delta_p(a), \quad (9)$$

for any $n \geq 1$ and $m \in \mathbb{N}$.

Proof. Let $\{x_n\} \in PP(a)$ and $(s, r) \in \mathbb{N}^2$. From the definition of $PP(a)$, for all $n \geq 1$, we have

$$\rho(x_{n+s} - Tx_{n-1+s}) = \rho(x_{n+r} - Tx_{n-1+r}) = \gamma(A, B), \quad (10)$$

which implies, since T is a proximal quasi-contraction, that

$$\begin{aligned} \rho(x_{n+s} - x_{n+r}) &\leq q \max \{ \rho(x_{n-1+s} - x_{n-1+r}), \\ &\rho(x_{n-1+s} - x_{n+s}), \\ &\rho(x_{n-1+r} - x_{n+r}), \\ &\rho(x_{n-1+s} - x_{n+r}), \\ &\rho(x_{n-1+r} - x_{n+s}) \} \\ &\leq q \delta_p(x_{n-1}). \end{aligned} \quad (11)$$

This implies immediately that

$$\delta_p(x_n) \leq q \delta_p(x_{n-1}), \quad (12)$$

for all $n \geq 1$. Hence, for any $n \in \mathbb{N}$, we have

$$\delta_p(x_n) \leq q^n \delta_p(a). \quad (13)$$

Using the above inequality, for all $n \geq 1$ and $m \in \mathbb{N}$, we have

$$\rho(x_n - x_{n+m}) \leq \delta_p(x_n) \leq q^n \delta_p(a). \quad (14)$$

$\square$

Lemma 9. Let (A, B) be a pair of subsets of a modular space X_ρ . Let $T : A \rightarrow B$ be a given nonself-mapping. Suppose that

- (i) A_0 is proximal T -orbitally ρ -complete;

- (ii) $T(A_0) \subseteq B_0$;
- (iii) $\exists a \in A_0$ such that $\delta_p(a) < \infty$;
- (iv) T is proximal quasi-contraction;
- (v) ρ satisfies the Fatou property.

Then, any sequence $\{x_n\} \in PP(a)$ ρ -converges to some $z \in A_0$ such that

$$\rho(x_n - z) \leq q^n \delta_p(a), \quad (15)$$

for all $n \geq 1$. Moreover, there exists $w \in A_0$ such that

$$\rho(w - Tz) = \gamma(A, B). \quad (16)$$

Proof. Let $\{x_n\} \in PP(a)$. From Lemma 8, we know that $\{x_n\}$ is ρ -Cauchy. Since A_0 is proximal T -orbitally ρ -complete, then there exists $z \in A_0$ such that $\{x_n\}$ ρ -converges to z . Again, by Lemma 8, we have

$$\rho(x_n - x_{n+m}) \leq q^n \delta_p(a), \quad (17)$$

for any $n \geq 1$ and $m \in \mathbb{N}$. Letting $m \rightarrow \infty$ in the above inequality and using the Fatou property, we obtain

$$\rho(x_n - z) \leq q^n \delta_p(a), \quad (18)$$

for all $n \geq 1$. Now, since $Tz \in B_0$, by the definition of B_0 , there exists some $w \in A_0$ such that $\rho(w - Tz) = \gamma(A, B)$. $\square$

Now, we are ready to state and prove our main result.

Theorem 10. Suppose that the assumptions of the previous lemma are satisfied. Assume $\rho(z - w) < \infty$ and $\rho(a - w) < \infty$. Then, the ρ -limit $z \in A_0$ of $\{x_n\} \in PP(a)$ is a best proximity point of T . Moreover, if $u \in A_0$ is any best proximity point of T such that $\rho(z - u) < \infty$, then one has $z = u$.

Proof. By Lemma 9, we have

$$\rho(w - Tz) = \gamma(A, B). \quad (19)$$

On the other hand, from the definition of $\{x_n\}$, we have

$$\rho(x_1 - Ta) = \gamma(A, B). \quad (20)$$

Since T is proximal quasi-contraction, we get that

$$\begin{aligned} \rho(w - x_1) &\leq q \max \{ \rho(a - z), \rho(z - w), \\ &\quad \rho(a - x_1), \rho(z - x_1), \rho(a - w) \}. \end{aligned} \quad (21)$$

Using Lemmas 8 and 9, we obtain that

$$\rho(w - x_1) \leq \max \{ q \delta_p(a), q \rho(z - w), q \rho(a - w) \}. \quad (22)$$

Again, from the definition of $\{x_n\}$, we have

$$\rho(x_2 - Tx_1) = \gamma(A, B). \quad (23)$$

Since T is proximal quasi-contraction, we get that

$$\begin{aligned} \rho(w - x_2) &\leq q \max \{ \rho(z - x_1), \rho(z - w), \rho(x_1 - x_2), \\ &\quad \rho(z - x_2), \rho(x_1 - w) \} \\ &\leq q \max \{ q \delta_p(a), \rho(z - w), \delta_p(x_1), \\ &\quad q^2 \delta_p(a), \rho(x_1 - w) \} \\ &\leq q \max \{ q \delta_p(a), \rho(z - w), q \delta_p(a), \\ &\quad q^2 \delta_p(a), \rho(x_1 - w) \} \\ &= q \max \{ q \delta_p(a), \rho(z - w), \rho(x_1 - w) \} \\ &\text{(from (22))} \leq \max \{ q^2 \delta_p(a), q \rho(z - w), q^2 \rho(a - w) \}. \end{aligned} \quad (24)$$

Thus, we proved that

$$\rho(w - x_2) \leq \max \{ q^2 \delta_p(a), q \rho(z - w), q^2 \rho(a - w) \}. \quad (25)$$

Continuing this process, by induction, we get that

$$\rho(w - x_n) \leq \max \{ q^n \delta_p(a), q \rho(z - w), q^n \rho(a - w) \}, \quad (26)$$

for all $n \geq 1$. Therefore, we have

$$\limsup_{n \rightarrow \infty} \rho(x_n - w) \leq q \rho(z - w). \quad (27)$$

Using the Fatou property, we get

$$\rho(z - w) \leq q \rho(z - w), \quad (28)$$

which implies, since $q < 1$, that $\rho(z - w) = 0$; that is, $z = w$. Thus, from (19), we get that

$$\rho(z - Tz) = \gamma(A, B). \quad (29)$$

Hence, z is a best proximity point of T .

Suppose now that $u \in A_0$ is a best proximity point of T such that $\rho(z - u) < \infty$. Since T is proximal quasi-contraction, we obtain that

$$\begin{aligned} \rho(z - u) &\leq q \max \{ \rho(z - u), \rho(z - z), \rho(u - u), \\ &\quad \rho(z - u), \rho(u - z) \} \\ &= q \rho(z - u). \end{aligned} \quad (30)$$

Since $q < 1$, we have $\rho(z - u) = 0$, which implies that $u = z$. $\square$

Consider now the case $A = B$. In this case, a best proximity point of $T : A \rightarrow B$ will be a fixed point of the self-mapping T .

Definition 11. We say that A is T -orbitally ρ -complete if $\{T^{ma}\}$ is a ρ -Cauchy for every $a \in A$, then it is ρ -convergent to an element of A .

Similarly to Ćirić [15] definition, Khamsi [16] introduced the concept of quasicontraction self-mappings in modular spaces.

Definition 12. The self-mapping $T : A \rightarrow A$ is said to be a quasicontraction if there exists a constant $q \in (0, 1)$ such that

$$\rho(Tx - Ty) \leq q \max \{ \rho(x - y), \rho(x - Tx), \rho(y - Ty), \rho(x - Ty), \rho(y - Tx) \}, \quad (31)$$

for all $x, y \in A$.

From Theorem 10, we can deduce the following result, that is, a slight extension of the fixed point theorem established by Khamsi in [16].

Corollary 13. Consider a self-mapping $T : A \rightarrow A$, where A is a nonempty subset of X_ρ . Suppose that the following conditions hold:

- (i) A is T -orbitally ρ -complete;
- (ii) $\exists a \in A$ such that $\sup \{ \rho(T^s a - T^r a) : s, r \in \mathbb{N} \} < \infty$;
- (iii) ρ satisfies the Fatou property;
- (iv) T is quasi-contraction.

Then, the sequence $\{T^{n_k}\}$ ρ -converges to some $z \in A$. Moreover, if $\rho(z - Tz) < \infty$ and $\rho(a - Tz) < \infty$, then z is a fixed point of T . If $u \in A$ is a fixed point of T with $\rho(z - u) < \infty$, then $u = z$.

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