Research Article

Representation Theorem for Generators of BSDEs Driven by G -Brownian Motion and Its Applications

Kun He¹ and Mingshang Hu²

¹ Department of Mathematics, Donghua University, 2999 North Renmin Road, Songjiang, Shanghai 201620, China

² School of Mathematics, Shandong University, 27 Shanda Nanlu, Jinan 250100, China

Correspondence should be addressed to Kun He; hekun@dhu.edu.cn

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We obtain a representation theorem for the generators of BSDEs driven by G -Brownian motions and then we use the representation theorem to get a converse comparison theorem for G -BSDEs and some equivalent results for nonlinear expectations generated by G -BSDEs.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and, for fixed $T \in [0, +\infty)$, let $(B_t)_{0 \leq t \leq T}$ be a standard Brownian motion and let $\mathcal{F}_t$ be the augmentation of $\sigma\{B_s, 0 \leq s \leq t\}$. Then Pardoux and Peng [1] introduced the backward stochastic differential equations (BSDEs) and proved the existence and uniqueness result of the BSDEs. In 1997, Peng [2] promoted g -expectations based on BSDEs. One of the important properties of g -expectations is comparison theorem or monotonicity. Chen [3] first considers a converse result of BSDEs under equal case. After that, Briand et al. [4] obtained a converse comparison theorem for BSDEs under general case. They also derived a representation theorem for the generator g . Following this paper, Jiang [5] discussed a more general representation theorem then, in his another paper [6], showed a more general converse comparison theorem. Here the representation theorem is an important method in solving the converse comparison problem and other problems (see Jiang [7]).

Peng [8–13] defined the G -expectations and G -Brownian motions (G -BMs) and proved the representation theorem of G -expectation by a set of singular probabilities, which differs from nonlinear g -expectations because g -expectations are equivalent with a group of absolutely continuous probabilities with respect to the probability measure P . Soner et al. [14] obtained an existence and uniqueness result of 2 BSDEs. Recently, Hu et al. [15] proved another existence and

uniqueness result on BSDEs driven by G -Brownian motions (G -BSDEs).

An important advantage of G -BSDEs is the easiness to define the nonlinear expectations. Hu et al. in [16] gave a comparison theorem for G -BSDEs and talked about the properties of corresponding nonlinear expectations. In this paper, we consider the representation theorem for generators of G -BSDEs and then consider the converse comparison theorem of G -BSDEs and some equivalent results for nonlinear expectations generated by G -BSDEs. In the following, in Section 2, we review some basic concepts and results about G -expectations. We give the representation theorem of G -BSDEs in Section 3. In Section 4, we consider the applications of representation theorem of G -BSDEs, which contain the converse comparison theorem and some equivalent results for nonlinear expectations generated by G -BSDEs.

2. Preliminaries

We review some basic notions and results of G -expectation, the related spaces of random variables, and the backward stochastic differential equations driven by a G -Brownian motion. The readers may refer to [10, 13, 15, 17–19] for more details.

Definition 1. Let Ω be a given set and let $\mathcal{H}$ be a vector lattice of real valued functions defined on Ω , namely, $c \in \mathcal{H}$ for each

constant c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of random variables. A sublinear expectation $\widehat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\widehat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, one has

- (a) monotonicity: if $X \geq Y$, then $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$;
- (b) constant preservation: $\widehat{\mathbb{E}}[c] = c$;
- (c) subadditivity: $\widehat{\mathbb{E}}[X + Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$;
- (d) positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X]$ for each $\lambda \geq 0$. $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sublinear expectation space.

Definition 2. Let X_1 and X_2 be two n -dimensional random vectors defined, respectively, in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $X_1 \stackrel{d}{=} X_2$, if $\widehat{\mathbb{E}}_1[\varphi(X_1)] = \widehat{\mathbb{E}}_2[\varphi(X_2)]$, for all $\varphi \in C_{b\text{-Lip}}(\mathbb{R}^n)$, where $C_{b\text{-Lip}}(\mathbb{R}^n)$ denotes the space of bounded and Lipschitz functions on $\mathbb{R}^n$.

Definition 3. In a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, a random vector $Y = (Y_1, \dots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $X = (X_1, \dots, X_m)$, $X_i \in \mathcal{H}$ under $\widehat{\mathbb{E}}[\cdot]$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{b\text{-Lip}}(\mathbb{R}^m \times \mathbb{R}^n)$ one has $\widehat{\mathbb{E}}[\varphi(X, Y)] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]$.

Definition 4 (G -normal distribution). A d -dimensional random vector $X = (X_1, \dots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called G -normally distributed if for each $a, b \geq 0$ one has

$$aX + b\bar{X} \stackrel{d}{=} \sqrt{a^2 + b^2} X, \quad (1)$$

where $\bar{X}$ is an independent copy of X ; that is, $\bar{X} \stackrel{d}{=} X$ and $\bar{X} \perp X$. Here, the letter G denotes the function

$$G(A) := \frac{1}{2} \widehat{\mathbb{E}}[\langle AX, X \rangle] : \mathbb{S}_d \longrightarrow \mathbb{R}, \quad (2)$$

where $\mathbb{S}_d$ denotes the collection of $d \times d$ symmetric matrices.

Peng [13] showed that $X = (X_1, \dots, X_d)$ is G -normally distributed if and only if for each $\varphi \in C_{b\text{-Lip}}(\mathbb{R}^d)$, $u(t, x) := \widehat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$, $(t, x) \in [0, \infty) \times \mathbb{R}^d$, is the solution of the following G -heat equation:

$$\partial_t u - G(D_x^2 u) = 0, \quad u(0, x) = \varphi(x). \quad (3)$$

The function $G(\cdot) : \mathbb{S}_d \rightarrow \mathbb{R}$ is a monotonic, sublinear mapping on $\mathbb{S}_d$ and $G(A) = (1/2)\widehat{\mathbb{E}}[\langle AX, X \rangle] \leq (1/2)|A|\widehat{\mathbb{E}}[|X|^2]$ implies that there exists a bounded, convex, and closed subset $\Gamma \subset \mathbb{S}_d^+$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}[\gamma A], \quad (4)$$

where $\mathbb{S}_d^+$ denotes the collection of nonnegative elements in $\mathbb{S}_d$.

In this paper, we only consider nondegenerate G -normal distribution; that is, there exists some $\underline{\sigma}^2 > 0$ such that $G(A) - G(B) \geq \underline{\sigma}^2 \text{tr}[A - B]$ for any $A \geq B$.

Definition 5. (i) Let $\Omega = C_0^d(\mathbb{R}^+)$ denote the space of $\mathbb{R}^d$ -valued continuous functions on $[0, \infty)$ with $\omega_0 = 0$ and let $B_t(\omega) = \omega_t$ be the canonical process. Set

$$L_{ip}(\Omega) := \left\{ \varphi(B_{t_1}, \dots, B_{t_n}) : n \geq 1, t_1, \dots, t_n \in [0, \infty), \right. \\ \left. \varphi \in C_{b\text{-Lip}}(\mathbb{R}^{d \times n}) \right\}. \quad (5)$$

Let $G : \mathbb{S}_d \rightarrow \mathbb{R}$ be a given monotonic and sublinear function. G -expectation is a sublinear expectation defined by

$$\widehat{\mathbb{E}}[X] = \widehat{\mathbb{E}}\left[\varphi\left(\sqrt{t_1 - t_0}\xi_1, \dots, \sqrt{t_m - t_{m-1}}\xi_m\right)\right], \quad (6)$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})$, where $\xi_1, \dots, \xi_n$ are identically distributed d -dimensional G -normally distributed random vectors in a sublinear expectation space $(\bar{\Omega}, \bar{\mathcal{H}}, \bar{\mathbb{E}})$ such that ξ_{i+1} is independent of $(\xi_1, \dots, \xi_i)$ for every $i = 1, \dots, m-1$. The corresponding canonical process $B_t = (B_t^i)_{i=1}^d$ is called a G -Brownian motion.

(ii) For each fixed $t \in [0, \infty)$, the conditional G -expectation $\widehat{\mathbb{E}}_t$ for $\xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}}) \in L_{ip}(\Omega)$, where without loss of generality we suppose $t_i = t$, is defined by

$$\widehat{\mathbb{E}}_t\left[\varphi\left(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}}\right)\right] \\ = \psi\left(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_i} - B_{t_{i-1}}\right), \quad (7)$$

where

$$\psi(x_1, \dots, x_i) \\ = \widehat{\mathbb{E}}\left[\varphi\left(x_1, \dots, x_i, B_{t_{i+1}} - B_{t_i}, \dots, B_{t_m} - B_{t_{m-1}}\right)\right]. \quad (8)$$

For each fixed $T > 0$, we set

$$L_{ip}(\Omega_T) := \left\{ \varphi(B_{t_1}, \dots, B_{t_n}) : n \geq 1, t_1, \dots, t_n \in [0, T], \right. \\ \left. \varphi \in C_{b\text{-Lip}}(\mathbb{R}^{d \times n}) \right\}. \quad (9)$$

For each $p \geq 1$, we denote by $L_G^p(\Omega)$ (resp., $L_G^p(\Omega_T)$) the completion of $L_{ip}(\Omega)$ (resp., $L_{ip}(\Omega_T)$) under the norm $\|\xi\|_{p,G} = (\widehat{\mathbb{E}}[|\xi|^p])^{1/p}$. It is easy to check that $L_G^q(\Omega) \subset L_G^p(\Omega)$ for $1 \leq p \leq q$ and $\widehat{\mathbb{E}}_t[\cdot]$ can be extended continuously to $L_G^1(\Omega)$.

For each fixed $\mathbf{a} \in \mathbb{R}^d$, $B_t^{\mathbf{a}} = \langle \mathbf{a}, B_t \rangle$ is a 1-dimensional $G_{\mathbf{a}}$ -Brownian motion, where $G_{\mathbf{a}}(\alpha) = (1/2)(\sigma_{\mathbf{a}\mathbf{a}^T}^2 \alpha^+ - \sigma_{-\mathbf{a}\mathbf{a}^T}^2 \alpha^-)$, $\sigma_{\mathbf{a}\mathbf{a}^T}^2 = 2G(\mathbf{a}\mathbf{a}^T)$ and $\sigma_{-\mathbf{a}\mathbf{a}^T}^2 = -2G(-\mathbf{a}\mathbf{a}^T)$. Let $\pi_t^N = \{t_0^N, \dots, t_N^N\}$, $N = 1, 2, \dots$, be a sequence of partitions of $[0, t]$ such that $\mu(\pi_t^N) = \max\{|t_{i+1}^N - t_i^N| : i = 0, \dots, N-1\} \rightarrow 0$; the quadratic variation process of $B^{\mathbf{a}}$ is defined by

$$\langle B^{\mathbf{a}} \rangle_t = \lim_{\mu(\pi_t^N) \rightarrow 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N}^{\mathbf{a}} - B_{t_j^N}^{\mathbf{a}})^2. \quad (10)$$

For each fixed $\mathbf{a}, \bar{\mathbf{a}} \in \mathbb{R}^d$, the mutual variation process of $B^{\mathbf{a}}$ and $B^{\bar{\mathbf{a}}}$ is defined by

$$\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}} \rangle_t = \frac{1}{4} [\langle B^{\mathbf{a}+\bar{\mathbf{a}}} \rangle_t - \langle B^{\mathbf{a}-\bar{\mathbf{a}}} \rangle_t]. \quad (11)$$

Definition 6. For fixed $T > 0$, let $M_G^0(0, T)$ be the collection of processes in the following form: for a given partition $\{t_0, \dots, t_N\} = \pi_T$ of $[0, T]$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j I_{[t_j, t_{j+1})}(t), \quad (12)$$

where $\xi_j \in L_{ip}(\Omega_{t_j})$, $j = 0, 1, 2, \dots, N-1$. For $p \geq 1$, one denotes by $H_G^p(0, T)$, $M_G^p(0, T)$ the completion of $M_G^0(0, T)$ under the norms $\|\eta\|_{H_G^p} = \{\widehat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$, $\|\eta\|_{M_G^p} = \{\widehat{\mathbb{E}}[\int_0^T |\eta_s|^p ds]\}^{1/p}$, respectively.

For each $\eta \in M_G^1(0, T)$, we can define the integrals $\int_0^T \eta_t dt$ and $\int_0^T \eta_t d\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}} \rangle_t$ for each $\mathbf{a}, \bar{\mathbf{a}} \in \mathbb{R}^d$. For each $\eta \in H_G^p(0, T; \mathbb{R}^d)$ with $p \geq 1$, we can define Itô's integral $\int_0^T \eta_t dB_t$.

Let $S_G^0(0, T) = \{h(t, B_{t_1 \wedge t}, \dots, B_{t_n \wedge t}) : t_1, \dots, t_n \in [0, T], h \in C_{b, \text{Lip}}(\mathbb{R}^{n+1})\}$. For $p \geq 1$ and $\eta \in S_G^0(0, T)$, set $\|\eta\|_{S_G^p} = \{\widehat{\mathbb{E}}[\sup_{t \in [0, T]} |\eta_t|^p]\}^{1/p}$. Denote by $S_G^p(0, T)$ the completion of $S_G^0(0, T)$ under the norm $\|\cdot\|_{S_G^p}$.

We consider the following type of G-BSDEs (in this paper, we always use Einstein convention):

$$\begin{aligned} Y_t = & \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g_{ij}(s, Y_s, Z_s) d\langle B^i, B^j \rangle_s \\ & - \int_t^T Z_s dB_s - (K_T - K_t), \end{aligned} \quad (13)$$

where

$$f(t, \omega, y, z), g_{ij}(t, \omega, y, z) : [0, T] \times \Omega_T \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}, \quad (14)$$

satisfy the following properties.

(H1) There exists some $\beta > 1$ such that for any $y, z, f(\cdot, \cdot, y, z), g_{ij}(\cdot, \cdot, y, z) \in M_G^\beta(0, T)$.

(H2) There exists some $L > 0$ such that

$$\begin{aligned} & |f(t, y, z) - f(t, y', z')| + \sum_{i,j=1}^d |g_{ij}(t, y, z) - g_{ij}(t, y', z')| \\ & \leq L(|y - y'| + |z - z'|). \end{aligned} \quad (15)$$

For simplicity, we denote by $\mathfrak{S}_G^\alpha(0, T)$ the collection of processes (Y, Z, K) such that $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T; \mathbb{R}^d)$, K is a decreasing G-martingale with $K_0 = 0$ and $K_T \in L_G^\alpha(\Omega_T)$.

Definition 7. Let $\xi \in L_G^\beta(\Omega_T)$ and f and g_{ij} satisfy (H1) and (H2) for some $\beta > 1$. A triplet of processes (Y, Z, K) is called a solution of (13) if for some $1 < \alpha \leq \beta$ the following properties hold:

$$(a) (Y, Z, K) \in \mathfrak{S}_G^\alpha(0, T);$$

$$(b) Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g_{ij}(s, Y_s, Z_s) d\langle B^i, B^j \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t).$$

Theorem 8 (see [15]). Assume that $\xi \in L_G^\beta(\Omega_T)$ and f and g_{ij} satisfy (H1) and (H2) for some $\beta > 1$. Then, (13) has a unique solution (Y, Z, K) . Moreover, for any $1 < \alpha < \beta$, one has $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T; \mathbb{R}^d)$, and $K_T \in L_G^\alpha(\Omega_T)$.

We have the following estimates.

Proposition 9 (see [15]). Let $\xi \in L_G^\beta(\Omega_T)$ and f, g_{ij} satisfy (H1) and (H2) for some $\beta > 1$. Assume that $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0, T)$ for some $1 < \alpha < \beta$ is a solution of (13). Then, there exists a constant $C_\alpha > 0$ depending on α, T, G, L such that

$$\begin{aligned} |Y_t|^\alpha & \leq C_\alpha \widehat{\mathbb{E}}_t \left[|\xi|^\alpha + \left(\int_t^T |h_s^0| ds \right)^\alpha \right], \\ \widehat{\mathbb{E}} \left[\left(\int_0^T |Z_s|^2 ds \right)^{\alpha/2} \right] & \leq C_\alpha \left\{ \widehat{\mathbb{E}} \left[\sup_{t \in [0, T]} |Y_t|^\alpha \right] + \left(\widehat{\mathbb{E}} \left[\sup_{t \in [0, T]} |Y_t|^\alpha \right] \right)^{1/2} \right. \\ & \quad \times \left(\widehat{\mathbb{E}} \left[\left(\int_0^T |h_s^0| ds \right)^\alpha \right] \right)^{1/2} \Bigg\}, \\ \widehat{\mathbb{E}} [K_t] & \leq C_\alpha \left\{ \widehat{\mathbb{E}} \left[\sup_{t \in [0, T]} |Y_t|^\alpha \right] + \widehat{\mathbb{E}} \left[\left(\int_0^T |h_s^0| ds \right)^\alpha \right] \right\}, \end{aligned} \quad (16)$$

where $h_s^0 = |f(s, 0, 0)| + \sum_{i,j=1}^d |g_{ij}(s, 0, 0)|$.

Proposition 10 (see [15, 20]). Let $\alpha \geq 1$ and $\delta > 0$ be fixed. Then, there exists a constant C depending on α and δ such that

$$\begin{aligned} & \widehat{\mathbb{E}} \left[\sup_{t \in [0, T]} \widehat{\mathbb{E}}_t [|\xi|^\alpha] \right] \\ & \leq C \left\{ \left(\widehat{\mathbb{E}} [|\xi|^{\alpha+\delta}] \right)^{\alpha/(\alpha+\delta)} + \widehat{\mathbb{E}} [|\xi|^{\alpha+\delta}] \right\}, \end{aligned} \quad (17)$$

$$\forall \xi \in L_G^{\alpha+\delta}(\Omega_T).$$

Theorem 11 (see [16]). Let (Y^l, Z^l, K^l) , $l = 1, 2$, be the solutions of the following G-BSDEs:

$$\begin{aligned} Y_t^l &= \xi + \int_t^T f(s, Y_s^l, Z_s^l) ds + \int_t^T g_{ij}(s, Y_s^l, Z_s^l) d\langle B^i, B^j \rangle_s \\ &\quad + V_T^l - V_t^l - \int_t^T Z_s^l dB_s - (K_T^l - K_t^l), \end{aligned} \quad (18)$$

where $\xi \in L_G^\beta(\Omega_T)$, f and g_{ij} satisfy (H1) and (H2) for some $\beta > 1$ and $(V_t^l)_{t \leq T}$ are RCLL processes in $M_G^\beta(0, T)$ such that $\widehat{\mathbb{E}}[\sup_{t \in [0, T]} |V_t^l|^\beta] < \infty$. If $V_t^1 - V_t^2$ is an increasing process, then $Y_t^1 \geq Y_t^2$ for $t \in [0, T]$.

In this paper, we also need the following assumptions for G-BSDE (13).

(H3) For each fixed $(\omega, y, z) \in \Omega_T \times \mathbb{R} \times \mathbb{R}^d$, $t \rightarrow f(t, \omega, y, z)$ and $t \rightarrow g_{ij}(t, \omega, y, z)$ are continuous.

(H4) For each fixed $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, $f(t, y, z)$, $g_{ij}(t, y, z) \in L_G^\beta(\Omega_t)$, and

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0+} \frac{1}{\varepsilon} \widehat{\mathbb{E}} \left[\int_t^{t+\varepsilon} \left(|f(u, y, z) - f(t, y, z)|^\beta \right. \right. \\ \left. \left. + \sum_{i,j=1}^d |g_{ij}(u, y, z) - g_{ij}(t, y, z)|^\beta \right) du \right] = 0. \end{aligned} \quad (19)$$

(H5) For each $(t, \omega, y) \in [0, T] \times \Omega_T \times \mathbb{R}$, $f(t, \omega, y, 0) = g_{ij}(t, \omega, y, 0) = 0$.

Assume that $\xi \in L_G^\beta(\Omega_T)$; f and g_{ij} satisfy (H1), (H2), and (H5) for some $\beta > 1$. Let $(Y^{T,\xi}, Z^{T,\xi}, K^{T,\xi})$ be the solution of G-BSDE (13) corresponding to ξ , f , and g_{ij} on $[0, T]$. It is easy to check that $Y^{T,\xi} = Y^{T',\xi}$ on $[0, T]$ for $T' > T$. Following [16], we can define consistent nonlinear expectation

$$\widehat{\mathbb{E}}_t[\xi] = Y_t^{T,\xi} \quad \text{for } t \in [0, T] \quad (20)$$

and set $\widehat{\mathbb{E}}[\xi] = \widehat{\mathbb{E}}_0[\xi] = Y_0^{T,\xi}$.

3. Representation Theorem of Generators of G-BSDEs

We consider the following type of G-FBSDEs:

$$\begin{aligned} X_s^{t,x} &= x + \int_t^s b(X_u^{t,x}) du \\ &\quad + \int_t^s h_{ij}(X_u^{t,x}) d\langle B^i, B^j \rangle_u + \int_t^s \sigma(X_u^{t,x}) dB_u, \end{aligned} \quad (21)$$

$$\begin{aligned} {}^\varepsilon Y_s^{t,x,y,p} &= y + \langle p, X_{t+\varepsilon}^{t,x} - x \rangle \\ &\quad + \int_s^{t+\varepsilon} f(u, {}^\varepsilon Y_u^{t,x,y,p}, {}^\varepsilon Z_u^{t,x,y,p}) du \\ &\quad + \int_s^{t+\varepsilon} g_{ij}(u, {}^\varepsilon Y_u^{t,x,y,p}, {}^\varepsilon Z_u^{t,x,y,p}) d\langle B^i, B^j \rangle_u \\ &\quad - \int_s^{t+\varepsilon} {}^\varepsilon Z_u^{t,x,y,p} dB_u - ({}^\varepsilon K_{t+\varepsilon}^{t,x,y,p} - {}^\varepsilon K_s^{t,x,y,p}), \end{aligned} \quad (22)$$

where $h_{ij} = h_{ji}$ and $g_{ij} = g_{ji}$, $1 \leq i, j \leq d$.

We now give the main result in this section.

Theorem 12. Let $b : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $h_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ be Lipschitz functions and let f and g_{ij} satisfy (H1), (H2), (H3), and (H4) for some $\beta > 1$. Then, for each $(t, x, y, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and $\alpha \in (1, \beta)$, one has

$$\begin{aligned} L_G^\alpha - \lim_{\varepsilon \rightarrow 0+} \frac{1}{\varepsilon} \{ {}^\varepsilon Y_t^{t,x,y,p} - y \} \\ = f(t, y, \sigma^T(x)p) + \langle p, b(x) \rangle \\ + 2G \left((g_{ij}(t, y, \sigma^T(x)p) + \langle p, h_{ij}(x) \rangle)_{i,j=1}^d \right). \end{aligned} \quad (23)$$

Proof. For each fixed $(t, x, y, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, we write $(Y^\varepsilon, Z^\varepsilon, K^\varepsilon)$ instead of $({}^\varepsilon Y^{t,x,y,p}, {}^\varepsilon Z^{t,x,y,p}, {}^\varepsilon K^{t,x,y,p})$ for simplicity. We have $\widehat{\mathbb{E}}[|X_{t+\varepsilon}^{t,x}|^\gamma] < \infty$ for each $\gamma \geq 1$ (see [16, 19]). Thus, by Theorem 8, G-BSDE (22) has a unique solution $(Y^\varepsilon, Z^\varepsilon, K^\varepsilon)$ and $Y_t^\varepsilon \in L_G^\alpha(\Omega_t)$. We set, for $s \in [t, t+\varepsilon]$,

$$\begin{aligned} \widetilde{Y}_s^\varepsilon &= Y_s^\varepsilon - (y + \langle p, X_s^{t,x} - x \rangle), \\ \widetilde{Z}_s^\varepsilon &= Z_s^\varepsilon - \sigma^T(X_s^{t,x})p, \quad \widetilde{K}_s^\varepsilon = K_s^\varepsilon. \end{aligned} \quad (24)$$

Applying Itô's formula to $\widetilde{Y}_s^\varepsilon$ on $[t, t+\varepsilon]$, it is easy to verify that $(\widetilde{Y}^\varepsilon, \widetilde{Z}^\varepsilon, \widetilde{K}^\varepsilon)$ solves the following G-BSDE:

$$\begin{aligned} \widetilde{Y}_s^\varepsilon &= \int_s^{t+\varepsilon} f(u, \widetilde{Y}_u^\varepsilon + y + \langle p, X_u^{t,x} - x \rangle, \\ &\quad \widetilde{Z}_u^\varepsilon + \sigma^T(X_u^{t,x})p) du \\ &\quad + \int_s^{t+\varepsilon} \langle p, b(X_u^{t,x}) \rangle du \\ &\quad + \int_s^{t+\varepsilon} g_{ij}(u, \widetilde{Y}_u^\varepsilon + y + \langle p, X_u^{t,x} - x \rangle, \\ &\quad \widetilde{Z}_u^\varepsilon + \sigma^T(X_u^{t,x})p) d\langle B^i, B^j \rangle_u \\ &\quad + \int_s^{t+\varepsilon} \langle p, h_{ij}(X_u^{t,x}) \rangle d\langle B^i, B^j \rangle_u \\ &\quad - \int_s^{t+\varepsilon} \widetilde{Z}_u^\varepsilon dB_u - (\widetilde{K}_{t+\varepsilon}^\varepsilon - \widetilde{K}_s^\varepsilon). \end{aligned} \quad (25)$$

From Proposition 9,

$$\begin{aligned} |\tilde{Y}_s^\varepsilon|^\alpha &\leq C_\alpha \widehat{\mathbb{E}}_s \left[\left(\int_s^{t+\varepsilon} \left(|f(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x})p)| \right. \right. \right. \\ &\quad \left. \left. \left. + |\langle p, b(X_u^{t,x}) \rangle| \right. \right. \right. \\ &\quad \left. \left. + \sum_{i,j=1}^d |g_{ij}(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x})p)| \right. \right. \\ &\quad \left. \left. \left. + |\langle p, h_{ij}(X_u^{t,x}) \rangle| \right) du \right)^\alpha \right], \\ \widehat{\mathbb{E}} \left[\left(\int_t^{t+\varepsilon} |\tilde{Z}_u^\varepsilon|^2 du \right)^{\alpha/2} \right] \\ &\leq C_\alpha \left\{ \widehat{\mathbb{E}} \left[\left(\int_t^{t+\varepsilon} \left(|f(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x})p)| \right. \right. \right. \right. \\ &\quad \left. \left. \left. + |\langle p, b(X_u^{t,x}) \rangle| + |\langle p, h_{ij}(X_u^{t,x}) \rangle| \right. \right. \right. \\ &\quad \left. \left. + \sum_{i,j=1}^d |g_{ij}(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x})p)| \right) du \right)^\alpha \right] \right. \\ &\quad \left. + \widehat{\mathbb{E}} \left[\sup_{s \in [t, t+\varepsilon]} |\tilde{Y}_s^\varepsilon|^\alpha \right] \right\}, \end{aligned} \quad (26)$$

hold for some constant $C_\alpha > 0$, only depending on α, T, G , and L . By Proposition 10 and the Lipschitz assumption, we obtain

$$\begin{aligned} \widehat{\mathbb{E}} \left[\sup_{s \in [t, t+\varepsilon]} |\tilde{Y}_s^\varepsilon|^\alpha + \left(\int_t^{t+\varepsilon} |\tilde{Z}_u^\varepsilon|^2 du \right)^{\alpha/2} \right] \\ \leq C_1 \varepsilon^\alpha \widehat{\mathbb{E}} \left[1 + \left(\frac{1}{\varepsilon} \int_t^{t+\varepsilon} \left(|f(u, 0, 0)|^\beta \right. \right. \right. \\ \left. \left. \left. + \sum_{i,j=1}^d |g_{ij}(u, 0, 0)|^\beta \right) du \right)^{\alpha/\beta} \right. \\ \left. + \sup_{s \in [t, t+\varepsilon]} |X_s^{t,x}|^\beta \right], \end{aligned} \quad (27)$$

where C_1 is a constant depending on $x, y, p, \alpha, \beta, T, G$, and L . Noting that $\widehat{\mathbb{E}}[\sup_{s \in [t, t+\varepsilon]} |X_s^{t,x}|^\beta] \leq C_2(1 + |x|^\beta)$ (see [16, 19]), where C_2 depends on T and L , and the following inequality holds:

$$\begin{aligned} \int_t^{t+\varepsilon} \left(|f(u, 0, 0)|^\beta + \sum_{i,j=1}^d |g_{ij}(u, 0, 0)|^\beta \right) du \\ \leq 2^{\beta-1} \left\{ \varepsilon \left(|f(t, 0, 0)|^\beta + \sum_{i,j=1}^d |g_{ij}(t, 0, 0)|^\beta \right) \right. \\ \left. + \int_t^{t+\varepsilon} \left(|f(u, 0, 0) - f(t, 0, 0)|^\beta \right. \right. \\ \left. \left. + \sum_{i,j=1}^d |g_{ij}(u, 0, 0) - g_{ij}(t, 0, 0)|^\beta \right) du \right\}. \end{aligned} \quad (28)$$

Together with assumption (H4), we get

$$\widehat{\mathbb{E}} \left[\sup_{s \in [t, t+\varepsilon]} |\tilde{Y}_s^\varepsilon|^\alpha + \left(\int_t^{t+\varepsilon} |\tilde{Z}_u^\varepsilon|^2 du \right)^{\alpha/2} \right] \leq C_3 \varepsilon^\alpha, \quad (29)$$

where C_3 depends on $x, y, p, \alpha, \beta, T, G$, and L . Now, we prove (23). Let us consider

$$\begin{aligned} \frac{1}{\varepsilon} \{Y_t^\varepsilon - y\} &= \frac{1}{\varepsilon} \tilde{Y}_t^\varepsilon = \frac{1}{\varepsilon} \widehat{\mathbb{E}}_t [\tilde{Y}_t^\varepsilon + \tilde{K}_{t+\varepsilon}^\varepsilon - \tilde{K}_t^\varepsilon] \\ &= \frac{1}{\varepsilon} \widehat{\mathbb{E}}_t \left[\int_t^{t+\varepsilon} f(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x})p) du \right. \\ &\quad \left. + \int_t^{t+\varepsilon} \langle p, b(X_u^{t,x}) \rangle du \right. \\ &\quad \left. + \int_t^{t+\varepsilon} g_{ij}(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x})p) \right. \\ &\quad \left. \times d\langle B^i, B^j \rangle_u \right. \\ &\quad \left. + \int_t^{t+\varepsilon} \langle p, h_{ij}(X_u^{t,x}) \rangle d\langle B^i, B^j \rangle_u \right] + L_\varepsilon, \end{aligned} \quad (30)$$

where

$$\begin{aligned} L_\varepsilon &= \frac{1}{\varepsilon} \left\{ \widehat{\mathbb{E}}_t \left[\int_t^{t+\varepsilon} f(u, \tilde{Y}_u^\varepsilon + y + \langle p, X_u^{t,x} - x \rangle, \right. \right. \\ &\quad \left. \left. \tilde{Z}_u^\varepsilon + \sigma^T(X_u^{t,x})p) du \right. \right. \\ &\quad \left. \left. + \int_t^{t+\varepsilon} \langle p, b(X_u^{t,x}) \rangle du \right. \right. \\ &\quad \left. \left. + \int_t^{t+\varepsilon} g_{ij}(u, \tilde{Y}_u^\varepsilon + y + \langle p, X_u^{t,x} - x \rangle, \right. \right. \\ &\quad \left. \left. \tilde{Z}_u^\varepsilon + \sigma^T(X_u^{t,x})p) d\langle B^i, B^j \rangle_u \right. \right. \\ &\quad \left. \left. + \int_t^{t+\varepsilon} \langle p, h_{ij}(X_u^{t,x}) \rangle d\langle B^i, B^j \rangle_u \right] \right. \\ &\quad \left. - \widehat{\mathbb{E}}_t \left[\int_t^{t+\varepsilon} f(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x})p) du \right. \right. \\ &\quad \left. \left. + \int_t^{t+\varepsilon} \langle p, b(X_u^{t,x}) \rangle du \right. \right. \\ &\quad \left. \left. + \int_t^{t+\varepsilon} g_{ij}(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x})p) \right. \right. \\ &\quad \left. \left. \times d\langle B^i, B^j \rangle_u \right. \right. \\ &\quad \left. \left. + \int_t^{t+\varepsilon} \langle p, h_{ij}(X_u^{t,x}) \rangle d\langle B^i, B^j \rangle_u \right] \right\}. \end{aligned} \quad (31)$$

It is easy to check that $|L_\varepsilon| \leq (C_4/\varepsilon)\widehat{\mathbb{E}}_t[\int_t^{t+\varepsilon} (|\widetilde{Y}_u^\varepsilon| + |\widetilde{Z}_u^\varepsilon|)du]$, where C_4 depends on G, L , and T . Thus, by (29), we get

$$\begin{aligned} \widehat{\mathbb{E}}[|L_\varepsilon|^\alpha] &\leq \frac{C_4^\alpha}{\varepsilon^\alpha} \widehat{\mathbb{E}}\left[\left(\int_t^{t+\varepsilon} (|\widetilde{Y}_u^\varepsilon| + |\widetilde{Z}_u^\varepsilon|)du\right)^\alpha\right] \\ &\leq \frac{2^{\alpha-1}C_4^\alpha}{\varepsilon^\alpha} \widehat{\mathbb{E}}\left[\left(\int_t^{t+\varepsilon} |\widetilde{Y}_u^\varepsilon|du\right)^\alpha + \left(\int_t^{t+\varepsilon} |\widetilde{Z}_u^\varepsilon|du\right)^\alpha\right] \\ &\leq 2^{\alpha-1}C_4^\alpha \left\{ \widehat{\mathbb{E}}\left[\sup_{s \in [t, t+\varepsilon]} |\widetilde{Y}_s^\varepsilon|^\alpha\right] + \varepsilon^{-\alpha/2} \widehat{\mathbb{E}}\left[\left(\int_t^{t+\varepsilon} |\widetilde{Z}_u^\varepsilon|^2 du\right)^{\alpha/2}\right] \right\} \\ &\leq 2^{\alpha-1}C_4^\alpha C_3 (\varepsilon^\alpha + \varepsilon^{\alpha/2}), \end{aligned} \quad (32)$$

which implies $L_G^\alpha - \lim_{\varepsilon \rightarrow 0+} L_\varepsilon = 0$. We set

$$\begin{aligned} M_\varepsilon &= \frac{1}{\varepsilon} \left\{ \widehat{\mathbb{E}}_t \left[\int_t^{t+\varepsilon} f(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x})p) du \right. \right. \\ &\quad + \int_t^{t+\varepsilon} \langle p, b(X_u^{t,x}) \rangle du \\ &\quad + \int_t^{t+\varepsilon} g_{ij}(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x})p) \\ &\quad \times d\langle B^i, B^j \rangle_u \\ &\quad + \int_t^{t+\varepsilon} \langle p, h_{ij}(X_u^{t,x}) \rangle d\langle B^i, B^j \rangle_u \Big] \\ &\quad - \widehat{\mathbb{E}}_t \left[\int_t^{t+\varepsilon} f(u, y, \sigma^T(x)p) du + \langle p, b(x) \rangle \varepsilon \right. \\ &\quad + \int_t^{t+\varepsilon} g_{ij}(u, y, \sigma^T(x)p) d\langle B^i, B^j \rangle_u \\ &\quad + \int_t^{t+\varepsilon} \langle p, h_{ij}(x) \rangle d\langle B^i, B^j \rangle_u \Big] \Big\}. \end{aligned} \quad (33)$$

By the Lipschitz condition, we can get $|M_\varepsilon| \leq (C_5/\varepsilon)\widehat{\mathbb{E}}_t[\int_t^{t+\varepsilon} |X_u^{t,x} - x|du]$, where C_5 depends on p, G, L , and T . Noting that $\widehat{\mathbb{E}}[\sup_{s \in [t, t+\varepsilon]} |X_s^{t,x} - x|^\alpha] \leq C_6(1 + |x|^\alpha)\varepsilon^{\alpha/2}$ (see [16, 19]), where C_6 depends on L, G , and α , we obtain

$$\begin{aligned} \widehat{\mathbb{E}}[|M_\varepsilon|^\alpha] &\leq C_5^\alpha \widehat{\mathbb{E}}\left[\sup_{s \in [t, t+\varepsilon]} |X_s^{t,x} - x|^\alpha\right] \\ &\leq C_5^\alpha C_6 (1 + |x|^\alpha) \varepsilon^{\alpha/2}, \end{aligned} \quad (34)$$

which implies $L_G^\alpha - \lim_{\varepsilon \rightarrow 0+} M_\varepsilon = 0$. Now, we set

$$\begin{aligned} N_\varepsilon &= \frac{1}{\varepsilon} \left\{ \widehat{\mathbb{E}}_t \left[\int_t^{t+\varepsilon} f(u, y, \sigma^T(x)p) du + \langle p, b(x) \rangle \varepsilon \right. \right. \\ &\quad + \int_t^{t+\varepsilon} g_{ij}(u, y, \sigma^T(x)p) d\langle B^i, B^j \rangle_u \\ &\quad + \int_t^{t+\varepsilon} \langle p, h_{ij}(x) \rangle d\langle B^i, B^j \rangle_u \Big] \\ &\quad - \widehat{\mathbb{E}}_t \left[\int_t^{t+\varepsilon} f(t, y, \sigma^T(x)p) du + \langle p, b(x) \rangle \varepsilon \right. \\ &\quad + \int_t^{t+\varepsilon} g_{ij}(t, y, \sigma^T(x)p) d\langle B^i, B^j \rangle_u \\ &\quad + \int_t^{t+\varepsilon} \langle p, h_{ij}(x) \rangle d\langle B^i, B^j \rangle_u \Big] \Big\}. \end{aligned} \quad (35)$$

It is easy to deduce that $|N_\varepsilon| \leq (C_7/\varepsilon)\widehat{\mathbb{E}}_t[\int_t^{t+\varepsilon} (|f(u, y, \sigma^T(x)p) - f(t, y, \sigma^T(x)p)| + \sum_{i,j=1}^d |g_{ij}(u, y, \sigma^T(x)p) - g_{ij}(t, y, \sigma^T(x)p)|)du]$, where C_7 depends on G . Then,

$$\begin{aligned} \widehat{\mathbb{E}}[|N_\varepsilon|^\alpha] &\leq C_7^\alpha \frac{1}{\varepsilon} \widehat{\mathbb{E}}_t \left[\int_t^{t+\varepsilon} \left(|f(u, y, \sigma^T(x)p) - f(t, y, \sigma^T(x)p)| \right. \right. \\ &\quad + \sum_{i,j=1}^d |g_{ij}(u, y, \sigma^T(x)p) \\ &\quad \left. \left. - g_{ij}(t, y, \sigma^T(x)p)| \right)^\alpha du \right] \\ &\leq C_7^\alpha \left(\frac{1}{\varepsilon} \widehat{\mathbb{E}}_t \left[\int_t^{t+\varepsilon} \left(|f(u, y, \sigma^T(x)p) \right. \right. \right. \right. \\ &\quad \left. \left. - f(t, y, \sigma^T(x)p)| \right. \right. \right. \\ &\quad + \sum_{i,j=1}^d |g_{ij}(u, y, \sigma^T(x)p) \\ &\quad \left. \left. - g_{ij}(t, y, \sigma^T(x)p)| \right)^\beta du \right] \right)^{\alpha/\beta}. \end{aligned} \quad (36)$$

Take limit on both sides of the above inequality and use assumption (H4); then, we have

$$L_G^\alpha - \lim_{\varepsilon \rightarrow 0+} N_\varepsilon = 0. \quad (37)$$

On the other hand,

$$\begin{aligned}
& \widehat{\mathbb{E}}_t \left[\int_t^{t+\varepsilon} f(t, y, \sigma^T(x)p) du + \langle p, b(x) \rangle \varepsilon \right. \\
& \quad + \int_t^{t+\varepsilon} g_{ij}(t, y, \sigma^T(x)p) d\langle B^i, B^j \rangle_u \\
& \quad \left. + \int_t^{t+\varepsilon} \langle p, h_{ij}(x) \rangle d\langle B^i, B^j \rangle_u \right] \\
& = f(t, y, \sigma^T(x)p) \varepsilon + \langle p, b(x) \rangle \varepsilon \\
& \quad + \widehat{\mathbb{E}}_t \left[(g_{ij}(t, y, \sigma^T(x)p) + \langle p, h_{ij}(x) \rangle) \right. \\
& \quad \quad \left. \times (\langle B^i, B^j \rangle_{t+\varepsilon} - \langle B^i, B^j \rangle_t) \right] \\
& = \left(f(t, y, \sigma^T(x)p) + \langle p, b(x) \rangle \right. \\
& \quad \left. + 2G \left((g_{ij}(t, y, \sigma^T(x)p) + \langle p, h_{ij}(x) \rangle)_{i,j=1}^d \right) \right) \varepsilon. \tag{38}
\end{aligned}$$

Then, we have

$$\begin{aligned}
& L_G^\alpha - \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \{Y_t^\varepsilon - y\} \\
& = f(t, y, \sigma^T(x)p) + \langle p, b(x) \rangle \\
& \quad + 2G \left((g_{ij}(t, y, \sigma^T(x)p) + \langle p, h_{ij}(x) \rangle)_{i,j=1}^d \right). \tag{39}
\end{aligned}$$

The proof is complete. $\square$

4. Some Applications

4.1. Converse Comparison Theorem for G-BSDEs. We consider the following G-BSDEs:

$$\begin{aligned}
Y_t^{l,\xi} &= \xi + \int_t^T f^l(s, Y_s^{l,\xi}, Z_s^{l,\xi}) ds \\
& \quad + \int_t^T g_{ij}^l(s, Y_s^{l,\xi}, Z_s^{l,\xi}) d\langle B^i, B^j \rangle_s \\
& \quad - \int_t^T Z_s^{l,\xi} dB_s - (K_T^{l,\xi} - K_t^{l,\xi}), \quad l = 1, 2, \tag{40}
\end{aligned}$$

where $g_{ij}^l = g_{ji}^l$.

We first generalized the comparison theorem in [16].

Proposition 13. Let f^l and g_{ij}^l satisfy (H1) and (H2) for some $\beta > 1$, $l = 1, 2$. If $f^2 - f^1 + 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}^d) \leq 0$, then, for each $\xi \in L_G^\beta(\Omega_T)$, one has $Y_t^{1,\xi} \geq Y_t^{2,\xi}$ for $t \in [0, T]$.

Proof. From the above G-BSDEs, we have

$$\begin{aligned}
Y_t^{2,\xi} &= \xi + \int_t^T f^2(s, Y_s^{2,\xi}, Z_s^{2,\xi}) ds \\
& \quad + \int_t^T g_{ij}^2(s, Y_s^{2,\xi}, Z_s^{2,\xi}) d\langle B^i, B^j \rangle_s \\
& \quad - \int_t^T Z_s^{2,\xi} dB_s - (K_T^{2,\xi} - K_t^{2,\xi}) \\
& = \xi + \int_t^T f^1(s, Y_s^{2,\xi}, Z_s^{2,\xi}) ds \\
& \quad + \int_t^T g_{ij}^1(s, Y_s^{2,\xi}, Z_s^{2,\xi}) d\langle B^i, B^j \rangle_s \\
& \quad + V_T - V_t - \int_t^T Z_s^{2,\xi} dB_s - (K_T^{2,\xi} - K_t^{2,\xi}), \tag{41}
\end{aligned}$$

where

$$\begin{aligned}
V_t &= \int_0^t (f^2 - f^1)(s, Y_s^{2,\xi}, Z_s^{2,\xi}) ds \\
& \quad + \int_0^t (g_{ij}^2 - g_{ij}^1)(s, Y_s^{2,\xi}, Z_s^{2,\xi}) d\langle B^i, B^j \rangle_s \\
& = \int_0^t (f^2 - f^1 + 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}^d))(s, Y_s^{2,\xi}, Z_s^{2,\xi}) ds \\
& \quad + \int_0^t (g_{ij}^2 - g_{ij}^1)(s, Y_s^{2,\xi}, Z_s^{2,\xi}) d\langle B^i, B^j \rangle_s \\
& \quad - \int_0^t 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}^d)(s, Y_s^{2,\xi}, Z_s^{2,\xi}) ds. \tag{42}
\end{aligned}$$

By the assumption, it is easy to check that $(V_t)_{t \leq T}$ is a decreasing process. Thus, using Theorem 11, we obtain $Y_t^{1,\xi} \geq Y_t^{2,\xi}$ for $t \in [0, T]$. $\square$

Remark 14. Suppose $d = 1$ and let $f^1 = 10|z|$, $f^2 = |z|$, $g^1 = |z|$, and $g^2 = 2|z|$. It is easy to check that $f^2 - f^1 + 2G(g^2 - g^1) \leq 0$. Thus, $f^2 - f^1 + 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}^d) \leq 0$ does not imply $f^2 \leq f^1$ and $(g_{ij}^2)_{i,j=1}^d \leq (g_{ij}^1)_{i,j=1}^d$.

Now, we give the converse comparison theorem.

Theorem 15. Let f^l and g_{ij}^l satisfy (H1), (H2), (H3), (H4), and (H5) for some $\beta > 1$, $l = 1, 2$. If $Y_t^{1,\xi} \geq Y_t^{2,\xi}$ for each $t \in [0, T]$ and $\xi \in L_G^\beta(\Omega_T)$, then $f^2 - f^1 + 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}^d) \leq 0$ q.s..

Proof. For simplicity, we take the notation $\widehat{\mathbb{E}}_t^l[\xi] = Y_t^{l,\xi}$, $l = 1, 2$. For each fixed $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, let us consider

$$\eta_\varepsilon = y + \langle z, h_{ij} \rangle (\langle B^i, B^j \rangle_{t+\varepsilon} - \langle B^i, B^j \rangle_t) + \langle z, B_{t+\varepsilon} - B_t \rangle, \tag{43}$$

where $h_{ij} = h_{ji} \in \mathbb{R}^d$. By Theorem 12, we have, for each $\alpha \in (1, \beta)$,

$$\begin{aligned} L_G^\alpha - \lim_{\varepsilon \rightarrow 0+\varepsilon} \frac{1}{\varepsilon} (\tilde{\mathbb{E}}_t^l[\eta_\varepsilon] - \gamma) \\ = f^l(t, y, z) + 2G\left((g_{ij}^l(t, y, z) + \langle z, h_{ij} \rangle)_{i,j=1}^d\right). \end{aligned} \quad (44)$$

Since $\tilde{\mathbb{E}}_t^1[\eta_\varepsilon] \geq \tilde{\mathbb{E}}_t^2[\eta_\varepsilon]$,

$$\begin{aligned} f^1(t, y, z) + 2G\left((g_{ij}^1(t, y, z) + \langle z, h_{ij} \rangle)_{i,j=1}^d\right) \\ \geq f^2(t, y, z) + 2G\left((g_{ij}^2(t, y, z) + \langle z, h_{ij} \rangle)_{i,j=1}^d\right) \text{ q.s.} \end{aligned} \quad (45)$$

Take a h_{ij} such that $\langle z, h_{ij} \rangle = -g_{ij}^1(t, y, z)$. Therefore, $\{f^2 - f^1 + 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}^d)\}(t, y, z) \leq 0$ q.s. By the assumptions (H2) and (H3), it is easy to deduce that $f^2 - f^1 + 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}^d) \leq 0$ q.s. $\square$

In the following, we use the notation $\tilde{\mathbb{E}}_t^l[\xi] = Y_t^{l,\xi}$, $l = 1, 2$.

Corollary 16. Let f^l and g_{ij}^l be deterministic functions and satisfy (H1), (H2), (H3), and (H5) for some $\beta > 1$, $l = 1, 2$. If $\tilde{\mathbb{E}}^1[\xi] \geq \tilde{\mathbb{E}}^2[\xi]$ for each $\xi \in L_G^\beta(\Omega_T)$, then $f^2 - f^1 + 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}^d) \leq 0$.

Proof. Taking η_ε as in Theorem 15, since f^l and g_{ij}^l are deterministic, we could get $\tilde{\mathbb{E}}_t^l[\eta_\varepsilon] = \tilde{\mathbb{E}}^l[\eta_\varepsilon]$, for $l = 1, 2$. And the proof in Theorem 15 still holds true. $\square$

4.2. Some Equivalent Relations. We consider the following G-BSDE:

$$\begin{aligned} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g_{ij}(s, Y_s, Z_s) d\langle B^i, B^j \rangle_s \\ - \int_t^T Z_s dB_s - (K_T - K_t), \end{aligned} \quad (46)$$

where $g_{ij} = g_{ji}$. We use the notation $\tilde{\mathbb{E}}_t[\xi] = Y_t$.

Proposition 17. Let f and g_{ij} satisfy (H1), (H2), (H3), (H4), and (H5) for some $\beta > 1$ and fix $\alpha \in (1, \beta)$. Then, one has

$$(1) \tilde{\mathbb{E}}_t[\xi + \eta] = \tilde{\mathbb{E}}_t[\xi] + \eta \text{ for } t \in [0, T], \xi \in L_G^\alpha(\Omega_T), \text{ and } \eta \in L_G^\alpha(\Omega_t) \text{ if and only if for each } t \in [0, T], y, y' \in \mathbb{R}, z \in \mathbb{R}^d,$$

$$\begin{aligned} f(t, y, z) - f(t, y', z) \\ + 2G\left((g_{ij}(t, y, z) - g_{ij}(t, y', z))_{i,j=1}^d\right) = 0; \end{aligned} \quad (47)$$

$$(2) \tilde{\mathbb{E}}_t[\xi + \eta] \leq \tilde{\mathbb{E}}_t[\xi] + \tilde{\mathbb{E}}_t[\eta] \text{ for } t \in [0, T], \xi \in L_G^\alpha(\Omega_T), \text{ and } \eta \in L_G^\alpha(\Omega_T) \text{ if and only if for each } t \in [0, T], y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d,$$

$$\begin{aligned} 0 \geq f(t, y + y', z + z') - f(t, y, z) - f(t, y', z') \\ + 2G\left((g_{ij}(t, y + y', z + z') - g_{ij}(t, y, z) \right. \\ \left. - g_{ij}(t, y', z'))_{i,j=1}^d\right); \end{aligned} \quad (48)$$

$$(3) \tilde{\mathbb{E}}_t[\lambda\xi + (1-\lambda)\eta] \leq \lambda\tilde{\mathbb{E}}_t[\xi] + (1-\lambda)\tilde{\mathbb{E}}_t[\eta] \text{ for } t \in [0, T], \lambda \in [0, 1], \xi \in L_G^\alpha(\Omega_T), \text{ and } \eta \in L_G^\alpha(\Omega_T) \text{ if and only if for each } t \in [0, T], y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d, \lambda \in [0, 1],$$

$$\begin{aligned} 0 \geq f(t, \lambda y + (1-\lambda)y', \lambda z + (1-\lambda)z') \\ - \lambda f(t, y, z) - (1-\lambda)f(t, y', z') \\ + 2G\left((g_{ij}(t, \lambda y + (1-\lambda)y', \lambda z + (1-\lambda)z') \right. \\ \left. - \lambda g_{ij}(t, y, z) - (1-\lambda)g_{ij}(t, y', z'))_{i,j=1}^d\right); \end{aligned} \quad (49)$$

$$(4) \tilde{\mathbb{E}}_t[\lambda\xi] = \lambda\tilde{\mathbb{E}}_t[\xi] \text{ for } t \in [0, T], \lambda \geq 0, \text{ and } \xi \in L_G^\alpha(\Omega_T) \text{ if and only if for each } t \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^d, \lambda \geq 0,$$

$$\begin{aligned} f(t, \lambda y, \lambda z) - \lambda f(t, y, z) \\ = 2G\left((\lambda g_{ij}(t, y, z) - g_{ij}(t, \lambda y, \lambda z))_{i,j=1}^d\right) \\ = -2G\left((g_{ij}(t, \lambda y, \lambda z) - \lambda g_{ij}(t, y, z))_{i,j=1}^d\right). \end{aligned} \quad (50)$$

Proof. (1) “ $\Rightarrow$ ” part. For each fixed $t \in [0, T]$, $y, y' \in \mathbb{R}, z \in \mathbb{R}^d$, we take

$$\begin{aligned} \xi_\varepsilon = y + \langle z, h_{ij} \rangle (\langle B^i, B^j \rangle_{t+\varepsilon} - \langle B^i, B^j \rangle_t) \\ + \langle z, B_{t+\varepsilon} - B_t \rangle, \quad \eta = y' - y, \end{aligned} \quad (51)$$

where $h_{ij} = h_{ji} \in \mathbb{R}^d$. Then, by Theorem 12 and $\tilde{\mathbb{E}}_t[\xi_\varepsilon + \eta] = \tilde{\mathbb{E}}_t[\xi_\varepsilon] + \eta$, we can obtain

$$\begin{aligned} f(t, y', z) + 2G\left((g_{ij}(t, y', z) + \langle z, h_{ij} \rangle)_{i,j=1}^d\right) \\ = f(t, y, z) + 2G\left((g_{ij}(t, y, z) + \langle z, h_{ij} \rangle)_{i,j=1}^d\right). \end{aligned} \quad (52)$$

We choose h_{ij} such that $g_{ij}(t, y', z) + \langle z, h_{ij} \rangle = 0$, which implies (47).

“ $\Leftarrow$ ” part. Let (Y, Z, K) be the solution of G-BSDE (46) corresponding to terminal condition ξ . We claim that $(Y_s + \eta, Z_s, K_s)_{s \in [t, T]}$ is the solution of G-BSDE (46) corresponding

to terminal condition $\xi + \eta$ on $[t, T]$. For this, we only need to check that, for $s \in [t, T]$,

$$\begin{aligned} & \int_s^T f(u, Y_u, Z_u) du + \int_s^T g_{ij}(u, Y_u, Z_u) d\langle B^i, B^j \rangle_u \\ &= \int_s^T f(u, Y_u + \eta, Z_u) du \\ &+ \int_s^T g_{ij}(u, Y_u + \eta, Z_u) d\langle B^i, B^j \rangle_u. \end{aligned} \quad (53)$$

By (47) we can get

$$\begin{aligned} & \int_s^T (g_{ij}(u, Y_u, Z_u) - g_{ij}(u, Y_u + \eta, Z_u)) d\langle B^i, B^j \rangle_u \\ & - 2 \int_s^T G\left((g_{ij}(u, Y_u, Z_u) - g_{ij}(u, Y_u + \eta, Z_u))_{i,j=1}^d\right) du \\ &= \int_s^T (g_{ij}(u, Y_u, Z_u) - g_{ij}(u, Y_u + \eta, Z_u)) d\langle B^i, B^j \rangle_u \\ &+ \int_s^T (f(u, Y_u, Z_u) - f(u, Y_u + \eta, Z_u)) du \leq 0, \\ & \int_s^T (g_{ij}(u, Y_u + \eta, Z_u) - g_{ij}(u, Y_u, Z_u)) d\langle B^i, B^j \rangle_u \\ & - 2 \int_s^T G\left((g_{ij}(u, Y_u + \eta, Z_u) - g_{ij}(u, Y_u, Z_u))_{i,j=1}^d\right) du \\ &= \int_s^T (g_{ij}(u, Y_u + \eta, Z_u) - g_{ij}(u, Y_u, Z_u)) d\langle B^i, B^j \rangle_u \\ &+ \int_s^T (f(u, Y_u + \eta, Z_u) - f(u, Y_u, Z_u)) du \leq 0, \end{aligned} \quad (54)$$

which implies (53). The proof of (1) is complete.

(2) “ $\Rightarrow$ ” part. For each fixed $t \in [0, T)$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, we consider $\xi_\varepsilon = y + \langle z, h_{ij} \rangle (\langle B^i, B^j \rangle_{t+\varepsilon} - \langle B^i, B^j \rangle_t) + \langle z, B_{t+\varepsilon} - B_t \rangle$ and $\eta_\varepsilon = y' + \langle z', h'_{ij} \rangle (\langle B^i, B^j \rangle_{t+\varepsilon} - \langle B^i, B^j \rangle_t) + \langle z', B_{t+\varepsilon} - B_t \rangle$, where $h_{ij} = h_{ji} \in \mathbb{R}^d$ and $h'_{ij} \in \mathbb{R}^d$. Then, by Theorem 12 and $\tilde{\mathbb{E}}_t[\xi_\varepsilon + \eta_\varepsilon] = \tilde{\mathbb{E}}_t[\xi_\varepsilon] + \tilde{\mathbb{E}}_t[\eta_\varepsilon]$, we obtain

$$\begin{aligned} & f(t, y + y', z + z') \\ &+ 2G\left((g_{ij}(t, y + y', z + z') \right. \\ &\quad \left. + \langle z, h_{ij} \rangle + \langle z', h'_{ij} \rangle)_{i,j=1}^d\right) \\ &\leq f(t, y, z) + f(t, y', z') \\ &+ 2G\left((g_{ij}(t, y, z) + \langle z, h_{ij} \rangle)_{i,j=1}^d\right) \\ &+ 2G\left((g_{ij}(t, y', z') + \langle z', h'_{ij} \rangle)_{i,j=1}^d\right). \end{aligned} \quad (55)$$

We choose h_{ij}, h'_{ij} such that $g_{ij}(t, y, z) + \langle z, h_{ij} \rangle = 0$ and $g_{ij}(t, y', z') + \langle z', h'_{ij} \rangle = 0$, which implies (48).

“ $\Leftarrow$ ” part. Let (Y, Z, K) and (Y', Z', K') be the solutions of G-BSDE (46) corresponding to terminal condition ξ and η , respectively. Then, $(Y + Y', Z + Z', K)$ solves the following G-BSDE:

$$\begin{aligned} Y_t + Y'_t &= \xi + \eta + \int_t^T f(s, Y_s + Y'_s, Z_s + Z'_s) ds \\ &+ \int_t^T g_{ij}(s, Y_s + Y'_s, Z_s + Z'_s) d\langle B^i, B^j \rangle_s \\ &+ V_T - V_t - \int_t^T (Z_s + Z'_s) dB_s - (K_T - K_t), \end{aligned} \quad (56)$$

where

$$\begin{aligned} V_t &= -K'_t - \int_0^t (f(s, Y_s + Y'_s, Z_s + Z'_s) \\ &\quad - f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)) ds \\ &\quad - \int_0^t (g_{ij}(s, Y_s + Y'_s, Z_s + Z'_s) - g_{ij}(s, Y_s, Z_s) \\ &\quad - g_{ij}(s, Y'_s, Z'_s)) d\langle B^i, B^j \rangle_s \\ &= -K'_t - \left\{ \int_0^t (g_{ij}(s, Y_s + Y'_s, Z_s + Z'_s) \right. \\ &\quad - g_{ij}(s, Y_s, Z_s) - g_{ij}(s, Y'_s, Z'_s)) d\langle B^i, B^j \rangle_s \\ &\quad - 2 \int_0^t G\left((g_{ij}(s, Y_s + Y'_s, Z_s + Z'_s) \right. \\ &\quad - g_{ij}(s, Y_s, Z_s) \\ &\quad \left. \left. - g_{ij}(s, Y'_s, Z'_s)\right)_{i,j=1}^d\right) ds \Big\} \\ &\quad - \int_0^t \left\{ f(s, Y_s + Y'_s, Z_s + Z'_s) \right. \\ &\quad - f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s) \\ &\quad + 2G\left((g_{ij}(s, Y_s + Y'_s, Z_s + Z'_s) - g_{ij}(s, Y_s, Z_s) \right. \\ &\quad \left. \left. - g_{ij}(s, Y'_s, Z'_s)\right)_{i,j=1}^d\right) \Big\} ds. \end{aligned} \quad (57)$$

By (48), it is easy to check that V_t is an increasing process. Then, by Theorem 11, we can get $\tilde{\mathbb{E}}_t[\xi + \eta] \leq \tilde{\mathbb{E}}_t[\xi] + \tilde{\mathbb{E}}_t[\eta]$. The proof of (2) is complete.

Finally, we could prove (3) as in (2) and (4) as in (1). $\square$

Proposition 18. One has the following.

- (1) If $G(A) + G(-A) > 0$ for any $A \in \mathbb{S}_d$ and $A \neq 0$, then (47) holds if and only if f and g_{ij} are independent of y .

- (2) If there exists an $A \in \mathbb{S}_d$ with $A \neq 0$ such that $G(A) + G(-A) = 0$ and $G(A) \neq 0$, then, for any fixed $g(t, y, z)$ satisfying (H1)–(H5), one has $f(t, y, z) = -2G(A)g(t, y, z)$ and $(g_{ij}(t, y, z))_{i,j=1}^d = g(t, y, z)A$ satisfying (47).

Proof. It is easy to verify (2), and we only need to prove (1). If (47) holds, it is easy to check that $G((g_{ij}(t, y, z) - g_{ij}(t, 0, z))_{i,j=1}^d) + G((g_{ij}(t, 0, z) - g_{ij}(t, y, z))_{i,j=1}^d) = 0$ holds. Then, from the assumption, we get $g_{ij}(t, y, z) = g_{ij}(t, 0, z)$. Therefore, by (47), we have $f(t, y, z) = f(t, 0, z)$, which implies that f and g_{ij} are independent of y . The converse part is obvious. $\square$

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References

- [1] É. Pardoux and S. G. Peng, "Adapted solution of a backward stochastic differential equation," *Systems & Control Letters*, vol. 14, no. 1, pp. 55–61, 1990.
- [2] S. Peng, "Backward SDE and related g -expectation," in *Backward Stochastic Differential Equations*, N. El Karoui and L. Mazliak, Eds., vol. 364 of *Pitman Research Notes in Mathematics Series*, pp. 141–159, 1997.
- [3] Z. J. Chen, "A property of backward stochastic differential equations," *Comptes Rendus de l'Académie des Sciences. Série I*, vol. 326, no. 4, pp. 483–488, 1998.
- [4] P. Briand, F. Coquet, Y. Hu, J. Mémin, and S. G. Peng, "A converse comparison theorem for BSDEs and related properties of g -expectation," *Electronic Communications in Probability*, vol. 5, pp. 101–117, 2000.
- [5] L. Jiang, "Representation theorems for generators of backward stochastic differential equations," *Comptes Rendus Mathématique. Académie des Sciences. Paris I*, vol. 340, no. 2, pp. 161–166, 2005.
- [6] L. Jiang, "Converse comparison theorems for backward stochastic differential equations," *Statistics & Probability Letters*, vol. 71, no. 2, pp. 173–183, 2005.
- [7] L. Jiang, "Convexity, translation invariance and subadditivity for g -expectations and related risk measures," *The Annals of Applied Probability*, vol. 18, no. 1, pp. 245–258, 2008.
- [8] S. Peng, "Filtration consistent nonlinear expectations and evaluations of contingent claims," *Acta Mathematicae Applicatae Sinica*, vol. 20, no. 2, pp. 191–214, 2004.
- [9] S. Peng, "Nonlinear expectations and nonlinear Markov chains," *Chinese Annals of Mathematics B*, vol. 26, no. 2, pp. 159–184, 2005.
- [10] S. Peng, "G-expectation, G-Brownian motion and related stochastic calculus of Itô type," in *Stochastic Analysis and Applications*, vol. 2 of *The Abel Symposium*, pp. 541–567, Springer, Berlin, Germany, 2007.
- [11] S. Peng, "G-Brownian motion and dynamic risk measure under volatility uncertainty," <http://arxiv.org/abs/0711.2834v1>.
- [12] S. Peng, "Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation," *Stochastic Processes and Their Applications*, vol. 118, no. 12, pp. 2223–2253, 2008.
- [13] S. Peng, "A new central limit theorem under sublinear expectations," <http://arxiv.org/abs/0803.2656v1>.
- [14] H. M. Soner, N. Touzi, and J. Zhang, "Wellposedness of second order backward SDEs," *Probability Theory and Related Fields*, vol. 153, no. 1–2, pp. 149–190, 2012.
- [15] M. S. Hu, S. L. Ji, S. G. Peng, and Y. S. Song, "Backward stochastic differential equations driven by G-brownian motion," <http://arxiv.org/abs/1206.5889v1>.
- [16] M. S. Hu, S. L. Ji, S. G. Peng, and Y. S. Song, "Comparison theorem, Feynman-Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion," <http://arxiv.org/abs/1212.5403v1>.
- [17] S. Peng, "G-G-Brownian motion and dynamic risk measure under volatility uncertainty," <http://arxiv.org/abs/0711.2834v1>.
- [18] S. G. Peng, "Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation," *Stochastic Processes and Their Applications*, vol. 118, no. 12, pp. 2223–2253, 2008.
- [19] S. G. Peng, "Nonlinear expectations and stochastic calculus under uncertainty," <http://arxiv.org/abs/1002.4546v1>.
- [20] Y. S. Song, "Some properties on G-evaluation and its applications to G-martingale decomposition," *Science China Mathematics*, vol. 54, no. 2, pp. 287–300, 2011.