Research Article

Best Possible Bounds for Neuman-Sándor Mean by the Identric, Quadratic and Contraharmonic Means

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We prove that the double inequalities $I^{\alpha_1}(a, b)Q^{1-\alpha_1}(a, b) < M(a, b) < I^{\beta_1}(a, b)Q^{1-\beta_1}(a, b)$, $I^{\alpha_2}(a, b)C^{1-\alpha_2}(a, b) < M(a, b) < I^{\beta_2}(a, b)C^{1-\beta_2}(a, b)$ hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \geq 1/2$, $\beta_1 \leq \log[\sqrt{2}\log(1 + \sqrt{2})]/(1 - \log \sqrt{2})$, $\alpha_2 \geq 5/7$, and $\beta_2 \leq \log[2\log(1 + \sqrt{2})]$, where $I(a, b)$, $M(a, b)$, $Q(a, b)$, and $C(a, b)$ are the identric, Neuman-Sándor, quadratic, and contraharmonic means of a and b , respectively.

1. Introduction

For $p \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$, the identric mean $I(a, b)$, Neuman-Sándor mean $M(a, b)$ [1], quadratic mean $Q(a, b)$, contraharmonic mean $C(a, b)$, and p th power mean $M_p(a, b)$ are defined by

$$\begin{aligned} I(a, b) &= \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{1/(b-a)}, \\ M(a, b) &= \frac{a-b}{2\sinh^{-1}[(a-b)/(a+b)]}, \\ Q(a, b) &= \sqrt{\frac{a^2 + b^2}{2}}, \quad C(a, b) = \frac{a^2 + b^2}{a+b}, \\ M_p(a, b) &= \begin{cases} \left(\frac{a^p + b^p}{2} \right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \end{aligned} \quad (1)$$

respectively, where $\sinh^{-1}(x) = \log(x + \sqrt{1 + x^2})$ is the inverse hyperbolic sine function.

Recently, the identric, Neuman-Sándor, quadratic, and contraharmonic means have attracted the interest of numerous eminent mathematicians. In particular, many remarkable inequalities for these means can be found in the literature [1–18].

Let $H(a, b) = 2ab/(a + b)$, $G(a, b) = \sqrt{ab}$, $L(a, b) = (b - a)/(\log b - \log a)$, $P(a, b) = (a - b)/(4 \arctan \sqrt{a/b} - \pi)$, $A(a, b) = (a + b)/2$, and $T(a, b) = (a - b)/[2 \arctan((a - b)/(a + b))]$ be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, and second Seiffert means of two distinct positive numbers a and b , respectively. Then it is well known that the inequalities

$$\begin{aligned} H(a, b) &= M_{-1}(a, b) < G(a, b) = M_0(a, b) < L(a, b) \\ &< P(a, b) < I(a, b) < A(a, b) < M_1(a, b) \\ &< M(a, b) < T(a, b) < Q(a, b) = M_2(a, b) \\ &< C(a, b) \end{aligned} \quad (2)$$

hold for all $a, b > 0$ with $a \neq b$.

Neuman and Sándor [1, 8] established that

$$\begin{aligned} A(a, b) < M(a, b) &< \frac{A(a, b)}{\log(1 + \sqrt{2})}, \\ \frac{\pi}{4} T(a, b) < M(a, b) &< T(a, b), \\ M(a, b) &< \frac{A^2(a, b)}{P(a, b)}, \\ \sqrt{A(a, b) T(a, b)} &< M(a, b) \\ &< \sqrt{\frac{A^2(a, b) + T^2(a, b)}{2}}, \\ M(a, b) &< \frac{2A(a, b) + Q(a, b)}{3} \end{aligned} \quad (3)$$

for all $a, b > 0$ with $a \neq b$.

Let $0 < a, b \leq 1/2$ with $a \neq b$, $a' = 1 - a$, and $b' = 1 - b$. Then the Ky Fan inequalities

$$\begin{aligned} \frac{G(a, b)}{G(a', b')} &< \frac{L(a, b)}{L(a', b')} < \frac{P(a, b)}{P(a', b')} < \frac{A(a, b)}{A(a', b')} \\ &< \frac{M(a, b)}{M(a', b')} < \frac{T(a, b)}{T(a', b')} \end{aligned} \quad (4)$$

were presented in [1].

Li et al. [19] found the best possible bounds for the Neuman-Sándor mean in terms of the generalized logarithmic mean $L_r(a, b)$. Neuman [20] and Zhao et al. [21] proved that the inequalities

$$\begin{aligned} \alpha Q(a, b) + (1 - \alpha) A(a, b) \\ &< M(a, b) < \beta Q(a, b) + (1 - \beta) A(a, b), \\ \lambda C(a, b) + (1 - \lambda) A(a, b) \\ &< M(a, b) < \mu C(a, b) + (1 - \mu) A(a, b), \\ \alpha_1 H(a, b) + (1 - \alpha_1) Q(a, b) \\ &< M(a, b) < \beta_1 H(a, b) + (1 - \beta_1) Q(a, b), \\ \alpha_2 C(a, b) + (1 - \alpha_2) Q(a, b) \\ &< M(a, b) < \beta_2 C(a, b) + (1 - \beta_2) Q(a, b) \end{aligned} \quad (5)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq [1 - \log(1 + \sqrt{2})]/[(\sqrt{2} - 1)\log(1 + \sqrt{2})]$, $\beta \geq 1/3$, $\lambda \leq [1 - \log(1 + \sqrt{2})]/\log(1 + \sqrt{2})$, $\mu \geq 1/6$, $\alpha_1 \geq 2/9$, $\beta_1 \leq 1 - 1/[\sqrt{2}\log(1 + \sqrt{2})]$, $\alpha_2 \geq 1/3$, and $\beta_2 \leq 1 - 1/[\sqrt{2}\log(1 + \sqrt{2})]$.

In [22], Chu and Long gave the best possible constants p, q, α , and β such that the double inequalities $M_p(a, b) < M(a, b) < M_q(a, b)$ and $\alpha I(a, b) < M(a, b) < \beta I(a, b)$ hold for all $a, b > 0$ with $a \neq b$.

The ratio of identric means leads to the weighted geometric mean

$$\frac{I(a^2, b^2)}{I(a, b)} = (a^a b^b)^{1/(a+b)}, \quad (6)$$

which has been investigated in [23–25]. Alzer [26] proved that the inequalities

$$\begin{aligned} \sqrt{A(a, b) G(a, b)} &< \sqrt{I(a, b) L(a, b)} \\ &< \frac{I(a, b) + L(a, b)}{2} < \frac{A(a, b) + G(a, b)}{2} \end{aligned} \quad (7)$$

hold for all $a, b > 0$ with $a \neq b$.

The following sharp bounds for I , $(IL)^{1/2}$, and $(I + L)/2$ in terms of the power mean and the convex combination of arithmetic and geometric means are given in [27] as

$$\begin{aligned} M_{2/3}(a, b) &< I(a, b) < M_{\log 2}(a, b), \\ M_0(a, b) &< \sqrt{I(a, b) L(a, b)} < M_{1/2}(a, b), \\ M_{\log 2/(1+\log 2)}(a, b) \\ &< \frac{I(a, b) + L(a, b)}{2} < M_{1/2}(a, b), \\ \frac{2}{3} A(a, b) + \frac{1}{3} G(a, b) \\ &< I(a, b) < \frac{2}{e} A(a, b) + \left(1 - \frac{2}{e}\right) G(a, b) \end{aligned} \quad (8)$$

for all $a, b > 0$ with $a \neq b$.

Chu et al. [28] presented the optimal constants $\alpha_1, \beta_1, \alpha_2$, and β_2 such that the double inequalities

$$\begin{aligned} \alpha_1 Q(a, b) + (1 - \alpha_1) A(a, b) \\ &< \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta \\ &< \beta_1 Q(a, b) + (1 - \beta_1) A(a, b), \\ Q^{\alpha_2}(a, b) A^{1-\alpha_2}(a, b) \\ &< \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta \\ &< Q^{\beta_2}(a, b) A^{1-\beta_2}(a, b) \end{aligned} \quad (9)$$

hold for all $a, b > 0$ with $a \neq b$.

The aim of this paper is to find the best possible constants $\alpha_1, \beta_1, \alpha_2$ and β_2 such that the double inequalities

$$\begin{aligned} I^{\alpha_1}(a, b) Q^{1-\alpha_1}(a, b) &< M(a, b) < I^{\beta_1}(a, b) Q^{1-\beta_1}(a, b), \\ I^{\alpha_2}(a, b) C^{1-\alpha_2}(a, b) &< M(a, b) < I^{\beta_2}(a, b) C^{1-\beta_2}(a, b) \end{aligned} \quad (10)$$

hold for all $a, b > 0$ with $a \neq b$. All numerical computations are carried out using MATHEMATICA software.

2. Lemmas

In order to prove our main results, we need several lemmas, which we present in this section.

Lemma 1. *The double inequality*

$$x + \frac{x^3}{3} - \frac{2x^5}{15} < \sqrt{1+x^2} \sinh^{-1}(x) < x + \frac{x^3}{3} - \frac{2x^5}{15} + \frac{8x^7}{105} \quad (11)$$

holds for $x \in (0, 1)$.

Proof. To prove Lemma 1, it suffices to prove that

$$\eta_1(x) = \sqrt{1+x^2} \sinh^{-1}(x) - \left(x + \frac{x^3}{3} - \frac{2x^5}{15}\right) > 0, \quad (12)$$

$$\eta_2(x) = \sqrt{1+x^2} \sinh^{-1}(x) - \left(x + \frac{x^3}{3} - \frac{2x^5}{15} + \frac{8x^7}{105}\right) < 0 \quad (13)$$

for $x \in (0, 1)$.

From the expressions of $\eta_1(x)$ and $\eta_2(x)$, we get

$$\begin{aligned} \eta_1(0) &= \eta_2(0) = 0, \\ \eta_1'(x) &= \frac{x\eta_1^*(x)}{\sqrt{1+x^2}}, \quad \eta_2'(x) = \frac{x\eta_2^*(x)}{\sqrt{1+x^2}}, \end{aligned} \quad (14)$$

where

$$\begin{aligned} \eta_1^*(x) &= \sinh^{-1}(x) - \left(x - \frac{2x^3}{3}\right) \sqrt{1+x^2}, \\ \eta_2^*(x) &= \sinh^{-1}(x) - \left(x - \frac{2x^3}{3} + \frac{8x^5}{15}\right) \sqrt{1+x^2}, \end{aligned} \quad (15)$$

$$\eta_1^*(0) = \eta_2^*(0) = 0,$$

$$\eta_1^{*'}(x) = \frac{8x^4}{3\sqrt{1+x^2}} > 0, \quad (16)$$

$$\eta_2^{*'}(x) = -\frac{16x^6}{5\sqrt{1+x^2}} < 0, \quad (17)$$

for $x \in (0, 1)$.

Therefore, inequality (12) follows from (14)–(16), and inequality (13) follows from (14)–(17). $\square$

Lemma 2. *Let*

$$L(x) = \log \left[\frac{(1+x)^{1+x}}{(1-x)^{1-x}} \right] - 2x - x \log(1-x^2). \quad (18)$$

Then

$$L(x) > \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} \quad (19)$$

for $x \in (0, 1)$, and

$$L(x) < \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + x^9 \quad (20)$$

for $x \in (0, 3/4)$.

Proof. To prove inequalities (19) and (20), it suffices to show that

$$\begin{aligned} \iota_1(x) &:= \log \left[\frac{(1+x)^{1+x}}{(1-x)^{1-x}} \right] - 2x - x \log(1-x^2) \\ &\quad - \left(\frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} \right) > 0 \end{aligned} \quad (21)$$

for $x \in (0, 1)$, and

$$\begin{aligned} \iota_2(x) &:= \log \left[\frac{(1+x)^{1+x}}{(1-x)^{1-x}} \right] - 2x - x \log(1-x^2) \\ &\quad - \left(\frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + x^9 \right) < 0 \end{aligned} \quad (22)$$

for $x \in (0, 3/4)$.

From (21) and (22), one has

$$\iota_1(0^+) = \iota_2(0^+) = 0, \quad (23)$$

$$\iota_1'(x) = \frac{2x^8}{1-x^2} > 0 \quad (24)$$

for $x \in (0, 1)$, and

$$\iota_2'(x) = -\frac{9x^8}{1-x^2} \left(\frac{7}{9} - x^2 \right) < 0 \quad (25)$$

for $x \in (0, 3/4)$.

Therefore, inequality (21) follows from (23) and (24), and inequality (22) follows from (23) and (25). $\square$

Lemma 3. *Let*

$$\Phi_1(x) = \frac{1}{\sqrt{1+x^2} \sinh^{-1}(x)} - \frac{1}{x(1+x^2)}. \quad (26)$$

Then the double inequality

$$\frac{2x}{3} - \frac{34x^3}{45} + \frac{x^5}{2} < \Phi_1(x) < \frac{2x}{3} - \frac{34x^3}{45} + \frac{4x^5}{5} \quad (27)$$

holds for $x \in (0, 0.7)$.

Proof. To prove inequality (27), it suffices to show that

$$\begin{aligned} \phi_1(x) &= x\sqrt{1+x^2} - \sinh^{-1}(x) \\ &\quad - x(1+x^2) \sinh^{-1}(x) \left(\frac{2x}{3} - \frac{34x^3}{45} + \frac{x^5}{2} \right) \\ &> 0, \end{aligned} \quad (28)$$

$$\begin{aligned} \phi_2(x) &= x\sqrt{1+x^2} - \sinh^{-1}(x) \\ &\quad - x(1+x^2) \sinh^{-1}(x) \left(\frac{2x}{3} - \frac{34x^3}{45} + \frac{4x^5}{5} \right) \\ &< 0 \end{aligned} \quad (29)$$

for $x \in (0, 0.7)$.

First, we prove inequality (28). From the expression of $\phi_1(x)$, we have

$$\phi_1(0) = 0, \quad \phi_1(0.7) = 0.0033 \dots, \quad (30)$$

$$\phi_1'(x) = \frac{x\phi_1^*(x)}{90\sqrt{1+x^2}}, \quad (31)$$

where

$$\begin{aligned} \phi_1^*(x) &= 120x + 8x^3 + 23x^5 - 45x^7 \\ &\quad - 2(60 - 16x^2 - 69x^4 + 180x^6) \\ &\quad \times \sqrt{1+x^2} \sinh^{-1}(x). \end{aligned} \quad (32)$$

Equation (32) leads to

$$\begin{aligned} \phi_1^*(0.6) &= 3.017 \dots, \quad \phi_1^*(0.7) = -3.551 \dots, \\ \phi_1^{**}(x) &= -\frac{x\phi_1^*(x)}{1+x^2}, \end{aligned} \quad (33)$$

where

$$\begin{aligned} \phi_1^{**}(x) &= -56x - 309x^3 + 422x^5 + 675x^7 \\ &\quad + 2(28 - 324x^2 + 735x^4 + 1260x^6) \\ &\quad \times \sqrt{1+x^2} \sinh^{-1}(x). \end{aligned} \quad (34)$$

Note that

$$60 - 16x^2 - 69x^4 + 180x^6 > 0 \quad (35)$$

for $x \in (0, 0.6]$, and

$$28 - 324x^2 + 735x^4 + 1260x^6 > 0 \quad (36)$$

for $x \in [0.6, 0.7]$.

It follows from (32) and (34)–(36) together with Lemma 1 that

$$\begin{aligned} \phi_1^*(x) &> 120x + 8x^3 + 23x^5 - 45x^7 \\ &\quad - 2(60 - 16x^2 - 69x^4 + 180x^6) \left(x + \frac{x^3}{3}\right) \\ &= \frac{x^5}{3} (515 - 1077x^2 - 360x^4) \\ &\geq \frac{x^5}{3} \left[515 - 1077 \times \frac{9}{25} - 360 \times \frac{81}{625}\right] \\ &= \frac{10078x^5}{375} > 0 \end{aligned} \quad (37)$$

for $x \in (0, 0.6]$, and

$$\begin{aligned} \phi_1^{**}(x) &> -56x - 309x^3 + 422x^5 + 675x^7 \\ &\quad + 2(28 - 324x^2 + 735x^4 + 1260x^6) \\ &\quad \times \left(x + \frac{x^3}{3} - \frac{2x^5}{15}\right) = \frac{x^3}{15} \\ &\quad \times (-14075 + 25028x^2 + 56571x^4 + 9660x^6 - 5040x^8) \\ &> \frac{x^3}{15} [-14075 + 25028 \times (0.6)^2 + 56571 \times (0.6)^4] \\ &= \frac{1416676x^3}{9375} > 0 \end{aligned} \quad (38)$$

for $x \in [0.6, 0.7]$.

From (33), (37), and (38), we clearly see that there exists $x_1 \in (0.6, 0.7)$ such that $\phi_1^*(x) > 0$ for $x \in (0, x_1)$ and $\phi_1^*(x) < 0$ for $x \in (x_1, 0.7)$. Then (31) leads to the conclusion that $\phi_1(x)$ is strictly increasing on $(0, x_1]$ and strictly decreasing on $[x_1, 0.7]$.

Therefore, inequality (28) follows from (30) and the piecewise monotonicity of $\phi_1(x)$.

Next, we prove inequality (29). From the expression of $\phi_2(x)$, we get

$$\begin{aligned} \phi_2(0) &= 0, \\ \phi_2'(x) &= -\frac{2x\phi_2^*(x)}{45\sqrt{1+x^2}}, \end{aligned} \quad (39)$$

where

$$\begin{aligned} \phi_2^*(x) &= x(18x^6 + x^4 - 2x^2 - 30) \\ &\quad + 2(15 - 4x^2 + 3x^4 + 72x^6) \\ &\quad \times \sqrt{1+x^2} \sinh^{-1}(x). \end{aligned} \quad (40)$$

It follows from Lemma 1 and (40) that

$$\begin{aligned} \phi_2^*(x) &> x(18x^6 + x^4 - 2x^2 - 30) \\ &\quad + 2(15 - 4x^2 + 3x^4 + 72x^6) \left(x + \frac{x^3}{3} - \frac{2x^5}{15}\right) \\ &= \frac{x^5}{15} (5 + 2476x^2 + 708x^4 - 288x^6) > 0 \end{aligned} \quad (41)$$

for $x \in (0, 0.7]$.

Therefore, inequality (29) follows from (39) together with (41). $\square$

Lemma 4. Let

$$\Phi_2(x) = \frac{1}{\sqrt{1+x^2} \sinh^{-1}(x)} - \frac{1-x^2}{x(1+x^2)}. \quad (42)$$

Then the double inequality

$$\frac{5x}{3} - \frac{79x^3}{45} + \frac{11x^5}{10} < \Phi_2(x) < \frac{5x}{3} - \frac{79x^3}{45} + \frac{9x^5}{5} \quad (43)$$

holds for $x \in (0, 3/4)$.

Proof. To prove Lemma 4, it suffices to prove that

$$\begin{aligned} \varphi_1(x) &:= x\sqrt{1+x^2} - (1-x^2)\sinh^{-1}(x) \\ &\quad - x(1+x^2)\sinh^{-1}(x)\left(\frac{5x}{3} - \frac{79x^3}{45} + \frac{11x^5}{10}\right) > 0, \end{aligned} \quad (44)$$

$$\begin{aligned} \varphi_2(x) &:= x\sqrt{1+x^2} - (1-x^2)\sinh^{-1}(x) \\ &\quad - x(1+x^2)\sinh^{-1}(x)\left(\frac{5x}{3} - \frac{79x^3}{45} + \frac{9x^5}{5}\right) < 0 \end{aligned} \quad (45)$$

for $x \in (0, 3/4)$.

We first prove inequality (44). From the expression of $\varphi_1(x)$, we obtain

$$\varphi_1(0) = 0, \quad \varphi_1\left(\frac{3}{4}\right) = 0.008457 \dots > 0, \quad (46)$$

$$\varphi_1'(x) = \frac{x\varphi_1^*(x)}{90\sqrt{1+x^2}}, \quad (47)$$

where

$$\begin{aligned} \varphi_1^*(x) &= 120x + 8x^3 + 59x^5 - 99x^7 - 2 \\ &\quad \times (60 - 16x^2 - 177x^4 + 396x^6)\sqrt{1+x^2}\sinh^{-1}(x). \end{aligned} \quad (48)$$

Equation (48) leads to

$$\varphi_1^*(0.66) = 6.02 \dots > 0, \quad \varphi_1^*\left(\frac{3}{4}\right) = -19.299 \dots < 0, \quad (49)$$

$$\varphi_1^{*'}(x) = -\frac{x\varphi_1^{**}(x)}{1+x^2}, \quad (50)$$

where

$$\begin{aligned} \varphi_1^{**}(x) &= -56x - 705x^3 + 836x^5 + 1485x^7 \\ &\quad + 14(4 - 108x^2 + 213x^4 + 396x^6) \\ &\quad \times \sqrt{1+x^2}\sinh^{-1}(x). \end{aligned} \quad (51)$$

Note that

$$\begin{aligned} 60 - 16x^2 - 177x^4 + 396x^6 \\ > 60 - 16 \times (0.66)^2 - 177 \times (0.66)^4 \\ = 19.4451 > 0 \end{aligned} \quad (52)$$

for $x \in (0, 0.66)$, and

$$\begin{aligned} 4 - 108x^2 + 213x^4 + 396x^6 \\ > 4 - 108 \times \left(\frac{3}{4}\right)^2 + 213 \times (0.66)^4 \\ &\quad + 396 \times (0.66)^6 = 16.3972 > 0 \end{aligned} \quad (53)$$

for $x \in [0.66, 3/4]$.

It follows from Lemma 1, (48), and (51)–(53) that

$$\begin{aligned} \varphi_1^*(x) &> 120x + 8x^3 + 59x^5 - 99x^7 \\ &\quad - 2(60 - 16x^2 - 177x^4 + 396x^6) \\ &\quad \times \left(x + \frac{x^3}{3} - \frac{2x^5}{15} + \frac{8x^7}{105}\right) \\ &= \frac{x^5}{105} [46165 - 82573x^2 - 32420x^4 \\ &\quad + 7584x^6 + 6336x^6(1-x^2)] \\ &> \frac{x^5}{105} [46165 - 82573 \times (0.66)^2 - 32420 \times (0.66)^4] \\ &= \frac{x^5}{105} \times 4044.5917 \dots > 0 \end{aligned} \quad (54)$$

for $x \in (0, 0.66)$, and

$$\begin{aligned} \varphi_1^{**}(x) &> -56x - 705x^3 + 836x^5 + 1485x^7 \\ &\quad + 14(4 - 108x^2 + 213x^4 + 396x^6) \left(x + \frac{x^3}{3} - \frac{2x^5}{15}\right) \\ &= \frac{x^3}{15} [-32975 + 49598x^2 + 123369x^4 \\ &\quad + 10668x^6 + 11088x^6(1-x^2)] \\ &> \frac{x^3}{15} [-32975 + 49598 \times (0.66)^2 + 123369 \times (0.66)^4] \\ &= \frac{x^3}{15} \times 12038.83 \dots > 0 \end{aligned} \quad (55)$$

for $x \in [0.66, 3/4]$.

From (50) and (55), we know that $\varphi_1^*(x)$ is strictly decreasing on $[0.66, 3/4]$, and this in conjunction with (49) and (54) leads to the conclusion that there exists $x_1 \in (0.66, 3/4)$ such that $\varphi_1^*(x) > 0$ for $x \in (0, x_1)$ and $\varphi_1^*(x) < 0$ for $x \in (x_1, 3/4)$. Then (47) implies that $\varphi_1(x)$ is strictly increasing on $(0, x_1]$ and strictly decreasing on $[x_1, 3/4]$. Therefore, inequality (44) follows from (46) and the piecewise monotonicity of $\varphi_1(x)$.

Next, we prove inequality (45). From the expression of $\varphi_2(x)$ one has

$$\begin{aligned}\varphi_2(0) &= 0, \\ \phi_2'(x) &= -\frac{x\phi_2^*(x)}{45\sqrt{1+x^2}},\end{aligned}\quad (56)$$

where

$$\begin{aligned}\phi_2^*(x) &= -60x - 4x^3 + 2x^5 + 81x^7 \\ &\quad + 4(15 - 4x^2 + 3x^4 + 162x^6) \\ &\quad \times \sqrt{1+x^2} \sinh^{-1}(x).\end{aligned}\quad (57)$$

It follows from Lemma 1 and (52) that

$$\begin{aligned}\phi_2^*(x) &> -60x - 4x^3 + 2x^5 + 81x^7 \\ &\quad + 4(15 - 4x^2 + 3x^4 + 162x^6) \left(x + \frac{x^3}{3} - \frac{2x^5}{15}\right) \\ &= \frac{x^5}{15} (10 + 11027x^2 + 3216x^4 - 1296x^6) > 0\end{aligned}\quad (58)$$

for $x \in (0, 3/4)$.

Therefore, inequality (45) follows from (56) together with (58). $\square$

Lemma 5. Let $L(x)$ be defined as in Lemma 2 and

$$\Upsilon_1(x) = \frac{L(x)}{2x^2} + \frac{x}{1+x^2}.\quad (59)$$

Then the double inequality

$$\frac{4x}{3} - \frac{4x^3}{5} + \frac{4x^5}{5} < \Upsilon_1(x) < \frac{4x}{3} - \frac{4x^3}{5} + \frac{8x^5}{7}\quad (60)$$

holds for $x \in (0, 0.7)$.

Proof. From Lemma 2, one has

$$\begin{aligned}\Upsilon_1(x) &- \left(\frac{4x}{3} - \frac{4x^3}{5} + \frac{4x^5}{5}\right) \\ &> \frac{1}{2x^2} \left(\frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7}\right) + \frac{x}{1+x^2} \\ &\quad - \left(\frac{4x}{3} - \frac{4x^3}{5} + \frac{4x^5}{5}\right)\end{aligned}$$

$$= \frac{23x^5}{35(1+x^2)} \left(\frac{12}{23} - x^2\right) > 0,$$

$$\begin{aligned}\Upsilon_1(x) &- \left(\frac{4x}{3} - \frac{4x^3}{5} + \frac{8x^5}{7}\right) \\ &< \frac{1}{2x^2} \left(\frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + x^9\right) \\ &\quad + \frac{x}{1+x^2} - \left(\frac{4x}{3} - \frac{4x^3}{5} + \frac{8x^5}{7}\right) \\ &= -\frac{x^7(1-x^2)}{2(1+x^2)} < 0\end{aligned}\quad (61)$$

for $x \in (0, 0.7)$.

Therefore, Lemma 5 follows easily from (61). $\square$

Lemma 6. Let $L(x)$ be defined as in Lemma 2 and

$$\Upsilon_2(x) = \frac{L(x)}{2x^2} + \frac{2x}{1+x^2}.\quad (62)$$

Then the double inequality

$$\frac{7x}{3} - \frac{9x^3}{5} + \frac{7x^5}{5} < \Upsilon_2(x) < \frac{7x}{3} - \frac{9x^3}{5} + \frac{15x^5}{7}\quad (63)$$

holds for $x \in (0, 3/4)$.

Proof. It follows from Lemma 2 that

$$\begin{aligned}\Upsilon_2(x) &- \left(\frac{7x}{3} - \frac{9x^3}{5} + \frac{7x^5}{5}\right) \\ &> \frac{1}{2x^2} \left(\frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7}\right) \\ &\quad + \frac{2x}{1+x^2} - \left(\frac{7x}{3} - \frac{9x^3}{5} + \frac{7x^5}{5}\right) \\ &= \frac{44x^5}{35(1+x^2)} \left(\frac{13}{22} - x^2\right) > 0, \\ \Upsilon_2(x) &- \left(\frac{7x}{3} - \frac{9x^3}{5} + \frac{15x^5}{7}\right) \\ &< \frac{1}{2x^2} \left(\frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + x^9\right) \\ &\quad + \frac{2x}{1+x^2} - \left(\frac{7x}{3} - \frac{9x^3}{5} + \frac{15x^5}{7}\right) \\ &= -\frac{x^7(3-x^2)}{2(1+x^2)} < 0\end{aligned}\quad (64)$$

for $x \in (0, 3/4)$.

Therefore, Lemma 6 follows from (64). $\square$

Lemma 7. *The inequality*

$$\frac{x^3}{\sqrt{1+x^2}} > [\sinh^{-1}(x)]^3 \quad (65)$$

holds for $x \in (0, 1)$.

Proof. Let

$$\zeta(x) = \frac{x^3}{\sqrt{1+x^2}} - [\sinh^{-1}(x)]^3. \quad (66)$$

Then

$$\begin{aligned} \zeta(0) &= 0, \\ \zeta'(x) &= \frac{\zeta_1(x)}{(1+x^2)^{3/2}}, \end{aligned} \quad (67)$$

where

$$\zeta_1(x) = x^2(3+2x^2) - 3[\sqrt{1+x^2}\sinh^{-1}(x)]^2. \quad (68)$$

It follows from Lemma 1 and (68) that

$$\begin{aligned} \zeta_1(x) &> x^2(3+2x^2) - 3\left(x + \frac{x^3}{3} - \frac{2x^5}{15} + \frac{8x^7}{105}\right)^2 \\ &= x^6 \left[\frac{37}{525} + \left(\frac{208}{525} + \frac{36x^2}{175} + \frac{64x^6}{3675} \right) \right. \\ &\quad \left. \times (1-x^2) + \frac{32x^6}{735} \right] > 0 \end{aligned} \quad (69)$$

for $x \in (0, 1)$.

Therefore, Lemma 7 follows from (67) together with (69). $\square$

Lemma 8. *Let*

$$\begin{aligned} \mu_1(x) &= \frac{1+3x^2}{(x+x^3)^2} - \frac{1}{(1+x^2)[\sinh^{-1}(x)]^2} \\ &\quad - \frac{x}{(1+x^2)^{3/2}\sinh^{-1}(x)}. \end{aligned} \quad (70)$$

Then $\mu_1(x) < 0.2$ for $x \in [0.7, 1)$.

Proof. Let

$$\begin{aligned} \omega_1(x) &= \frac{1}{x^2} - \frac{1}{[\sinh^{-1}(x)]^2}, \\ \omega_2(x) &= \frac{2}{\sqrt{1+x^2}} - \frac{x}{\sinh^{-1}(x)}. \end{aligned} \quad (71)$$

Then

$$\mu_1(x) = \frac{\omega_1(x)}{1+x^2} + \frac{\omega_2(x)}{(1+x^2)^{3/2}}. \quad (72)$$

Lemma 7 and $x > \sinh^{-1}(x)$ give $\omega_1(x) < 0$ and

$$\omega_1'(x) = \frac{2}{x^3[\sinh^{-1}(x)]^3} \left[\frac{x^3}{\sqrt{1+x^2}} - (\sinh^{-1}(x))^3 \right] > 0 \quad (73)$$

for $x \in (0, 1)$. This in turn implies that

$$\left[\frac{\omega_1(x)}{1+x^2} \right]' = \frac{\omega_1'(x)(1+x^2) - 2x\omega_1(x)}{(1+x^2)^2} > 0 \quad (74)$$

for $x \in (0, 1)$.

On the other hand, from the expression of $\omega_2(x)$, we get

$$\begin{aligned} \omega_2(1) &= 0.2796 \dots > 0, \\ \omega_2'(x) &= -\frac{2x}{(1+x^2)^{3/2}} + \frac{\omega_2^*(x)}{[\sinh^{-1}(x)]^2}, \end{aligned} \quad (75)$$

where

$$\begin{aligned} \omega_2^*(x) &= \frac{x}{\sqrt{1+x^2}} - \sinh^{-1}(x), \\ \omega_2^*(0) &= 0, \end{aligned} \quad (76)$$

$$\omega_2^{*'}(x) = -\frac{x^2}{(1+x^2)^{3/2}} < 0$$

for $x \in (0, 1)$.

From (75)–(76), we clearly see that $\omega_2'(x) < 0$ and $\omega_2(x) > 0$ for $x \in (0, 1)$. This in turn implies that

$$\begin{aligned} \left[\frac{\omega_2(x)}{(1+x^2)^{3/2}} \right]' &= \frac{\omega_2'(x)(1+x^2)^{3/2} - 3x\sqrt{1+x^2}\omega_2(x)}{(1+x^2)^3} \\ &< 0 \end{aligned} \quad (77)$$

for $x \in (0, 1)$.

Equation (72) together with inequalities (74) and (77) lead to the conclusion that

$$\begin{aligned} \mu_1(x) &\leq \frac{\omega_1(1)}{2} + \frac{\omega_2(0.7)}{[1+(0.7)^2]^{3/2}} \\ &= 0.167 \dots < 0.2 \end{aligned} \quad (78)$$

for $x \in [0.7, 1)$. $\square$

Lemma 9. *Let*

$$\begin{aligned} \mu_2(x) &= \frac{1+4x^2-x^4}{(x+x^3)^2} - \frac{1}{(1+x^2)[\sinh^{-1}(x)]^2} \\ &\quad - \frac{x}{(1+x^2)^{3/2}\sinh^{-1}(x)}. \end{aligned} \quad (79)$$

Then $\mu_2(x) < 0.51$ for $x \in [0.65, 1)$.

Proof. Let

$$\tau_1(x) = \frac{1}{x^2} - \frac{1}{[\sinh^{-1}(x)]^2} = \mu_1(x), \quad (80)$$

$$\tau_2(x) = \frac{3-x^2}{\sqrt{1+x^2}} - \frac{x}{\sinh^{-1}(x)},$$

then

$$\mu_2(x) = \frac{\tau_1(x)}{1+x^2} + \frac{\tau_2(x)}{(1+x^2)^{3/2}}. \quad (81)$$

From (74), we clearly see that

$$\left[\frac{\tau_1(x)}{1+x^2} \right]' = \left[\frac{\omega_1(x)}{1+x^2} \right]' > 0 \quad (82)$$

for $x \in (0, 1)$.

On the other hand, from the expression of $\tau_2(x)$ together with Lemma 1, we get

$$\tau_2(1) = 0.2796 \dots > 0,$$

$$\tau_2'(x) = -\frac{1}{\sinh^{-1}(x)} - \frac{x\tau_2^*(x)}{(1+x^2)^{3/2}[\sinh^{-1}(x)]^2},$$

$$\tau_2^*(x) = (5+x^2)[\sinh^{-1}(x)]^2 - (1+x^2), \quad (83)$$

$$\tau_2^*(0.65) = 0.6033 \dots,$$

$$\begin{aligned} \tau_2^{*'}(x) &= 2x[\sinh^{-1}(x)]^2 \\ &+ 2 \left[\frac{5+x^2}{1+x^2} \sqrt{1+x^2} \sinh^{-1}(x) - x \right] > 0 \end{aligned}$$

for $x \in (0, 1)$.

From (83), we clearly see that $\tau_2'(x) < 0$ and $\tau_2(x) > 0$ for $x \in [0.65, 1)$. This in turn implies that

$$\left[\frac{\tau_2(x)}{(1+x^2)^{3/2}} \right]' = \frac{\tau_2'(x)(1+x^2)^{3/2} - 3x\sqrt{1+x^2}\tau_2(x)}{(1+x^2)^3} < 0 \quad (84)$$

for $x \in [0.65, 1)$.

Equation (81) together with inequalities (82) and (84) lead to the conclusion that

$$\mu_2(x) \leq \frac{\tau_1(1)}{2} + \frac{\tau_2(0.65)}{[1+(0.65)^2]^{3/2}} = 0.503 \dots < 0.51 \quad (85)$$

for $x \in [0.65, 1)$. $\square$

Lemma 10. Let $L(x)$ be defined as in Lemma 2 and

$$\nu_1(x) = \frac{2(1+x^4)}{(1-x^2)(1+x^2)^2} - \frac{L(x)}{x^3}. \quad (86)$$

Then $\nu_1(x) > 1.2$ for $x \in [0.7, 1)$.

Proof. Differentiating $\nu_1(x)$ yields

$$\nu_1'(x) = \frac{3L(x)}{x^4} - \frac{2+8x^2-20x^4-6x^8}{x(1-x^2)^2(1+x^2)^3}. \quad (87)$$

It follows from (19) and (87) that

$$\begin{aligned} \nu_1'(x) &> \frac{1}{x} \left[3 \left(\frac{2}{3} + \frac{2x^2}{5} + \frac{2x^4}{7} \right) - \frac{2+8x^2-20x^4-6x^8}{(1-x^2)^2(1+x^2)^3} \right] \\ &= \frac{2x(-84+316x^2-97x^4+68x^6+26x^8+36x^{10}+15x^{12})}{35(1-x^2)^2(1+x^2)^3} \\ &> \frac{2x}{35(1-x^2)^2(1+x^2)^3} \left[-84+316 \times (0.7)^2 - \frac{349}{5} \right. \\ &\quad \left. + 68x^4 \left(x^2 - \frac{2}{5} \right) \right] \\ &> \frac{2x}{35(1-x^2)^2(1+x^2)^3} > 0 \end{aligned} \quad (88)$$

for $x \in [0.7, 1)$.

Therefore, $\nu_1(x) \geq \nu_1(0.7) = 1.214 \dots > 1.2$ for $x \in [0.7, 1)$ follows from (88). $\square$

Lemma 11. Let $L(x)$ be defined as in Lemma 2 and

$$\nu_2(x) = \frac{3-2x^2+3x^4}{(1-x^2)(1+x^2)^2} - \frac{L(x)}{x^3}. \quad (89)$$

Then $\nu_2(x) > 1.38$ for $x \in [0.65, 1)$.

Proof. Differentiating $\nu_2(x)$ yields

$$\nu_2'(x) = \frac{3L(x)}{x^4} - \frac{2(1+7x^2-17x^4+5x^6-4x^8)}{x(1-x^2)^2(1+x^2)^3}. \quad (90)$$

It follows from (19) and (90) together with the monotonicity of the function $561x^2 - 272x^4$ on $[0.65, 1)$ that

$$\begin{aligned} \nu_2'(x) &> \frac{1}{x} \left[3 \left(\frac{2}{3} + \frac{2x^2}{5} + \frac{2x^4}{7} \right) - \frac{2(1+7x^2-17x^4+5x^6-4x^8)}{(1-x^2)^2(1+x^2)^3} \right] \\ &= \frac{2x(-189+561x^2-272x^4+103x^6+26x^8+36x^{10}+15x^{12})}{35(1-x^2)^2(1+x^2)^3} \\ &> \frac{2x[-189+561 \times (0.65)^2 - 272 \times (0.65)^4 + 103 \times (0.65)^6]}{35(1-x^2)^2(1+x^2)^3} \\ &= \frac{2x \times 7.23 \dots}{35(1-x^2)^2(1+x^2)^3} > 0 \end{aligned} \quad (91)$$

for $x \in [0.65, 1)$.

Equation (91) leads to the conclusion that $\nu_2(x) \geq \nu_2(0.65) = 1.389 \dots > 1.38$ for $x \in [0.65, 1)$. $\square$

Lemma 12. Let $\Phi_1(x)$ and $Y_1(x)$ be defined, respectively, as in Lemmas 3 and 5, and $\Theta_1(x; p) = \Phi_1(x) - pY_1(x)$. Then $\Theta_1(x; p)$ is strictly decreasing on $[0.7, 1)$ if $p > 1/6$.

Proof. Differentiating $\Theta_1(x; p)$ with respect to x and making use of Lemmas 8 and 10, we get

$$\begin{aligned} \frac{d\Theta_1(x; p)}{dx} &= \Phi_1'(x) - pY_1'(x) = \mu_1(x) - p\nu_1(x) \\ &< 0.2 - \frac{1}{6} \times 1.2 = 0 \end{aligned} \quad (92)$$

for $x \in [0.7, 1)$ and $p > 1/6$. This in turn implies that $\Theta_1(x; p)$ is strictly decreasing on $[0.7, 1)$ if $p > 1/6$. $\square$

Lemma 13. Let $\Phi_2(x)$ and $Y_2(x)$ be defined, respectively, as in Lemmas 4 and 6, and $\Theta_2(x; q) = \Phi_2(x) - qY_2(x)$. Then $\Theta_2(x; q)$ is strictly decreasing on $[0.65, 1)$ if $q > 2/5$.

Proof. Differentiating $\Theta_2(x; q)$ with respect to x and making use of Lemmas 9 and 11, we have

$$\begin{aligned} \frac{d\Theta_2(x; q)}{dx} &= \Phi_2'(x) - qY_2'(x) = \mu_2(x) - q\nu_2(x) \\ &< 0.51 - \frac{2}{5} \times 1.38 = -0.042 < 0 \end{aligned} \quad (93)$$

for $x \in [0.65, 1)$ and $q > 2/5$. This in turn implies that $\Theta_2(x; q)$ is strictly decreasing on $[0.65, 1)$ if $q > 2/5$. $\square$

3. Main Results

Theorem 14. The double inequality

$$I^{\alpha_1}(a, b) Q^{1-\alpha_1}(a, b) < M(a, b) < I^{\beta_1}(a, b) Q^{1-\beta_1}(a, b) \quad (94)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\beta_1 \leq \log[\sqrt{2} \log(1 + \sqrt{2})]/(1 - \log \sqrt{2}) = 0.337 \dots$ and $\alpha_1 \geq 1/2$.

Proof. Since $I(a, b)$, $M(a, b)$, and $Q(a, b)$ are symmetric and homogeneous of degree one, then without loss of generality,

we assume that $a > b$. Let $p \in (0, 1)$, $x = (a - b)/(a + b)$, and $\lambda_1 = \log[\sqrt{2} \log(1 + \sqrt{2})]/(1 - \log \sqrt{2})$. Then $x \in (0, 1)$, and

$$\frac{I(a, b)}{A(a, b)} = \frac{1}{e} \left[\frac{(1+x)^{1+x}}{(1-x)^{1-x}} \right]^{1/2x}, \quad (95)$$

$$\frac{M(a, b)}{A(a, b)} = \frac{x}{\sinh^{-1}(x)}, \quad \frac{Q(a, b)}{A(a, b)} = \sqrt{1+x^2},$$

$$\begin{aligned} &\frac{\log[Q(a, b)] - \log[M(a, b)]}{\log[Q(a, b)] - \log[I(a, b)]} \\ &= \frac{\log \sqrt{1+x^2} - \log x + \log[\sinh^{-1}(x)]}{\log \sqrt{1+x^2} - \log[(1+x)^{1+x}/(1-x)^{1-x}]/(2x) + 1}, \end{aligned} \quad (96)$$

$$\begin{aligned} &\lim_{x \rightarrow 0^+} \frac{\log \sqrt{1+x^2} - \log x + \log[\sinh^{-1}(x)]}{\log \sqrt{1+x^2} - \log[(1+x)^{1+x}/(1-x)^{1-x}]/(2x) + 1} \\ &= \frac{1}{2}, \end{aligned} \quad (97)$$

$$\begin{aligned} &\lim_{x \rightarrow 1^-} \frac{\log \sqrt{1+x^2} - \log x + \log[\sinh^{-1}(x)]}{\log \sqrt{1+x^2} - \log[(1+x)^{1+x}/(1-x)^{1-x}]/(2x) + 1} \\ &= \lambda_1. \end{aligned} \quad (98)$$

The difference between the convex combination of $\log[I(a, b)]$, $\log[Q(a, b)]$ and $\log[M(a, b)]$ is as follows:

$$\begin{aligned} &p \log[I(a, b)] + (1-p) \log[Q(a, b)] - \log[M(a, b)] \\ &= \frac{p}{2x} \log \left[\frac{(1+x)^{1+x}}{(1-x)^{1-x}} \right] - p \\ &\quad + (1-p) \log \sqrt{1+x^2} - \log \left[\frac{x}{\sinh^{-1}(x)} \right] := D_p(x). \end{aligned} \quad (99)$$

Equation (99) leads to

$$\begin{aligned} D_p(0^+) &= 0, \\ D_p(1^-) &= \log[\sqrt{2} \log(1 + \sqrt{2})] - p(1 - \log \sqrt{2}), \\ D_{\lambda_1}(1^-) &= 0, \end{aligned} \quad (100)$$

$$\begin{aligned} D'_p(x) &= -\frac{1+px^2}{x+x^3} + \frac{1}{\sqrt{1+x^2} \sinh^{-1}(x)} - \frac{L(x)}{2x^2} \\ &= \Phi_1(x) - pY_1(x) = \Theta_1(x; p), \end{aligned} \quad (101)$$

where $L(x)$, $\Phi_1(x)$, $Y_1(x)$, and $\Theta_1(x; p)$ are defined as in Lemmas 2, 3, 5, and 12, respectively.

It follows from (101) together with Lemmas 3 and 5 that

$$\begin{aligned} D'_{1/2}(x) &< \frac{2x}{3} - \frac{34x^3}{45} + \frac{4x^5}{5} - \frac{1}{2} \left(\frac{4x}{3} - \frac{4x^3}{5} + \frac{4x^5}{5} \right) \\ &= -\frac{2x^2}{5} \left(\frac{8}{9} - x^2 \right) < 0 \end{aligned} \quad (102)$$

for $x \in (0, 0.7)$. Moreover, we see clearly, from Lemma 12, that $D'_{1/2}(x)$ is strictly decreasing on $[0.7, 1)$ and so $D'_{1/2}(x) < D'_{1/2}(0.7) = -0.109 \dots < 0$ for $x \in [0.7, 1)$. This in conjunction with (100) and (102) implies that

$$D_{1/2}(x) < 0 \quad (103)$$

for $x \in (0, 1)$.

On the other hand, (101) and Lemmas 3 and 5 together with the monotonicity of the function $-2(17 - 18\lambda_1)x^2/45 + (7 - 16\lambda_1)x^4/14$ on $(0, 0.7)$ lead to

$$\begin{aligned} D'_{\lambda_1}(x) &> \frac{2x}{3} - \frac{34x^3}{45} + \frac{x^5}{2} - \lambda_1 \left(\frac{4x}{3} - \frac{4x^3}{5} + \frac{8x^5}{7} \right) \\ &= x \left[\frac{2(1 - 2\lambda_1)}{3} - \frac{2(17 - 18\lambda_1)}{45}x^2 + \frac{7 - 16\lambda_1}{14}x^4 \right] \\ &> x \left[\frac{2(1 - 2\lambda_1)}{3} - \frac{2(17 - 18\lambda_1)}{45} \times (0.7)^2 \right. \\ &\quad \left. + \frac{7 - 16\lambda_1}{14} \times (0.7)^4 \right] \\ &= \frac{(74969 - 218832\lambda_1)x}{180000} > 0 \end{aligned} \quad (104)$$

for $x \in (0, 0.7)$.

It follows from Lemma 12 that $D'_{\lambda_1}(x)$ is strictly decreasing on $[0.7, 1)$. Note that

$$D'_{\lambda_1}(0.7) = 0.0229 \dots > 0, \quad D'_{\lambda_1}(1^-) = -\infty. \quad (105)$$

From (104) and (105) together with the monotonicity of $D'_{\lambda_1}(x)$ on $[0.7, 1)$, we clearly see that there exists $c_1 \in (0.7, 1)$ such that $D_{\lambda_1}(x)$ is strictly increasing on $(0, c_1]$ and strictly decreasing on $[c_1, 1)$. This in conjunction with (100) implies that

$$D_{\lambda_1}(x) > 0 \quad (106)$$

for $x \in (0, 1)$.

Equation (99) together with inequalities (103) and (106) gives rise to

$$\begin{aligned} M(a, b) &> I^{1/2}(a, b) Q^{1/2}(a, b), \\ M(a, b) &< I^{\lambda_1}(a, b) Q^{1-\lambda_1}(a, b). \end{aligned} \quad (107)$$

Therefore, Theorem 14 follows from (107) together with the following statements.

(i) If $\alpha_1 < 1/2$, then (96) and (97) imply that there exists $\delta_1 \in (0, 1)$ such that $M(a, b) < I^{\alpha_1}(a, b) Q^{1-\alpha_1}(a, b)$ for all $a, b > 0$ with $(a - b)/(a + b) \in (0, \delta_1)$.

(ii) If $\beta_1 > \lambda_1$, then (96) and (98) imply that there exists $\delta_2 \in (0, 1)$ such that $M(a, b) > I^{\beta_1}(a, b) Q^{1-\beta_1}(a, b)$ for all $a, b > 0$ with $(a - b)/(a + b) \in (1 - \delta_2, 1)$. $\square$

Theorem 15. *The double inequality*

$$I^{\alpha_2}(a, b) C^{1-\alpha_2}(a, b) < M(a, b) < I^{\beta_2}(a, b) C^{1-\beta_2}(a, b) \quad (108)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \geq 5/7$ and $\beta_2 \leq \log[2 \log(1 + \sqrt{2})] = 0.566 \dots$.

Proof. We will follow the same idea in the proof of Theorem 14. Since $I(a, b)$, $M(a, b)$, and $C(a, b)$ are symmetric and homogeneous of degree one. Without loss of generality, we assume that $a > b$. Let $q \in (0, 1)$, $\lambda_2 = \log[2 \log(1 + \sqrt{2})]$, and $x = (a - b)/(a + b)$. Then $x \in (0, 1)$.

Making use of (95) together with $C(a, b)/A(a, b) = 1 + x^2$ gives

$$\begin{aligned} &\frac{\log[C(a, b)] - \log[M(a, b)]}{\log[C(a, b)] - \log[I(a, b)]} \\ &= \frac{\log(1 + x^2) - \log x + \log[\sinh^{-1}(x)]}{\log(1 + x^2) - \log[(1 + x)^{1+x}/(1 - x)^{1-x}]/(2x) + 1}, \end{aligned} \quad (109)$$

$$\begin{aligned} &\lim_{x \rightarrow 0^+} \frac{\log(1 + x^2) - \log x + \log[\sinh^{-1}(x)]}{\log(1 + x^2) - \log[(1 + x)^{1+x}/(1 - x)^{1-x}]/(2x) + 1} \\ &= \frac{5}{7}, \end{aligned} \quad (110)$$

$$\begin{aligned} &\lim_{x \rightarrow 1^-} \frac{\log(1 + x^2) - \log x + \log[\sinh^{-1}(x)]}{\log(1 + x^2) - \log[(1 + x)^{1+x}/(1 - x)^{1-x}]/(2x) + 1} \\ &= \lambda_2. \end{aligned} \quad (111)$$

The difference between the convex combination of $\log[I(a, b)]$, $\log[C(a, b)]$ and $\log[M(a, b)]$ is as follows:

$$\begin{aligned} &q \log[I(a, b)] + (1 - q) \log[C(a, b)] - \log[M(a, b)] \\ &= \frac{q}{2x} \log \left[\frac{(1 + x)^{1+x}}{(1 - x)^{1-x}} \right] - q + (1 - q) \log(1 + x^2) \\ &\quad - \log \left[\frac{x}{\sinh^{-1}(x)} \right] := E_q(x). \end{aligned} \quad (112)$$

Equation (112) leads to

$$E_q(0^+) = 0, \quad E_q(1^-) = \log[2 \log(1 + \sqrt{2})] - q, \quad (113)$$

$$E_{\lambda_2}(1^-) = 0,$$

$$\begin{aligned} E'_q(x) &= -\frac{1-x^2+2qx^2}{x+x^3} + \frac{1}{\sqrt{1+x^2}\sinh^{-1}(x)} - \frac{L(x)}{2x^2} \\ &= \Phi_2(x) - qY_2(x) = \Theta_2(x; q), \end{aligned} \quad (114)$$

where $L(x)$, $\Phi_2(x)$, $Y_2(x)$, and $\Theta_2(x; q)$ are defined as in Lemmas 2, 4, 6, and 13, respectively.

It follows from Lemmas 4, 6, and 13 together with (114) that

$$\begin{aligned} E'_{5/7}(x) &< \left(\frac{5x}{3} - \frac{79x^3}{45} + \frac{9x^5}{5} \right) - \frac{5}{7} \left(\frac{7x}{3} - \frac{9x^3}{5} + \frac{7x^5}{5} \right) \\ &= -\frac{4x^2}{5} \left(\frac{37}{63} - x^2 \right) < 0 \end{aligned} \quad (115)$$

for $x \in (0, 0.65)$ and $E'_{5/7}(x)$ is strictly decreasing on $[0.65, 1)$. Thus, we have $E'_{5/7}(x) < E'_{5/7}(0.65) = -0.117 \dots$ for $x \in [0.65, 1)$. This in conjunction with (113) and (115) implies that

$$E_{5/7}(x) < 0 \quad (116)$$

for $x \in (0, 1)$.

On the other hand, Lemmas 4, 6, and 13 together with (114) lead to

$$\begin{aligned} E'_{\lambda_2}(x) &> \left(\frac{5x}{3} - \frac{79x^3}{45} + \frac{11x^5}{10} \right) - \lambda_2 \left(\frac{7x}{3} - \frac{9x^3}{5} + \frac{15x^5}{7} \right) \\ &= x \left[\frac{5-7\lambda_2}{3} - \frac{79-81\lambda_2}{45}x^2 - \frac{150\lambda_2-77}{70}x^4 \right] \\ &> x \left[\frac{5-7\lambda_2}{3} - \frac{79-81\lambda_2}{45} \times (0.65)^2 \right. \\ &\quad \left. - \frac{150\lambda_2-77}{70} \times (0.65)^4 \right] \\ &= \frac{113027173 - 197098950\lambda_2}{100800000}x > 0 \end{aligned} \quad (117)$$

for $x \in (0, 0.65)$ and $E'_{\lambda_2}(x)$ is strictly decreasing on $[0.65, 1)$. Note that

$$E'_{\lambda_2}(0.65) = 0.0609 \dots, \quad E'_{\lambda_2}(1^-) = -\infty. \quad (118)$$

From (117) and (118) together with the monotonicity of $E'_{\lambda_2}(x)$ on $[0.65, 1)$, we clearly see that there exists $c_2 \in (0.65, 1)$

such that $E_{\lambda_2}(x)$ is strictly increasing on $(0, c_2]$ and strictly decreasing on $[c_2, 1)$. This in conjunction with (113) implies that

$$E_{\lambda_2}(x) > 0 \quad (119)$$

for $x \in (0, 1)$.

Equation (112) together with inequalities (116) and (119) lead to the conclusion that

$$\begin{aligned} M(a, b) &> I^{5/7}(a, b) C^{2/7}(a, b), \\ M(a, b) &< I^{\lambda_2}(a, b) C^{1-\lambda_2}(a, b). \end{aligned} \quad (120)$$

Therefore, Theorem 15 follows from (120) together with the following statements.

- (i) If $\alpha_2 < 5/7$, then (109) and (110) imply that there exists $\delta_3 \in (0, 1)$ such that $M(a, b) < I^{\alpha_2}(a, b) C^{1-\alpha_2}(a, b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (0, \delta_3)$.
- (ii) If $\beta_2 > \lambda_2$, then (109) and (111) imply that there exists $\delta_4 \in (0, 1)$ such that $M(a, b) > I^{\beta_2}(a, b) C^{1-\beta_2}(a, b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (1-\delta_4, 1)$. $\square$

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