Research Article

The Upwind Finite Volume Element Method for Two-Dimensional Burgers Equation

Qing Yang

School of Mathematical Sciences, Shandong Normal University, Jinan 250014, China

Correspondence should be addressed to Qing Yang; sd_yangq@163.com

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A finite volume element method for approximating the solution to two-dimensional Burgers equation is presented. Upwind technique is applied to handle the nonlinear convection term. We present the semi-discrete scheme and fully discrete scheme, respectively. We show that the schemes are convergent to order one in space in $L^2$-norm. Numerical experiment is presented finally to validate the theoretical analysis.

1. Introduction

We consider the following two-dimensional Burgers equation [1–3]:

\begin{align}
&\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x_1} + v \frac{\partial u}{\partial x_2} = \zeta \Delta u, \quad x = (x_1, x_2) \in \Omega, \quad t \in J = (0, T],
&(a)
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x_1} + v \frac{\partial v}{\partial x_2} = \zeta \Delta v, \quad (x, t) \in \Omega \times J,
&(b)
\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2} = \phi (x), \quad (x, 0) = \psi (x), \quad x \in \Omega,
&(c)
\frac{\partial u}{\partial x_1} = g_1, \quad v = g_2, \quad (x, t) \in \partial \Omega \times J,
&(d)
\end{align}

for the unknown functions $u$ and $v$ in a bounded spatial domain $\Omega \subset \mathbb{R}^2$, over a time interval $[0, T]$. The coefficient $\zeta$ is a positive number.

Burgers equation is the simplest nonlinear convection-diffusion model [1]. It is often used in modeling such physical phenomena as turbulence, shocks, and so forth. The study of Burgers equation has been a very active area because of its importance.

It is well known that strictly parabolic discretization schemes applied to Burgers equation do not work well when it is advection dominated. Effective discretization schemes recognize to some extent the hyperbolic nature of the equation.

The finite volume element method (FVEM) [4–12] is an important discretization technique for partial differential equations, especially those that arise from physical conservation laws. FVEM has ability to be faithful to the physics in general and conservation in particular, to produce simple stencils, and to treat effectively Neumann boundary conditions and nonuniform grids, and so forth.

Liang [11, 12] combined the upwind technique and the FVEM to handle the linear convection-dominated problems. In this paper, we consider upwind finite volume element method for the approximation of (1). Upwind approximation is applied to handle the nonlinear convection term. The semi-discrete and fully discrete schemes are defined, respectively. We prove that they are both convergent to order one in space. Numerical experiments are presented finally to validate the theoretical analysis.

In this paper, we use the following Sobolev spaces and the norms associated with these spaces:

\begin{align}
L^2 (\Omega) &= \left\{ f \colon \int_{\Omega} |f|^2 \, dx < \infty \right\}, \quad \| f \|_2 = \left[ \int_{\Omega} |f|^2 \, dx \right]^{1/2}, \\
L^\infty (\Omega) &= \left\{ f \colon \esssup_{\Omega} |f| < \infty \right\}, \quad \| f \|_\infty = \esssup_{\Omega} |f|,
\end{align}
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\[ H^m(\Omega) = \left\{ f : \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \in L^2(\Omega), |\alpha| \leq m \right\}, \]
\[ m \geq 0, \]
\[ \|f\|_m = \left( \sum_{|\alpha| \leq m} \left\| \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \right\|^2_{L^2(\Omega)} \right)^{1/2}, \]
\[ W^m_{co}(\Omega) = \left\{ f : \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \in L^\infty(\Omega), |\alpha| \leq m \right\}, \]
\[ m \geq 0, \]
\[ \|f\|_{m,\infty} = \max_{|\alpha| \leq m} \left\| \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \right\|_{L^\infty(\Omega)}, \]  

(2)

In particular, \( H^0(\Omega) = L^2(\Omega) \), \( W^0_{co}(\Omega) = L^\infty(\Omega) \). Let \([a, b] \subset [0, T]\) and let \( X \) be any of the spaces just defined. If \( f(x, t) \) represents functions on \( \Omega \times [a, b] \), we set

\[ H^m(a, b; X) = \left\{ f : \int_a^b \left\| \frac{\partial^{|\alpha|} f}{\partial t^\alpha} (x, t) \right\|^2_x \, dt \leq m, \alpha \leq m \right\}, \]
\[ \|f\|_{H^m(a, b; X)} = \left( \int_a^b \left\| \frac{\partial^{|\alpha|} f}{\partial t^\alpha} (x, t) \right\|^2_x \, dt \right)^{1/2}, m \geq 0, \]
\[ W^m_{co}(a, b; X) = \left\{ f : \text{ess sup}_{[a, b]} \left\| \frac{\partial^{|\alpha|} f}{\partial t^\alpha} (x, t) \right\|_{L^\infty(\Omega)} < \infty, \alpha \leq m \right\}, \]
\[ \|f\|_{W^m_{co}(a, b; X)} = \max_{0 \leq \alpha \leq m} \text{ess sup}_{[a, b]} \left\| \frac{\partial^{|\alpha|} f}{\partial t^\alpha} (x, t) \right\|_{L^\infty(\Omega)}, m \geq 0, \]
\[ L^2(a, b; X) = H^0(a, b; X), \]
\[ L^\infty(a, b; X) = W^0_{co}(a, b; X). \]

(3)

If \([a, b] = [0, T]\), we drop it from the notation. We also drop \( \Omega \); thus, we write \( L^\infty(W^1_{co}) \) for \( L^\infty(0, T; W^1_{co}(\Omega)) \).

If \( w = (w_1, w_2) \) is a vector function, we say that \( w \in X \) if \( w_1 \in X \) and \( w_2 \in X \).

An outline of the paper follows. In the next section we define the upwind finite volume element schemes for (1). Some lemmas are presented in Section 3. We derive the \( L^2 \)-norm error estimates for the semi-discrete scheme and the fully discrete scheme in Sections 4 and 5, respectively. Finally in Section 6, we give some numerical experiments.

Throughout the paper we will denote by \( C \) and \( C_i \) (\( i = 1, 2, \ldots \)) generic constants independent of the mesh parameters, which may take different values in different occurrences.

2. The Approximation Schemes

In order to rewrite (1) as the vector form we define some vector notations. The gradient of a vector function \( w = (w_1, w_2) : \mathbb{R}^2 \to \mathbb{R}^2 \) is a matrix, and the divergence of a matrix function \( A = (a_{ij})_{1 \leq i, j \leq 2} : \mathbb{R}^2 \to \mathbb{R}^2 \times 2 \) is a vector

\[ \nabla w = \left( \frac{\partial w_1}{\partial x_1}, \frac{\partial w_1}{\partial x_2}, \frac{\partial w_2}{\partial x_1}, \frac{\partial w_2}{\partial x_2} \right), \]
\[ \nabla \cdot A = \left( \sum_{j=1}^{2} \frac{\partial a_{ij}}{\partial x_j}, \sum_{j=1}^{2} \frac{\partial a_{2j}}{\partial x_j} \right). \]

(4)

Consequently, we have for a vector function \( w = (w_1, w_2) \)

\[ \Delta w = \nabla \cdot \nabla w = (\Delta w_1, \Delta w_2). \]

(5)

Let \( \theta = (u, v) \), \( \theta_0(x) = (u_0(x), v_0(x)) \), and let \( g = (g_1, g_2) \); then the system (1) can be written as the following vector form:

\[ (a) \quad \frac{\partial \theta}{\partial t} + \theta \cdot \nabla \theta - \zeta \Delta \theta = 0, \quad (x, t) \in \Omega \times J, \]
\[ (b) \quad \theta(x, 0) = \theta_0(x), \quad x \in \Omega, \]
\[ (c) \quad \theta(x, t) = g, \quad (x, t) \in \partial \Omega \times J, \]

where

\[ \theta \cdot \nabla \theta = u \frac{\partial \theta}{\partial x_1} + v \frac{\partial \theta}{\partial x_2}. \]

(7)

Let \( \mathcal{T}_h = \{ K \} \) be a triangulation of the domain \( \Omega \), and as usual, we assume the triangles \( K \) to be shape regular. Denote by \( \Omega_h = \{ P \} \) the set of the vertices of all the triangles \( K \), and let \( \Omega_h = \Omega_h \setminus \partial \Omega \). For a given triangulation \( \mathcal{T}_h \), we construct a dual mesh \( \mathcal{T}_h^* \) whose elements are called control volumes. Each triangle \( K \in \mathcal{T}_h \) can be divided into three subdomains by connecting an inner point of the triangle to the midpoints of the three edges. Around each \( P_i \in \Omega_h \), we associate a control volume \( K_i^* = K_i^* \), which consists of the union of subregions having \( P_i \) as a vertex. For a vertex \( P_i \in \partial \Omega \), we can define its control volume in a similar way. Then we define the dual partition \( \mathcal{T}_h^* = \{ K_i^*, P_i \in \Omega_h \} \) to be the union of all the control volumes. Usually we can choose the inner points as the barycenters of the circum centers, and in the later case we assume that all the inner angles of each triangle are not larger than \( \pi/2 \). We will use the barycenters dual mesh in this paper, while, with some trivial changes, our analysis can be also applied to the case when the circum centers are used.

We now characterize the finite-dimensional spaces which will be employed in approximating (6). For the sake of simplicity, we will assume that \( g_1 = g_2 = 0 \). We define the following finite dimensional spaces:

\[ P_h = \left\{ w_h \in H^1_h(\Omega), w_h|_K \in \mathcal{P}_1(K), K \in \mathcal{T}_h \right\}, \]
\[ Y_h = \left\{ \phi_h \in L^2(\Omega), \phi_h|_{K_i^*} \in \mathcal{P}_0(K_i^*), \right\}, \]
\[ P_i \in \Omega_h; \phi_h|_{K_i^*} = 0, P_i \in \partial \Omega \}, \]
\[ U_h = P_h^2, \quad V_h = Y_h^2, \]  

(8)
where \( \mathcal{P}_l(V) \) \((l = 0, 1)\) denotes the set of polynomials on \( V \) with a degree of not more than \( l \).

Multiplying (6a) by test function \( z \in V_h \) and integrating by parts yield

\[
\left( \frac{\partial}{\partial t}, z \right) + B_1(\theta; \theta, z) + B_2(\theta; \theta, z) + A(\theta, z) = 0, \quad \forall z \in V_h, \tag{9}
\]

where

\[
B_1(\varphi; w, z) = -\sum_{P \in \Omega_h} z(P) \cdot \int_{K_i'} (\nabla \cdot w) \, \varphi \, dx,
\]

\[
B_2(\varphi; w, z) = \sum_{P \in \Omega_h} z(P) \cdot \int_{\partial K_i'} (\varphi \cdot v) \, w \, ds,
\]

\[
A(\varphi, w) = \sum_{P \in \Omega_h} z(P) \cdot \int_{\partial K_i'} (v \cdot \nabla w) \, ds
\]

here \( v \) is the unit outward normal vector of \( \partial K_i' \).

Now we approximate \( B_2(\varphi; w, z) \) by using the upwind technique.

Let \( \Lambda_i = \{ j : P_j \) is adjoint with \( P_i \} \). Assuming that \( j \in \Lambda_i \), let \( \Gamma_{ij} = \partial K_i' \cap \partial K_j' \) and \( y_{ij} \) is the length of \( \Gamma_{ij} \). Denote by \( v_{ij} \) the unit outward normal vector of \( \Gamma_{ij} \) when \( \Gamma_{ij} \) is regarded as the boundary of \( K_i' \). Define

\[
\beta_{ij}(\varphi) = \int_{\Gamma_{ij}} \varphi \cdot v_{ij} \, ds. \tag{11}
\]

Let

\[
\beta_{ij}^+(\varphi) = \max\left( \beta_{ij}(\varphi), 0 \right), \quad \beta_{ij}^-(\varphi) = \max\left( -\beta_{ij}(\varphi), 0 \right).
\]

\[
\int_{\partial K_i'} (\varphi \cdot v) \, w \, ds = \sum_{j \in \Lambda_i} \left\{ \beta_{ij}^+(\varphi) \, w(P_j) - \beta_{ij}^-(\varphi) \, w(P_j) \right\}.
\]

The upwind discretization of the nonlinear term \( B_2(\varphi; w, z) \) is defined by the form

\[
B_{2h}(\varphi; w, z) = \sum_{P \in \Omega_h} \sum_{j \in \Lambda_i} \left\{ \beta_{ij}^-(\varphi) \, w(P_j) - \beta_{ij}^+(\varphi) \, w(P_j) \right\} \cdot z(P_j). \tag{13}
\]

Using the heaviside function

\[
H(r) = \begin{cases} 
1, & r \geq 0, \\
0, & r < 0,
\end{cases}
\]

we can write \( B_{2h}(\varphi; w, z) \) as

\[
B_{2h}(\varphi; w, z) = \sum_{P \in \Omega_h} \sum_{j \in \Lambda_i} \beta_{ij}(\varphi) 
\times \left\{ H \left( \beta_{ij}(\varphi) \right) \, w(P_j) + \left( 1 - H \left( \beta_{ij}(\varphi) \right) \right) \, w(P_j) \right\} 
\cdot z(P_j). \tag{15}
\]

Introduce the interpolation operators \( \Pi_h : H^1(\Omega) \to P_h \) and \( \Pi_{h}^{*} : P_h \to Y_h \), respectively. For \( w = (w_1, w_2) \), define \( \Pi_h w = (\Pi_h w_1, \Pi_h w_2) \) and \( \Pi_{h}^{*} w = (\Pi_{h}^{*} w_1, \Pi_{h}^{*} w_2) \). Assuming that \( w \in H^2(\Omega)^2 \), we can easily get the following interpolation estimates:

\[
\|w - \Pi_h w\|_2 \leq C_2 \|w\|_2, \quad s = 0, 1. \tag{16}
\]

The semi-discrete upwind finite volume scheme of (6) is as follows: find \( \theta_h : [0, T] \to U_h \) such that

\[
\left( \frac{\partial \theta_h}{\partial t}, \Pi_h^* z_h \right) + B_1(\theta_h; \theta_h, \Pi_h^* z_h) + B_{2h}(\theta_h; \theta_h, \Pi_h^* z_h) + A(\theta_h, \Pi_h^* z_h) = 0, \quad \forall z_h \in U_h, \tag{17}
\]

where \( \theta_{0h}(x) \) is the interpolation projection of \( \theta_0(x) \), that is,

\[
\theta_{0h}(x) = \Pi_h \theta_0(x).
\]

Partition \([0, T] \) into \( 0 = t^0 < t^1 < \cdots < t^{n+1} = T \), with \( t^r = t^{r-1} + \tau \). Our analysis is valid for variable time steps, but we drop the superscript from \( t \) for convenience. For functions \( f \) on \( \Omega \times J \), we write \( f^r(x) \) for \( f(x, t^r) \). By approximating \( \partial \theta_h / \partial t \) at the time \( t = t_n \) with the backward difference \( \partial \theta_h^+/\partial \tau = (\theta_h^{n+1} - \theta_h^n) / \tau \), we define the fully discrete upwind finite volume scheme for (6) as follows: find \( \theta_h^n \in U_h \) such that

\[
\left( \partial \theta_h^+/\partial \tau, \Pi_h^* z_h \right) + B_{1h}(\theta_h^n; \theta_h^n, \Pi_h^* z_h) + B_{2h}(\theta_h^n; \theta_h^n, \Pi_h^* z_h) + A(\theta_h^n, \Pi_h^* z_h) = 0, \quad n \geq 1, \quad \forall z_h \in U_h,
\]

\[
\theta_h^n = \theta_{0h}. \tag{18}
\]

3. Some Lemmas

Now we present several Lemmas. Let \( w_h = (w_1, w_2) \in U_h \), \( \bar{w}_h = (\bar{w}_1, \bar{w}_2) \in U_h \).

**Lemma 1.** (i) \( \Pi_h^* \) is a self-adjoint operator, that is,

\[
(w_h, \Pi_h^* \bar{w}_h) = (\bar{w}_h, \Pi_h^* w_h), \quad \forall w_h, \bar{w}_h \in U_h. \tag{19}
\]

(ii) Let \( \|w_h\| = (w_h, \Pi_h^* w_h)^{1/2} \). Then, for some positive constants \( C_1 \) and \( C_2 \) that are independent of \( h \),

\[
C_1 \|w_h\| \leq \|w_h\| \leq C_2 \|w_h\|, \quad \forall w_h \in U_h. \tag{20}
\]

**Proof.** It is easy to know that

\[
(w_h, \Pi_h^* \bar{w}_h) = (w_1, \Pi_h^* \bar{w}_1) + (w_2, \Pi_h^* \bar{w}_2),
\]

\[
\|w_h\|^2 = (w_h, \Pi_h^* w_h) = (w_1, \Pi_h^* w_1) + (w_2, \Pi_h^* w_2).
\]

From [4] we know that for \( w_1, w_2, \bar{w}_1, \bar{w}_2 \in P_h \),

\[
(w_1, \Pi_h^* \bar{w}_1) = (\bar{w}_1, \Pi_h^* w_1), \quad (w_2, \Pi_h^* \bar{w}_2) = (\bar{w}_2, \Pi_h^* w_2), \quad C_1 \|w_h\|^2 \leq (w_1, \Pi_h^* w_1) \leq C_2 \|w_1\|^2,
\]

\[
C_1 \|w_2\|^2 \leq (w_2, \Pi_h^* w_2) \leq C_2 \|w_2\|^2.
\]

\[
\tag{22}
\]
Lemma 2. For the bilinear form $A(\cdot, \Pi_h^*)$, one has the following conclusions:

(i) For $w_h, \tilde{w}_h \in U_h$, one has

$$A(w_h, \Pi_h^* \tilde{w}_h) = A(\tilde{w}_h, \Pi_h^* w_h).$$  \hspace{1cm} (23)

(ii) There exists a positive constant $C$ such that

$$|A(w - \Pi_h w, \Pi_h^* \tilde{w}_h)| \leq Ch\|w\|_2 \|\tilde{w}_h\|_1, \quad \forall w \in H^2(\Omega)^2, \tilde{w}_h \in U_h.$$  \hspace{1cm} (24)

(iii) There exists a positive constant $\alpha$ such that

$$A(w_h, \Pi_h^* w_h) \geq \alpha\|w_h\|^2_1, \quad \forall w_h \in U_h.$$  \hspace{1cm} (25)

Proof. For $\phi, \psi \in P_h$, define the bilinear form

$$a(\phi, \Pi_h^* \psi) = - \sum_{P_i \in T_h} \psi(P_i) \int_{\partial K_i^*} \zeta \nabla \phi \cdot \nu \, ds.$$  \hspace{1cm} (26)

Then, we get

$$A(w_h, \Pi_h^* \tilde{w}_h) = - \sum_{P_i \in T_h} \tilde{w}_i(P_i) \int_{\partial K_i^*} \zeta \nabla w_i \cdot \nu \, ds$$

$$= - \sum_{P_i \in T_h} \tilde{w}_2(P_i) \int_{\partial K_i^*} \zeta \nabla w_2 \cdot \nu \, ds$$

$$= a(w_1, \Pi_h^* \tilde{w}_1) + a(w_2, \Pi_h^* \tilde{w}_2).$$

By combining the above results and the corresponding conclusions for $a(\cdot, \Pi_h^*)$ in [4], we can obtain (23)–(25). \hfill \Box

Lemma 3. For $\varphi \in (W^{1,0}_0(\Omega))^2$, $\theta \in (H^0(\Omega))^2$, $\varphi_h \in U_h$, and $z_h \in U_h$, one has

$$|B_2(\varphi; \Pi_h^* z_h) - B_{2h}(\varphi; \Pi_h^* \Pi_h^* z_h)| \leq |z_h| \left[ h\|\varphi\|_1 + \|\theta\|_1 \|\varphi - \varphi_h\| + h\|\varphi - \varphi_h\|_1 \right].$$  \hspace{1cm} (28)

Proof. First we have

$$B_2(\varphi; \Pi_h^* z_h) - B_{2h}(\varphi; \Pi_h^* \Pi_h^* z_h)$$

$$= B_2(\varphi; \Pi_h^* z_h) - B_{2h}(\varphi; \Pi_h^* z_h)$$

$$+ B_{2h}(\varphi; \Pi_h^* z_h) - B_{2h}(\varphi; \Pi_h^* \Pi_h^* z_h).$$  \hspace{1cm} (29)

Noting that $\theta(P_i) = \Pi_h \theta(P_i), P_i \in \Omega_h$, we can easily deduce

$$B_{2h}(\varphi; \Pi_h^* \Pi_h^* z_h) = B_{2h}(\varphi; \Pi_h^* z_h) - B_{2h}(\varphi; \Pi_h^* \Pi_h^* z_h) = 0$$  \hspace{1cm} (30)

by the definition of $B_{2h}(\cdot, \Pi_h^*)$. Now we only need to bound

$$B_2(\varphi; \theta, \Pi_h^* z_h) = B_{2h}(\varphi; \theta, \Pi_h^* z_h)$$

$$= \sum_{P_i \in \Omega_h} z_h(P_i) \cdot \sum_{j \in \Lambda_i} (\varphi \cdot v_{ij}) \theta ds$$

$$- \sum_{P_i \in \Omega_h} z_h(P_i) \cdot \sum_{j \in \Lambda_i} \beta_{ij} \theta ds$$

$$\times \left[ H(\beta_{ij}(\varphi_h)) \theta(P_i) + (1 - H(\beta_{ij}(\varphi_h))) \theta(P_j) \right]$$

$$= \sum_{P_i \in \Omega_h} z_h(P_i)$$

$$\cdot \sum_{j \in \Lambda_i} \int_{\Gamma_{ij}} (\varphi \cdot v_{ij})$$

$$\times \left[ \theta - H(\beta_{ij}(\varphi_h)) \theta(P_i) + \left(1 - H(\beta_{ij}(\varphi_h)) \right) \theta(P_j) \right] ds.$$  \hspace{1cm} (31)

We denote the last two terms on the right-hand side of (31) by $I_1$ and $I_2$, respectively. We now turn to analyze the two terms. Noting that $\beta_{ij} = -\beta_{ji}$, we rewrite $I_1$ as

$$I_1 = \frac{1}{2} \sum_{K \in \mathcal{T}_h, i \in \Lambda_K} \left[ z_h(P_i) - z_h(P_j) \right]$$

$$\int_{\Gamma_{ij}} (\varphi \cdot v_{ij}) \left[ H(\beta_{ij}(\varphi_h)) \theta(\varphi(P_i)) + \left(1 - H(\beta_{ij}(\varphi_h)) \right) \theta(\varphi(P_j)) \right] ds.$$  \hspace{1cm} (32)

here $\Lambda_K$ is the set of vertex of $K$. From the Taylor's Formula and the linear property of $z_h = (z_1, z_2)$, we obtain that

$$|z_h(P_i) - z_h(P_j)|^2 \leq h^2 \left[ |\nabla z_1|^2 + |\nabla z_2|^2 \right] = h^2 |z_i|^2_{1,k}.$$  \hspace{1cm} (33)
Applying the trace inequality, we get
\[
\left\{ \int_{\Gamma_{ij} \cap K} \left| \theta - \theta (P_i) \right|^2 \, ds \right\}^{1/2} 
\leq C h^{1/2} \left\{ \int_{\Gamma_{ij} \cap K} \left[ |u - u (P_i)|^2 + |v - v (P_i)|^2 \right] \, ds \right\}^{1/2} 
\leq C h \left\{ \left( h^{-1} |u - u (P_i)|_{0,K} + |u - u (P_i)|_{1,K} \right)^2 
+ \left( h^{-1} |v - v (P_i)|_{0,K} + |v - v (P_i)|_{1,K} \right)^2 \right\}^{1/2} 
\leq C h \left( |u|_{1,K}^2 + |v|_{1,K}^2 \right)^{1/2} = C h |\theta|_{1,K}. 
\] (34)

Similarly, we can deduce that
\[
\int_{\Gamma_{ij} \cap K} \left| \theta - \theta (P_i) \right| \, ds \leq C h |\theta|_{1,K}. 
\] (35)

We conclude that
\[
|I_1| \leq C h \| \phi \|_{\infty} |\theta|_1. 
\] (36)

The similar argument yields the estimate
\[
|I_2| = \left| \frac{1}{2} \sum_{K \in T_h} \sum_{i \in I_K} \int_{\Gamma_{ij} \cap K} \left[ (\phi - \phi_h) \cdot n \right] \, ds \times \left( H \left( \beta_y (\phi_h) \right) (P_i) + (1 - H \left( \beta_y (\phi_h) \right)) (P_j) \right) \cdot |z_h (P_i) - z_h (P_j)| \right| 
\leq C \| \theta \|_{\infty} |\theta|_{1} \left( \| \phi - \phi_h \| + h \| \phi - \phi_h \|_1 \right). 
\] (37)

Substituting the estimates (36) and (37) into (31), we obtain
\[
|B_2 (\phi; \theta, \Pi_h^* z_h) - B_{2h} (\phi_h; \theta, \Pi_h^* z_h)| 
\leq C |z_h| \left\{ h \| \phi \|_{\infty} |\theta|_1 + \| \theta \|_{\infty} \right\} \times \left( \| \phi - \phi_h \| + h \| \phi - \phi_h \|_1 \right). 
\] (38)

This yields the desired result immediately. \( \square \)

4. Error Bounds for Semi-Discrete Scheme

**Theorem 4.** Assume that \( \theta \) and \( \theta_h \) are solutions to (6) and (17), respectively. Also assumes that \( \theta \) is regular enough. Then there exists a positive constant \( C \) such that
\[
\| \theta - \theta_h \| \leq C h, 
\] (39)
where \( C \) depends on principally \( \| \theta \|_2, \| \theta \|_{L^\infty(U_h \cap \Omega)}, \) and \( \| \theta \|_{L^2(\Omega P^2)} \).

**Proof.** We derive the following error equation from (6) and (17):
\[
\left( \frac{\partial \theta}{\partial t} - \frac{\partial \theta_h}{\partial t}, \Pi_h^* z_h \right) + B_1 \left( \theta; \theta, \Pi_h^* z_h \right) - B_1 \left( \theta_h; \theta, \Pi_h^* z_h \right) 
+ B_2 \left( \theta; \theta, \Pi_h^* z_h \right) - B_{2h} \left( \theta_h; \theta, \Pi_h^* z_h \right) 
+ A \left( \theta - \theta_h, \Pi_h^* z_h \right) = 0. 
\] (40)

Let \( \rho = \theta - \Pi_h \theta, \) \( \xi = \Pi_h \theta - \theta_h \). We rewrite the previously mentioned equation as
\[
\left( \frac{\partial \xi}{\partial t}, \Pi_h^* \xi \right) + B_1 \left( \theta; \theta, \Pi_h^* \xi \right) - B_1 \left( \theta_h; \theta, \Pi_h^* \xi \right) 
+ B_2 \left( \theta; \theta, \Pi_h^* \xi \right) - B_{2h} \left( \theta_h; \theta, \Pi_h^* \xi \right) + A \left( \xi, \Pi_h^* \xi \right) 
= - \left( \frac{\partial \rho}{\partial t}, \Pi_h^* \xi \right) - A \left( \rho, \Pi_h^* \xi \right). 
\] (41)

We choose \( z_h = \xi \) in (41) to get
\[
\left( \frac{\partial \xi}{\partial t}, \Pi_h^* \xi \right) + A \left( \xi, \Pi_h^* \xi \right) 
= - \left( \frac{\partial \rho}{\partial t}, \Pi_h^* \xi \right) - A \left( \rho, \Pi_h^* \xi \right) 
\] (42)
\[
= \left[ B_1 \left( \theta; \theta, \Pi_h^* \xi \right) - B_1 \left( \theta_h; \theta, \Pi_h^* \xi \right) \right] 
- \left[ B_2 \left( \theta; \theta, \Pi_h^* \xi \right) - B_{2h} \left( \theta_h; \theta, \Pi_h^* \xi \right) \right]. 
\]

Using Lemmas 1, 2 and Young’s inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \| \xi \|_2^2 + \alpha \| \xi \|_2^2 \leq C \left( \left\| \frac{\partial \rho}{\partial t} \right\|_2^2 + \| \xi \|_2^2 + \| \theta \|_{\infty}^2 \right) + \epsilon \| \xi \|_2^2 \] (43)
\[
+ \left| B_1 \left( \theta; \theta, \Pi_h^* \xi \right) - B_1 \left( \theta_h; \theta, \Pi_h^* \xi \right) \right| 
+ \left| B_2 \left( \theta; \theta, \Pi_h^* \xi \right) - B_{2h} \left( \theta_h; \theta, \Pi_h^* \xi \right) \right|. 
\]

Now we bound the last two terms on the right-hand side of (43). We need the following induction hypothesis:
\[
\left( \log \frac{1}{h} \right)^{1/2} \| \xi \|_2 (s) \rightarrow 0, \quad h \rightarrow 0, \quad 0 \leq s < t, \quad 0 < t \leq T. 
\] (44)

We know from [13] that
\[
\| \phi \|_{\infty} \leq C \left( \log \frac{1}{h} \right)^{1/2} \| \phi \|_1, \quad \forall \phi \in P_h. 
\] (45)

This implies that
\[
\| \phi \|_{\infty} \leq C \left( \log \frac{1}{h} \right)^{1/2} \| \phi \|_1, \quad \forall \phi \in U_h. 
\] (46)
Also we have the following inverse inequality:
\[
\|\varphi\|_1 \leq C h^{-1} \|\varphi\|, \quad \forall \varphi \in U_h.
\]

Using (46), we get
\[
|B_1 (\theta; \theta, \Pi_h^* \xi) - B_1 (\theta; \theta, \theta, \Pi_h^* \xi)|
\leq \sum_{\tilde{P} \in \Omega_h} |\xi (\tilde{P})| \int_{|\tilde{P}|} |(\nabla \cdot \theta) - (\nabla \cdot \theta_h)| \, dx
\leq \sum_{\tilde{P} \in \Omega_h} |\xi (\tilde{P})| \int_{|\tilde{P}|} |(\nabla \cdot \theta) - (\nabla \cdot \theta_h) + |(\nabla \cdot \theta) - (\nabla \cdot \theta_h)| \times |(\xi + \Pi_h \theta)| \, dx
\leq C \left( \|\nabla \cdot \theta\|_\infty \|\theta - \theta_h\| \|\xi\| + \|\nabla \cdot \theta\| \|\theta - \theta_h\| \times \|\xi\| \right)
\leq C \left( \|\theta\|_1 + \|\xi\|_1 + (\log \frac{1}{h}) \frac{1}{2} \|\xi\|_1
\times (\|\theta\|_1 + \|\xi\|_1) \right) \|\xi\|
\leq C \left( \|\theta\|_1 + \|\xi\|_1 + (\log \frac{1}{h}) \frac{1}{2} \|\xi\|_1 \right)
\leq C \left( \|\theta\|_1 + \|\xi\|_1 + (\log \frac{1}{h}) \frac{1}{2} \|\xi\|_1 \right) + \epsilon \|\xi\|_1^2.
\]

Next, we write
\[
|B_2 (\theta; \theta, \Pi_h^* \xi) - B_2 (\theta; \theta, \theta, \Pi_h^* \xi)|
\leq |B_2 (\theta; \theta, \Pi_h^* \xi) - B_2 (\theta; \theta, \theta, \Pi_h^* \xi)|
+ |B_2 (\theta; \theta, \Pi_h^* \xi)| = D_1 + D_2.
\]

By Choosing \(\varphi = \theta, \varphi_h = \theta_h\), and \(z_h = \xi\) in Lemma 3, using (47) and the Young's inequality, we can obtain
\[
D_1 \leq C \left| \frac{1}{2} \|\theta\|_1 \right| \|\nabla \theta\|_\infty \|\theta - \theta_h\| + \|\nabla \theta_h\|_\infty \|\xi\|_1 + \|\theta\|_1 \times \|\theta - \theta_h\| + \frac{1}{2} \|\theta\|_1 \|\xi\|_1
\leq C \left( \|\theta\|_1 + \|\theta\|_1 \right) \|\theta\|_1 + \|\xi\|_1
\leq C \left( \|\theta\|_1 + \|\theta\|_1 \right) \|\theta\|_1 + \|\xi\|_1
\leq C \left( \|\theta\|_1 + \|\theta\|_1 \right) \|\theta\|_1 + \|\theta\|_1 \|\xi\|_1
\leq C \left( \|\theta\|_1 + \|\theta\|_1 \right) \|\theta\|_1 + \|\theta\|_1 \|\xi\|_1.
\]

By an argument like (36) and then by (46) and (47), we have
\[
D_2 = \left| \sum_{\tilde{P} \in \Omega_h} \sum_{\tilde{I} \in \tilde{\Lambda}_h} \int_{|\tilde{I}|} \left( \theta_h \cdot \nu_{ij} \right) \, ds
\times \left( H (\beta_{ij}) \xi (\tilde{P}) + (1 - H (\beta_{ij})) \xi (\tilde{P}) \right) \right|
\leq \frac{1}{2} \sum_{K \in \mathcal{T}_h} \sum_{i,j \in \mathcal{A}_K} \|\xi (\tilde{P}) - \xi (\tilde{P})\|
\times \left( H (\beta_{ij}) \xi (\tilde{P}) + (1 - H (\beta_{ij})) \xi (\tilde{P}) \right)
\times \int_{|\tilde{I}|} \left| \xi \cdot \nu_{ij} \right| \, ds
+ \frac{1}{2} \sum_{K \in \mathcal{T}_h} \sum_{i,j \in \mathcal{A}_K} \|\xi (\tilde{P}) - \xi (\tilde{P})\|
\times \left( H (\beta_{ij}) \xi (\tilde{P}) + (1 - H (\beta_{ij})) \xi (\tilde{P}) \right)
\times \int_{|\tilde{I}|} \left| \xi \cdot \nu_{ij} \right| \, ds
\leq C \left( \|\xi\|_1 + \|\xi\|_1 \right) \|\xi\|_1 + \|\Pi_h \theta\|_\infty \|\xi\|_1 + \|\xi\|_1
\leq C \left( \|\xi\|_1 + \|\xi\|_1 \right) \|\xi\|_1 + \|\Pi_h \theta\|_\infty \|\xi\|_1 + \|\xi\|_1.
\]

Substituting (50) and (51) into (49), we get
\[
|B_2 (\theta; \theta, \Pi_h^* \xi) - B_2 (\theta; \theta, \theta, \Pi_h^* \xi)|
\leq C \left( \|\theta\|_1 + \|\theta\|_1 \right) \|\theta\|_1 + \|\Pi_h \theta\|_\infty \|\xi\|_1 + \|\xi\|_1
\leq C \left( \|\theta\|_1 + \|\theta\|_1 \right) \|\theta\|_1 + \|\Pi_h \theta\|_\infty \|\xi\|_1 + \|\xi\|_1
\leq C \left( \|\theta\|_1 + \|\theta\|_1 \right) \|\theta\|_1 + \|\Pi_h \theta\|_\infty \|\xi\|_1 + \|\xi\|_1.
\]

Make (43), (48), and (52) together to obtain
\[
\frac{1}{2} \frac{d}{dt} \|\xi\|_1^2 + \alpha \|\xi\|_1^2
\leq C \left( \|\theta\|_1^2 + \left| \frac{\partial \rho}{\partial t} \right| \|\phi\|_1^2 \right)
\leq C \left( \|\theta\|_1^2 + \left| \frac{\partial \rho}{\partial t} \right| \|\phi\|_1^2 \right)
\leq C \left( \|\theta\|_1^2 + \left| \frac{\partial \rho}{\partial t} \right| \|\phi\|_1^2 \right).
\]

Integrating the previously mentioned equation from 0 to t and noting (44), we obtain that
\[
\frac{1}{2} \left( \|\xi\|_1^2 (t) - \|\xi\|_1^2 (0) \right) + \alpha \int_0^t \|\xi\|_1^2 \, dt
\leq \left( \|\theta\|_1^2 + \left| \frac{\partial \rho}{\partial t} \right| \|\phi\|_1^2 \right)
\leq \left( \|\theta\|_1^2 + \left| \frac{\partial \rho}{\partial t} \right| \|\phi\|_1^2 \right)
\leq \left( \|\theta\|_1^2 + \left| \frac{\partial \rho}{\partial t} \right| \|\phi\|_1^2 \right).
\]

for sufficiently small h and \(\epsilon\). By using Lemma 2(ii) and the Gronwall's inequality, we have that
\[
\|\xi\|_1^2 (t) + \int_0^t \|\xi\|_1^2 \, dt
\leq C \left( \|\theta\|_1^2 + \left| \frac{\partial \rho}{\partial t} \right| \|\phi\|_1^2 \right).
\]

\[
\|\xi\|_1^2 (t) + \int_0^t \|\xi\|_1^2 \, dt
\leq C \left( \|\theta\|_1^2 + \left| \frac{\partial \rho}{\partial t} \right| \|\phi\|_1^2 \right).
\]
It follows from the interpolation estimates that
\[
\|\xi\|^2 (t) + \int_0^T \|\xi\|^2 dt \\
\leq C \left\{ h^2 \left( \|\theta_0\|^2 + \int_0^T \|\frac{\partial \theta}{\partial t}\|^2 dt \right) + h \int_0^T \|\frac{\partial \theta}{\partial t}\|^2 dt \right\}. 
\]
(56)

Now we prove the induction hypothesis (44). Noting that \(\|\xi\|(0) = 0\), we know that (44) holds obviously for \(t = 0\). It follows from (56) that
\[
\left( \log \frac{1}{h} \right)^{1/2} \|\xi\|(t) \leq C h \left( \log \frac{1}{h} \right)^{1/2} \rightarrow 0, \quad h \rightarrow 0. 
\]
(57)

Then (44) holds for any \(t \in [0, T]\).

By (16), we have
\[
\|\rho\| \leq C h^2 \|\theta\|_2 \leq C h^2 \left\{ \|\theta_0\|_2 + \int_0^T \|\theta\|_2 dt \right\}. 
\]
(58)

From the triangle inequality, we obtain
\[
\|\theta - \theta_n\| \leq C h, 
\]
(59)

where \(C\) depends on \(\|\theta_0\|_2\), \(\|\theta\|_{L^\infty([0,T];H)}\), and \(\|\theta\|_{H^1([H])}\). We now complete the proof of the theorem. \(\square\)

5. Error Bound for the Fully Discrete Scheme

Theorem 5. Assume that \(\theta\) satisfies the necessary regularity and the discretization parameters obey the relation \(\tau = O(h)\). Then the error of the approximation (18) of (6) satisfies
\[
\max_{0 \leq t \leq T/\tau} \|\theta^n - \theta_h^n\| \leq C \{ h + \tau \}, 
\]
(60)

where \(C\) depends on \(\|\theta_0\|_2\), \(\|\theta\|_{L^\infty([0,T];H)}\), \(\|\theta\|_{H^1([H])}\), and \(\|\theta\|_{H^2([H])}\).

Proof. Subtract (18) from (9) to obtain that
\[
\left( \frac{\partial \theta^n}{\partial t} - \frac{\partial \theta_h^n}{\partial t}, \Pi_h^* z_h \right) + B_1 (\theta^n; \theta^n, \Pi_h^* z_h) \\
- B_1 (\theta_h^{n-1}; \theta_h^n, \Pi_h^* z_h) + B_2 (\theta^n; \theta^n, \Pi_h^* z_h) \\
- B_{2h} (\theta_h^{n-1}; \theta_h^n, \Pi_h^* z_h) + A (\theta^n - \theta_h^n, \Pi_h^* z_h) = 0. 
\]
(61)

Choose \(z_h = \xi^n\) to obtain that
\[
(\partial \xi^n, \Pi_h^* \xi^n) + A (\xi^n, \Pi_h^* \xi^n) \\
= \left( \frac{\partial \theta^n}{\partial t} - \frac{\partial \theta_h^n}{\partial t}, \Pi_h^* \xi^n \right) - (\partial \rho^n, \Pi_h^* \xi^n) - A (\rho^n, \Pi_h^* \xi^n) \\
- \left[ B_1 (\theta^n; \theta^n, \Pi_h^* \xi^n) - B_1 (\theta_h^{n-1}; \theta_h^n, \Pi_h^* \xi^n) \right] \\
- \left[ B_2 (\theta^n; \theta^n, \Pi_h^* \xi^n) - B_{2h} (\theta_h^{n-1}; \theta_h^n, \Pi_h^* \xi^n) \right]. 
\]
(62)

For the left-hand side of (62), from Lemmas I and 2, we have
\[
(\partial \xi^n, \Pi_h^* \xi^n) = \frac{1}{\tau} (\xi^n - \xi^{n-1}, \Pi_h^* \xi^n) \\
= \frac{1}{2\tau} (\xi^n - \xi^{n-1}, \Pi_h^* (\xi^n + \xi^{n-1})) \\
+ \frac{1}{2\tau} (\xi^n - \xi^{n-1}, \Pi_h^* (\xi^n - \xi^{n-1})) \\
\geq \frac{1}{2\tau} \left( \|\xi^n\|^2 - \|\xi^{n-1}\|^2 \right), \\
A (\xi^n, \Pi_h^* \xi^n) \geq a\|\xi^n\|^2. 
\]

We denote terms on the right-hand side of (62) by \(T_1, \ldots, T_5\). Then, (62) can be rewritten as
\[
\frac{1}{2\tau} \left( \|\xi^n\|^2 - \|\xi^{n-1}\|^2 \right) + a\|\xi^n\|^2 \leq T_1 + \cdots + T_5. 
\]
(64)

Now we estimate the terms \(T_1, \ldots, T_5\) one by one. From the Taylor’s formula, we have
\[
\frac{\partial \theta^n}{\partial t} - \frac{\partial \theta_h^n}{\partial t} = \frac{1}{\tau} \int_{t-\tau}^t \frac{\partial^2 \theta}{\partial t^2} dt. 
\]
(65)

It follows that
\[
\left| T_1 \right| \leq C \left\{ \frac{\partial \theta^n}{\partial t} - \frac{\partial \theta_h^n}{\partial t} \right\} \|\xi^n\| \\
\leq C \left\{ \tau \int_{t-\tau}^t \|\frac{\partial \theta}{\partial t}\|^2 dt + \|\xi^n\|^2 \right\}. 
\]
(66)

For the next two terms, we have
\[
\left| T_2 \right| \leq C \left\| \frac{\partial \rho^n}{\partial t} - \frac{\partial \theta^n}{\partial t} \right\| \|\xi^n\| \\
\leq C \left\{ \tau^{-1} \int_{t-\tau}^t \|\frac{\partial \rho}{\partial t}\|^2 dt + \|\xi^n\|^2 \right\}, 
\]
(67)

\[
\left| T_3 \right| \leq C h \|\theta^n\|_2 \|\xi^n\|, \\
\leq C h^2 \|\theta^n\|_2^2 + \epsilon \|\xi^n\|^2. 
\]

We make the following induction hypothesis:
\[
\left\| \xi^{n-1} \right\| \left( \log \frac{1}{h} \right)^{1/2} \rightarrow 0, \quad h \rightarrow 0, \quad 1 \leq n \leq L. 
\]
(68)
For $T_4$, using the similar argument as (48) and noting (68), we deduce that

$$|T_4| = \left| B_1 \left( \theta^n; \theta^n, \Pi_h \kappa^n \right) - B_2 \left( \theta_h^{-1}, \theta^n; \Pi_h \kappa^n \right) \right|$$

$$\leq \sum_{P \in \mathcal{T}_h} \left| \xi^n (P) \right| \int_{K_i} \left( |\nabla \cdot \theta | \theta^n - (\nabla \cdot \theta_h^{-1}) \theta_h^{-1} | \right) \, dx$$

$$\leq \sum_{P \in \mathcal{T}_h} \left| \xi^n (P) \right| \int_{K_i} \left( \left| \nabla \cdot \theta | \theta^n - \theta_h^{-1} \right| + \left| \nabla \cdot \theta | - \nabla \cdot \theta_h^{-1} \right| \right) \, dx$$

$$\leq C \left\{ \left[ \int_{\Gamma} \left( \rho^n \right)^2 + \left( \xi^n \right)^2 \right]^{1/2} + \left( \| \nabla \cdot \theta \|_{\infty} \right) \left( \| \theta^n \|_{\infty} + \| \theta_h^{-1} \|_{\infty} \right) \right\}$$

$$\leq C \left\{ \left[ \int_{\Gamma} \left( \rho^n \right)^2 + \left( \xi^n \right)^2 \right]^{1/2} + \left( \| \nabla \cdot \theta \|_{\infty} \right) \left( \| \theta^n \|_{\infty} + \| \theta_h^{-1} \|_{\infty} \right) \right\}$$

Now, we write

$$|T_5| = \left| B_2 \left( \theta^n; \theta^n, \Pi_h \kappa^n \right) - B_2^n \left( \theta_h^{-1}; \theta^n, \Pi_h \kappa^n \right) \right|$$

$$\leq \left| B_2 \left( \theta^n; \theta^n, \Pi_h \kappa^n \right) - B_2^n \left( \theta_h^{-1}; \theta^n, \Pi_h \kappa^n \right) \right|$$

$$\leq \left| B_2 \left( \theta^n; \theta^n, \Pi_h \kappa^n \right) \right|$$

$$\leq E_1 + E_2.$$
Table 1: Numerical results for $\zeta = 1$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$1/8$</th>
<th>$1/16$</th>
<th>$1/32$</th>
<th>$1/64$</th>
</tr>
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<td>$|u-u_h|_h$</td>
<td>$1.18416\times 10^{-7}$</td>
<td>$5.33942\times 10^{-8}$</td>
<td>$2.52582\times 10^{-8}$</td>
<td>$1.22755\times 10^{-8}$</td>
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<td>Rate</td>
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<td>1.08</td>
<td>1.05</td>
<td></td>
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<tr>
<td>$|v-v_h|_h$</td>
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<td>$2.36565\times 10^{-8}$</td>
<td>$9.49449\times 10^{-9}$</td>
<td>$4.21298\times 10^{-9}$</td>
</tr>
<tr>
<td>Rate</td>
<td>1.51</td>
<td>1.32</td>
<td>1.17</td>
<td></td>
</tr>
</tbody>
</table>

Figure 1: The exact solution $u$ when $\zeta = 1$ at $t = 1.0$.

Figure 2: The numerical solution $u_h$ when $\zeta = 1$ at $t = 1.0$, for $h = 1/32$.

Figure 3: The exact solution $u$ when $\zeta = 0.01$ at $t = 1.0$.

By choosing $h$ and $\epsilon$ small enough and noting Lemma 2(ii), we have

$$
\|\xi^l\|^2 + \tau \sum_{n=1}^L \|\xi^n\|^2_1
\leq C \left( h^2 \|\theta^0\|^2_2 + \tau \sum_{n=0}^{L} \|\rho^n\|^2 \right)
+ C \tau \sum_{n=1}^L \|\xi^n\|^2 + C \tau \int_0^{t_1} \left( \|\theta_t^2\|^2 + \|\theta_t\|^2_{L_1} \right) dt
+ C \int_0^{t_1} \|\rho_t\|^2 dt.
$$

(74)

Applying the Gronwall inequality and the interpolation theory, we deduce that

$$
\|\xi^l\|^2 \leq Cr^2 \int_0^T \left( \|\theta_t^2\|^2 + \|\theta_t\|^2_{L_1} \right) dt
+ C \tau^2 \left( \|\theta^0\|^2 + \int_0^T \|\theta_t\|^2_{L_2} dt \right).
$$

(75)

Thus we know that (68) holds for any $1 \leq L \leq N$. Using triangular inequality and the interpolation theory completes the proof.
6. Numerical Example

In this section, we will show the affectivity of our method by numerical experiments. The exact solutions to problem (I) can be obtained by employing Cole-Hopf transformation. For \( \Omega = \{(x_1, x_2) : 0 \leq x_1, x_2 \leq 1\} \), we consider the following solutions:

\[
\begin{align*}
    u &= \frac{3}{4} - \frac{1}{4 \left(1 + \exp \left(\eta \left(-4x_1 + 4x_2 - t\right)/32\right)\right)}, \\
    v &= \frac{3}{4} + \frac{1}{4 \left(1 + \exp \left(\eta \left(-4x_1 + 4x_2 - t\right)/32\right)\right)},
\end{align*}
\]

(77)

where \( \eta = 1/\zeta \). We present numerical results for the \( L^2 \)-norm estimates of \( u - u_h \) and \( v - v_h \). In Tables 1 and 2, we present the numerical results for \( \zeta = 1 \) and \( \zeta = 0.01 \), respectively. In all runs, we use the uniform mesh step \( h = \Delta t \) and choose the time \( t = 1 \). As seen in these tables, in all cases the errors decrease by a factor of about two as \( h \) decreases by the factor of two. This indicates that all \( L^2 \)-norm error estimates are of first-order convergence, which is consistent with our theoretical analysis.

When \( \zeta = 1.0 \) and \( \zeta = 0.01 \), the figures of the exact solutions \( u \) and the numerical solutions \( u_h \) at \( t = 1 \) for \( h = 1/32 \) are given in Figures 1, 2, 3, and 4. In order to show that our
method keeps stable when $\zeta$ is smaller, we also give the comparison figures of exact solution $u$ and numerical solution $u_h$ for $\zeta = 0.001$ in Figures 5 and 6. The comparison figure of numerical solution by using finite volume element method (FVEM) without upwinding is given in Figure 7, which show that the approximation produces unacceptable nonphysical oscillations.

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**References**


