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Research Article

Inverse Problems for the Quadratic Pencil of the Sturm-Liouville Equations with Impulse

Rauf Kh. Amırov[1] and A. Adiloglu Nabıev[2]

[1] *Department of Mathematics, Faculty of Sciences, Cumhuriyet University, 58140 Sivas, Turkey*
[2] *Department of Secondary Science and Mathematics Education, Faculty of Education, Cumhuriyet University, 58140 Sivas, Turkey*

Correspondence should be addressed to Rauf Kh. Amırov; emirov@cumhuriyet.edu.tr

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In this study some inverse problems for a boundary value problem generated with a quadratic pencil of Sturm-Liouville equations with impulse on a finite interval are considered. Some useful integral representations for the linearly independent solutions of a quadratic pencil of Sturm-Liouville equation have been derived and using these, important spectral properties of the boundary value problem are investigated; the asymptotic formulas for eigenvalues, eigenfunctions, and normalizing numbers are obtained. The uniqueness theorems for the inverse problems of reconstruction of the boundary value problem from the Weyl function, from the spectral data, and from two spectra are proved.

1. Introduction

The theory of inverse problems for differential operators occupies an important position in the current developments of the spectral theory of linear operators. Inverse problems of spectral analysis consist in the recovery of operators from their spectral data. One takes for the main spectral data, for instance, one, two, or more spectra, the spectral function, the spectrum, and the normalizing constants, the Weyl function. Different statements of inverse problems are possible depending on the selected spectral data. The already existing literature on the theory of inverse problems of spectral analysis is abundant. The most comprehensive account of the current state of this theory and its applications can be found in the monographs of Marchenko [1], Levitan [2], Beals et al. [3], and Yurko [4].

In the present work we consider some inverse problems for the boundary value problem generated by the differential equation

$$L_\lambda y := y'' + \left[\lambda^2 - 2\lambda p(x) - q(x)\right] y = 0, \qquad x \in [0,a) \cup (a,\pi] \tag{1}$$

with the boundary conditions

$$U(y) := y'(0) = 0, \qquad V(y) := y(\pi) = 0 \tag{2}$$

and with the jump conditions

$$y(a+0) = \alpha y(a-0), \qquad y'(a+0) = \alpha^{-1} y'(a-0), \tag{3}$$

where λ is the spectral parameter, $p(x) \in W_2^1[0,\pi]$, $q(x) \in L_2[0,\pi]$ are real functions, α is a real number, and $\alpha > 0$, $\alpha \neq 1$, $a \in (\pi/2, \pi)$. Here we denote by $W_2^m[0,\pi]$ the space of functions $f(x)$, $x \in [0,\pi]$, such that the derivatives $f^{(m)}(x)(m = \overline{0, n-1})$ are absolutely continuous and $f^{(n)}(x) \in L_2[0,\pi]$.

There exist many papers containing a fairly comprehensive analysis of direct and inverse problems of spectral analysis of the Sturm-Liouville equation

$$ly := -y'' + q(x) y = \lambda^2 y, \tag{4}$$

a special case ($p(x) \equiv 0$) of (1). For instance, inverse problems for a regular Sturm-Liouville operator with separated boundary conditions have been investigated in [5] (see also [1–4]).

Some versions of inverse problems for (1) which is a natural generalization of the Sturm-Liouville equation were

fully studied in [6–14]. Namely, the inverse problems for a pencil L_λ on the half axis and the entire axis were considered in [6–8], where the scattering data, the spectral function, and the Weyl function, respectively, were taken for the spectral data. The problem of the recovery of (1) from the spectra of two boundary value problems with certain separated boundary conditions was solved in [9]. The analysis of inverse spectral problems for (1) with other kinds of separated boundary conditions as well as with periodic and antiperiodic boundary conditions was the subject of [10] (see also [11]) where the corresponding results of the monograph [1] were extended to the case $p(x) \neq 0$. The inverse periodic problem for the pencil L_λ was solved in [12] using another approach. We also point out the paper [14], in which the uniqueness of the recovery of the pencil L_λ from three spectra was investigated.

Boundary value problems with discontinuities inside the interval often appear in mathematics, physics, and other fields of natural sciences. The inverse problems of reconstructing the material properties of a medium from data collected outside of the medium give solutions to many important problems in engineering and geosciences. For example, in electronics, the problem of constructing parameters of heterogeneous electronic lines is reduced to a discontinuous inverse problem [15, 16]. The reduced mathematical model exhibits the boundary value problem for the equation of type (1) with given spectral information which is described by the desirable amplitude and phase characteristics. Note that the problem of reconstructing the permittivity and conductivity profiles of a one-dimensional discontinuous medium is also closed to the spectral information [17, 18]. Geophysical models for oscillations of the Earth are also reduced to boundary value problems with discontinuity in an interior point [19].

Direct and inverse spectral problems for differential operators without discontinuities have been extensively studied by many authors [20–25]. Some classes of direct and inverse problems for discontinuous boundary value problems in various statements have been considered in [18, 26–32] and other works. Boundary value problems with singularity have been studied in [33–37], and for further discussion see the references therein. Note that the inverse spectral problem for the boundary problem (1)–(3) has never been considered before.

In what follows we denote the boundary value problem (1)–(3) by $L(a,\alpha)$. In Section 2 we derive some integral representations for the linearly independent solutions of (1), and using these, we investigate important spectral properties of the boundary value problem $L(a,\alpha)$. In Section 3 the asymptotic formulas for eigenvalues, eigenfunctions, and normalizing numbers of $L(a,\alpha)$ are obtained. Finally, in Section 4 three inverse problems of reconstructing the boundary value problem $L(\alpha,a)$ from the Weyl function, from the spectral data, and from two spectra are considered and the uniqueness theorems are proved.

2. Integral Representations of Solutions and the Spectral Characteristics

Let $f_\nu(x,\lambda)$ $(\nu = 1,2)$ be solution of (1) under the initial condition

$$f_\nu(0,\lambda) = 1, \qquad f'_\nu(0,\lambda) = \lambda w_\nu \tag{5}$$

and discontinuity conditions (3), where $w_\nu = (-1)^{\nu+1} i$.

It is obvious that the functions $f_\nu(x,\lambda)$ satisfy the integral equations

$$\begin{aligned} f_\nu(x,\lambda) &= l^+(x) e^{\lambda w_\nu x} + l^-(x) e^{\lambda w_\nu (2a-x)} \\ &\quad + l^+(x) \int_0^a \frac{\sin\lambda(x-t)}{\lambda} \{q(t)+2\lambda p(t)\} f_\nu(t,\lambda)\,dt \\ &\quad - l^-(x) \int_0^a \frac{\sin\lambda(x+t-2a)}{\lambda} \{q(t)+2\lambda p(t)\} f_\nu(t,\lambda)\,dt \\ &\quad + \int_a^x \frac{\sin\lambda(x-t)}{\lambda} \{q(t)+2\lambda p(t)\} f_\nu(t,\lambda)\,dt, \end{aligned} \tag{6}$$

where $l^\pm(x) = (1/2)(l(x) \pm (1/l(x))$ and

$$l(x) = \begin{cases} 1, & 0 \le x < a \\ \alpha, & a < x \le \pi. \end{cases} \tag{7}$$

Using the integral equations (6) and standard successive approximation methods [7, 9, 11], the following theorem is proved.

Theorem 1. *If $q(x) \in L_2[-b,b]$, $p(x) \in W_2^1[-b,b]$ $(0 < b < \pi)$, then the solution $f_\nu(x,\lambda)$ has the form*

$$f_\nu(x,\lambda) = f_{0\nu}(x,\lambda) + \int_{-x}^{x} A_\nu(x,t) e^{\lambda w_\nu t}\,dt, \tag{8}$$

where

$$\begin{gathered} f_{0\nu}(x,\lambda) = l^+(x) e^{\lambda w_\nu x} R_\nu^+(x) + l^-(x) e^{\lambda w_\nu (2a-x)} R_\nu^-(x), \\ R_\nu^\pm(x) = e^{\mp w_\nu \beta^\pm(x)}, \quad \beta^\pm(x) = \int_{(a\mp a)/2}^{x} p(t)\,dt, \end{gathered} \tag{9}$$

and the function $A_\nu(x,t)$ satisfies the inequality

$$\int_{-x}^{x} |A_\nu(x,t)|\,dt \le e^{c\sigma(x)} - 1 \tag{10}$$

with

$$\begin{gathered} \sigma(x) = \int_0^x [(x-t)|q(t)| + 2|p(t)|]\,dt, \\ c = 2(\alpha^+ + |\alpha^-| + 1), \qquad \alpha^\pm = \frac{1}{2}\left(\alpha \pm \frac{1}{\alpha}\right). \end{gathered} \tag{11}$$

Let $\varphi(x,\lambda)$ be the solution of (1) that satisfies the initial conditions

$$\varphi(0,\lambda)=1, \qquad \varphi'(0,\lambda)=0, \tag{12}$$

and the jump condition (3).

Then by using Theorem 1, we can formulate the following assertion.

Theorem 2. *Let $q(x)\in L_2[0,\pi]$, $p(x)\in W_2^1[0,\pi]$. Then there are the functions $A(x,t)$, $B(x,t)$ whose first order partial derivatives are summable on $[0,\pi]$ for each $x\in[0,\pi]$ such that the representation*

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+\int_0^x A(x,t)\cos\lambda t\,dt+\int_0^x B(x,t)\sin\lambda t\,dt \tag{13}$$

is satisfied, where

$$\begin{aligned}\varphi_0(x,\lambda)&=l^+(x)\cos\left[\lambda x-\beta^+(x)\right]\\&\quad+l^-(x)\cos\left[\lambda(2a-x)-\beta^-(x)\right].\end{aligned} \tag{14}$$

Moreover, the relations

$$\begin{aligned}&\alpha^+\beta^+(x)=\alpha^+xp(0)\\&\qquad+2\int_0^x\left[A(\xi,\xi)\sin\beta^+(\xi)-B(\xi,\xi)\cos\beta^+(\xi)\right]d\xi,\\&2\frac{d}{dx}\left[A(x,x)\cos\beta^+(x)+B(x,x)\sin\beta^+(x)\right]\\&\qquad=\alpha^+\left[q(x)+p^2(x)\right],\\&2\frac{d}{dx}\left[A(x,t)\cos\beta^-(x)-B(x,t)\sin\beta^-(x)\right]\Big|_{t=2a-x-0}^{t=2a-x+0}\\&\qquad=\alpha^-\left[q(x)+p^2(x)\right],\\&A_t(x,t)\big|_{t=0}=B(x,0)=0\end{aligned} \tag{15}$$

are held.

If one additionally supposes that $q(x)\in W_2^1[0,\pi]$, $p(x)\in W_2^2[0,\pi]$, then the functions $A(x,t)$ and $B(x,t)$ satisfy the system of partial differential equations

$$\begin{aligned}&A_{xx}(x,t)-q(x)A(x,t)-2p(x)B_t(x,t)=A_{tt}(x,t),\\&B_{xx}(x,t)-q(x)B(x,t)+2p(x)A_t(x,t)=B_{tt}(x,t).\end{aligned} \tag{16}$$

Conversely, if the second order derivatives of functions $A(x,t)$, $B(x,t)$ are summable on $[0,\pi]$ for each $x\in[0,\pi]$ and $A(x,t)$, $B(x,t)$ satisfy equalities (16) and conditions (15), then the function $\varphi(x,\lambda)$ which is defined by (13) is a solution of (1) satisfying initial conditions (12) and discontinuity conditions (3).

One here supposes that the function $q(x)$ satisfies the additional condition

$$\int_0^\pi\left\{\left|y'(x)\right|^2+q(x)\left|y(x)\right|^2\right\}dx>0 \tag{17}$$

for all $y(x)\in W_2^2[0,a)\cup(a,\pi]$ such that $y(x)\neq 0$ and

$$y'(0)\overline{y(0)}-y'(\pi)\overline{y(\pi)}=0. \tag{18}$$

Definition 3. A complex number λ_0 is called an eigenvalue of the boundary value problem $L(\alpha,a)$ if (1) with $\lambda=\lambda_0$ has a nontrivial solution $y_0(x)$ satisfying the boundary conditions (2) and the jump conditions (3). In this case $y_0(x)$ is called the eigenfunction of the problem $L(\alpha,a)$ corresponding to the eigenvalue λ_0. The number of linearly independent solutions of the problem $L(\alpha,a)$ for a given eigenvalue λ_0 is called the multiplicity of λ_0.

The following lemmas can be proved analogously to the corresponding assertions in [11].

Lemma 4. *The eigenvalues of the boundary value problem $L(\alpha,a)$ are real, nonzero, and simple.*

Proof. We define a linear operator L_0 in the Hilbert space $L_2[0,\pi]$ as follows. The domain $D(L_0)$ consists of all functions $y(x)\in W_2^2[0,\pi]$ satisfying the boundary conditions (2) and the jump conditions (3). For $y\in D(L_0)$, we set $L_0y=-y''+q(x)y$. Integration by part yields

$$\begin{aligned}(L_0y,y)&=\int_0^\pi L_0y\overline{y}(x)\,dx\\&=\int_0^\pi\left(\left|y'(x)\right|^2+q(x)\left|y(x)\right|^2\right)dx.\end{aligned} \tag{19}$$

Since condition (17) holds, it follows that $(L_0y,y)>0$.

Let λ_0 be an eigenvalue of the boundary value problem $L(\alpha,a)$ and $y_0(x)$ an eigenfunction corresponding to this eigenvalue and normalized by the condition $(y_0,y_0)=1$. By taking the inner product of both sides of the relation $y_0''(x)+[\lambda_0^2-2\lambda_0p(x)-q(x)]y_0(x)=0$ by $y_0(x)$, we obtain $\lambda_0^2-2\lambda_0(py_0,y_0)-(L_0y_0,y_0)=0$ and hence

$$\lambda_0=(py_0,y_0)\pm\sqrt{(py_0,y_0)^2+(L_0y_0,y_0)}. \tag{19$'$}$$

The desired assertion follows from the last relation by virtue of $(L_0y_0,y_0)>0$ with regard to the fact that $p(x)$ is real.

Let us show that λ_0 is a simple eigenvalue. Assume that this is not true. Suppose that $y_1(x)$ and $y_2(x)$ are linearly independent eigenfunctions corresponding to the eigenvalue λ_0. Then for a given value of λ_0, each solution $y_0(x)$ of (1) will be given as linear combination of solutions $y_1(x)$ and $y_2(x)$. Moreover it will satisfy boundary conditions (2) and discontinuity conditions (3). However, it is impossible. □

Lemma 5. *The problem* (1)–(3) *does not have associated functions.*

Proof. Let $y_0(x)$ be an eigenfunction corresponding to eigenvalue λ_0 and normalized by the condition $(y_0,y_0)=1$ of the problem (1)–(3). Suppose that $y_1(x)$ is an associated function of eigenfunction $y_0(x)$, that is, the following equalities hold:

$$\begin{aligned}&\lambda_0^2y_0-2\lambda_0p(x)y_0-L_0y_0=0,\\&\lambda_0^2y_1-2\lambda_0p(x)y_1-L_0y_1+2\left(\lambda_0-p(x)\right)y_0=0.\end{aligned} \tag{20}$$

If these equations are multiplied by $y_1(x)$ and $y_0(x)$, respectively, as inner product, subtracting them side by side and taking into our account that operator L_0 is symmetric, the function $p(x)$ and λ_0 are real, we get $\lambda_0 = (py_0, y_0)$. Due to the condition (6), $\lambda_0 = (py_0, y_0)$ does not agree with $(19')$. Therefore, the assertion is not true. □

Lemma 6. *Eigenfunctions corresponding to different eigenvalues of the problem $L(\alpha, a)$ are orthogonal in the sense of the equality*

$$(\lambda_1 + \lambda_2)(y_1, y_2) - 2(py_1, y_2) = 0, \tag{21}$$

where $(\cdot,\cdot)$ denotes the inner product in $L_2[0,\pi]$.

Lemma 7. *Let $y(x,\lambda)$ be a solution of* (1) *satisfying the condition* (18) *and the jump conditions* (3). *Then λ is real and nonzero and*

$$\int_0^\pi (\lambda - p(x))\left|y(x,\lambda)\right|^2 dx \neq 0. \tag{22}$$

Moreover, the sign of the left-hand side of (22) *is similar to the sign of λ.*

3. Properties of the Spectrum

In this section we investigate some spectral properties of the boundary value problem $L(\alpha, a)$.

Let $\psi(x,\lambda)$ be a solution of (1) with the conditions $\psi(\pi,\lambda) = 0$, $\psi'(\pi,\lambda) = 1$ and the jump conditions (3). It is clear that function $\psi(x,\lambda)$ is entire in λ for each fixed x.

Denote $\Delta(\lambda) = \langle\psi(x,\lambda), \varphi(x,\lambda)\rangle$, where $\langle y, z\rangle := y'z - yz'$. By virtue of Liouville's formula, the Wronskian $\langle\psi(x,\lambda), \varphi(x,\lambda)\rangle$ does not depend on x. The function $\Delta(\lambda)$ is called the characteristic function of $L(\alpha, a)$. Obviously, the function $\Delta(\lambda)$ is entire in λ and it has at most a countable set of zeros $\{\lambda_n\}$.

Lemma 8. *The zeros $\{\lambda_n\}$ of the characteristic function $\Delta(\lambda)$ coincide with the eigenvalues of the boundary value problem $L(\alpha, a)$. The functions $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are eigenfunctions corresponding to the eigenvalue λ_n, and there exists a sequence $\{\beta_n\}$ such that*

$$\psi(x,\lambda_n) = \beta_n\varphi(x,\lambda_n), \quad \beta_n \neq 0. \tag{23}$$

Proof. Let $\Delta(\lambda_0) = 0$. Then by virtue of $\langle\psi(x,\lambda_0), \varphi(x,\lambda_0)\rangle = 0$, $\varphi(x,\lambda_0) = C\psi(x,\lambda_0)$ for some constant C. Hence λ_0 is an eigenvalue and $\varphi(x,\lambda_0)$, $\psi(x,\lambda_0)$ are eigenfunctions related to λ_0.

Conversely, let λ_0 be an eigenvalue of $L(\alpha, a)$, show that $\Delta(\lambda_0) = 0$. Assuming the converse suppose that $\Delta(\lambda_0) \neq 0$. In this case the functions $\varphi(x,\lambda_0)$ and $\psi(x,\lambda_0)$ are linearly independent. Then $y(x,\lambda_0) = c_1\varphi(x,\lambda_0) + c_2\psi(x,\lambda_0)$ is a general solution of the problem $L(\alpha, a)$. If $c_1 \neq 0$, we can write

$$\varphi(x,\lambda_0) = \frac{1}{c_1}y(x,\lambda_0) - \frac{c_2}{c_1}\psi(x,\lambda_0). \tag{24}$$

Then we have

$$\begin{aligned}
&\langle\varphi(x,\lambda_0), \psi(x,\lambda_0)\rangle \\
&\quad = \frac{1}{c_1}\left[y'(\pi,\lambda_0)\psi(\pi,\lambda_0) - y(\pi,\lambda_0)\psi'(\pi,\lambda_0)\right] \\
&\quad = -\frac{1}{c_1}y(\pi,\lambda_0) \\
&\quad = 0
\end{aligned} \tag{25}$$

which is a contradiction. □

Note that we have also proved that for each eigenvalue there exists only one eigenfunction (up to a multiplicative constant). Therefore there exists sequence β_n such that $\psi(x,\lambda_n) = \beta_n\varphi(x,\lambda_n)$.

Let us denote

$$\alpha_n := \int_0^\pi \varphi^2(x,\lambda_n)\,dx - \frac{1}{\lambda_n}\int_0^\pi p(x)\varphi^2(x,\lambda_n)\,dx. \tag{26}$$

The numbers $\{\alpha_n\}$ are called normalized numbers of the boundary value problem $L(\alpha, a)$.

Lemma 9. *The equality $\dot\Delta(\lambda_n) = -2\lambda_n\alpha_n\beta_n$ holds. Here $\dot\Delta(\lambda) = (d/d\lambda)\Delta(\lambda)$.*

Proof. If we differentiate the equalities

$$\begin{aligned}
&-\varphi''(x,\lambda) + [2\lambda p(x) + q(x)]\varphi(x,\lambda) = \lambda^2\varphi(x,\lambda), \\
&-\psi''(x,\lambda) + [2\lambda p(x) + q(x)]\psi(x,\lambda) = \lambda^2\psi(x,\lambda)
\end{aligned} \tag{27}$$

with respect to λ, we get

$$\begin{aligned}
&-\dot\varphi''(x,\lambda) + [2\lambda p(x) + q(x)]\dot\varphi(x,\lambda) \\
&\quad = \lambda^2\dot\varphi(x,\lambda) + 2[\lambda - p(x)]\varphi(x,\lambda), \\
&-\dot\psi''(x,\lambda) + [2\lambda p(x) + q(x)]\dot\psi(x,\lambda) \\
&\quad = \lambda^2\dot\psi(x,\lambda) + 2[\lambda - p(x)]\psi(x,\lambda).
\end{aligned} \tag{28}$$

By virtue of these equalities we have

$$\begin{aligned}
&\frac{d}{dx}\left\{\varphi(x,\lambda)\dot\psi'(x,\lambda) - \varphi'(x,\lambda)\dot\psi(x,\lambda)\right\} \\
&\quad = 2[\lambda - p(x)]\varphi(x,\lambda)\psi(x,\lambda), \\
&\frac{d}{dx}\left\{\dot\varphi(x,\lambda)\psi'(x,\lambda) - \dot\varphi'(x,\lambda)\psi(x,\lambda)\right\} \\
&\quad = 2[\lambda - p(x)]\varphi(x,\lambda)\psi(x,\lambda).
\end{aligned} \tag{29}$$

If the last equations are integrated from x to π and from 0 to x, respectively, by the discontinuity conditions we obtain

$$\begin{aligned}&\left(\varphi(\xi,\lambda)\dot{\psi}'(\xi,\lambda)-\varphi'(\xi,\lambda)\dot{\psi}(\xi,\lambda)\right)\Big|_x^\pi\\&\quad=-2\int_x^\pi(\lambda-p(\xi))\varphi(\xi,\lambda)\psi(\xi,\lambda)\,d\xi,\\&\left(\dot{\varphi}(\xi,\lambda)\psi'(\xi,\lambda)-\dot{\varphi}'(\xi,\lambda)\psi(\xi,\lambda)\right)\Big|_0^x\\&\quad=-2\int_0^x(\lambda-p(\xi))\varphi(\xi,\lambda)\psi(\xi,\lambda)\,d\xi.\end{aligned}\tag{30}$$

If we add the last equalities side by side, we get

$$\begin{aligned}&W[\dot{\varphi}(\xi,\lambda),\psi(\xi,\lambda)]+W[\varphi(\xi,\lambda),\dot{\psi}(\xi,\lambda)]\\&\quad=-2\int_0^\pi(\lambda-p(\xi))\varphi(\xi,\lambda)\psi(\xi,\lambda)\,d\xi\end{aligned}\tag{31}$$

or

$$\dot{\Delta}(\lambda)=-2\int_0^\pi(\lambda-p(\xi))\varphi(\xi,\lambda)\psi(\xi,\lambda)\,d\xi.\tag{32}$$

For $\lambda\to\lambda_n$, this yields

$$\begin{aligned}\dot{\Delta}(\lambda_n)&=-2\int_0^\pi(\lambda_n-p(\xi))\varphi(\xi,\lambda_n)\psi(\xi,\lambda_n)\,d\xi\\&=-2\beta_n\int_0^\pi(\lambda_n-p(\xi))\varphi^2(\xi,\lambda_n)\,d\xi\\&=-2\lambda_n\beta_n\left[\int_0^\pi\varphi^2(\xi,\lambda_n)\,d\xi\right.\\&\qquad\left.-\frac{1}{\lambda_n}\int_0^\pi p(\xi)\varphi^2(\xi,\lambda_n)\,d\xi\right]\\&=-2\lambda_n\beta_n\alpha_n.\end{aligned}\tag{33}$$

The lemma is proved. □

Let $\Delta_0(\lambda)=\alpha^+\cos[\lambda\pi-\beta^+(\pi)]+\alpha^-\cos[\lambda(2a-\pi)+\beta^-(\pi)]$ and $\{\lambda_n^0\}$ are zeros of $\Delta_0(\lambda)$.

Lemma 10. *The roots of the characteristic equation $\Delta_0(\lambda)=0$ are separate, that is*

$$\inf_{n\neq m}\left|\lambda_n^0-\lambda_m^0\right|=\beta>0.\tag{34}$$

Proof. Let $\lambda\pi-\beta^+(\pi)=x$. Then, $\lambda(2a-\pi)+\beta^-(\pi)=kx+b$, where $k=(2a-\pi)/\pi$, $b=\beta^+(\pi)((2a-\pi)/\pi)+\beta^-(\pi)$. Since $a\in(\pi/2,\pi)$, then $k\in(0,1)$. Using these notations we can rewrite the equation $\Delta_0(\lambda)=0$ in the following form:

$$A\cos x=\cos(kx+b).\tag{35}$$

Here $A=-(\alpha^+/\alpha^-)$ which implies that $|A|>1$. Preliminarily show that there are no multiple roots of (35). Assuming the converse we suppose x_0 to be a multiple root of (35). Then

$$A\sin x_0=k\sin(kx_0+b)\tag{36}$$

holds. Now (35) and (36) imply that $A^2=1-(1-k^2)\sin^2(kx_0+b)\leq 1$ which is a contradiction. Therefore, (35) has no multiple roots.

Further assuming (34) not to be true let $\{x_p'\}$ and $\{x_p''\}$ be increasing sequences of roots of (35) such that $x_p'\neq x_p''$ and

$$\lim_{p\to\infty}\left|x_p'-x_p''\right|=0.\tag{37}$$

If we assume that $x_p'=2n_p\pi+r_p'$, where $n\in\mathbb{N}$ and $\{r_p'\}$ is a bounded sequence ($0<r_p'<2\pi$), then from (37) we find that $x_p''=2n_p\pi+r_p''$, where $\{r_p''\}$ is a bounded sequence such that $\lim_{p\to\infty}|r_p'-r_p''|=0$. It is obvious that $kx_p'=2\pi[kn_p]+s_p'$, $kx_p''=2\pi[kn_p]+s_p''$, where $s_p'=2\pi\{kn_p\}+r_p'k$, $s_p''=2\pi\{kn_p\}+r_p''k$ and $\lim_{p\to\infty}|s_p'-s_p''|=0$. Here $[\cdot]$ and $\{\cdot\}$ denote the integer and fractional parts of a real number, respectively. Since sequences $\{r_p'\}_{p\geq1}$, $\{r_p''\}_{p\geq1}$, $\{s_p'\}_{p\geq1}$ and $\{s_p''\}_{p\geq1}$ will be bounded, without loss of generality we can assume that these sequences are convergent. Then let

$$\lim_{p\to\infty}r_p'=\lim_{p\to\infty}r_p''=x_0,\tag{38}$$

$$\lim_{p\to\infty}s_p'=\lim_{p\to\infty}s_p''=y_0.\tag{39}$$

Therefore, we can write the equality $A\cos x_p'=\cos(kx_p'+b)$ as

$$A\cos r_p'=\cos\left(s_p'+b\right).\tag{40}$$

Then by virtue of (38) and (39), from (40) we get

$$A\cos x_0=\cos\left(y_0+b\right).\tag{41}$$

Similarly we can obtain

$$A\cos r_p''=\cos\left(s_p''+b\right).\tag{42}$$

Further, from (40) and (42), we have

$$A\sin\frac{r_p'+r_p''}{2}\sin\frac{r_p'-r_p''}{2}=\sin\left(\frac{s_p'+s_p''}{2}+b\right)\sin\frac{s_p'-s_p''}{2}.\tag{43}$$

Let us write this equality as

$$\begin{aligned}&A\sin\frac{r_p'+r_p''}{2}\sin\frac{r_p'-r_p''}{2}\\&\quad=\sin\left(\frac{s_p'+s_p''}{2}+b\right)\sin\frac{k\left(r_p'-r_p''\right)}{2}.\end{aligned}\tag{44}$$

Now dividing both sides of equality (44) by $(r_p'-r_p'')/2\neq0$ and taking limit as $p\to\infty$, by virtue of (3) and (39), we obtain

$$A\sin x_0=k\sin\left(y_0+b\right).\tag{45}$$

Finally, from (41) and (45), we conclude that $A^2=1-(1-k^2)\sin^2(y_0+b)\leq1$ which is a contradiction. Hence roots of equation $\Delta_0(\lambda)=0$ are separate. The lemma is proved. □

Denote

$$\Gamma_n = \left\{\lambda : |\lambda| = \left|\lambda_n^0\right| + \frac{\beta}{2}\right\}, \quad n = 0, 1, \ldots,$$
$$G_\delta := \left\{\lambda : \left|\lambda - \lambda_n^0\right| \geq \delta\right\} \quad (\delta > 0), \tag{46}$$

where δ is sufficiently small positive number $(\delta \ll \beta/2)$.

Lemma 11. *For sufficiently large values of n, one has*

$$\left|\Delta(\lambda) - \Delta_o(\lambda)\right| < \frac{C_\delta}{2} e^{|\operatorname{Im}\lambda|\pi}, \quad \lambda \in \Gamma_n. \tag{47}$$

Proof. As it is shown in [38], $|\Delta_o(\lambda)| \geq C_\delta e^{|\operatorname{Im}\lambda|\pi}$ for all $\lambda \in \overline{G_\delta}$, where $C_\delta > 0$ is some constant. On the other hand, since

$$\begin{aligned}
&\lim_{|\lambda|\to\infty} e^{-|\operatorname{Im}\lambda|\pi}\left(\Delta(\lambda) - \Delta_o(\lambda)\right) \\
&\quad = \lim_{|\lambda|\to\infty} e^{-|\operatorname{Im}\lambda|\pi}\left(\int_0^\pi \widetilde{A}(\pi, t)\cos\lambda t\, dt\right. \\
&\qquad\qquad \left. + \int_0^\pi \widetilde{B}(\pi, t)\sin\lambda t\, dt\right) \\
&\quad = 0
\end{aligned} \tag{48}$$

for sufficiently large values of n (see [1]) we get (47). The lemma is proved. □

Lemma 12. *The problem $L(\alpha, a)$ has countable set of eigenvalues. If one denotes by $\lambda_1, \lambda_2, \ldots$ the positive eigenvalues arranged in increasing order and by $\lambda_{-1}, \lambda_{-2}, \ldots$ the negative eigenvalues arranged in decreasing order, then eigenvalues of the problem $L(\alpha, a)$ have the asymptotic behavior*

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{\delta_n}{\lambda_n^0}, \quad n \longrightarrow \pm\infty, \tag{49}$$

where $\delta_n \in l_2$ and d_n is a bounded sequence, $\lambda_n^0 = n + (1/\pi)\beta^+(\pi) + h_n$, $\sup_n|h_n| < \infty$.

Proof. According to Lemma 11, if n is a sufficiently large natural number and $\lambda \in \Gamma_n$, we have $|\Delta_o(\lambda)| \geq C_\delta e^{|\operatorname{Im}\lambda|\pi} > (C_\delta/2)e^{|\operatorname{Im}\lambda|\pi} > |\Delta(\lambda) - \Delta_o(\lambda)|$. Applying Rouche's theorem we conclude that for sufficiently large n inside the contour Γ_n the functions $\Delta_o(\lambda)$ and $\Delta_o(\lambda) + \{\Delta(\lambda) - \Delta_o(\lambda)\} = \Delta(\lambda)$ have the same number of zeros counting their multiplicities. That is, there are exactly $(n+1)$ zeros $\lambda_0, \lambda_1, \ldots, \lambda_n$. in Γ_n. Analogously, it is shown by Rouche's theorem that, for sufficiently large values of n, the function $\Delta(\lambda)$ has a unique zero inside each circle $|\lambda - \lambda_n^0| < \delta$. Since $\delta > 0$ is arbitrary, it follows that $\lambda_n = \lambda_n^0 + \varepsilon_n$, where $\lim_{n\to\infty}\varepsilon_n = 0$. Further according to $\Delta(\lambda_n) = 0$, we have

$$\begin{aligned}
&\Delta_0\left(\lambda_n^0 + \varepsilon_n\right) + \int_0^\pi A(\pi, t)\cos\left(\lambda_n^0 + \varepsilon_n\right) t\, dt \\
&\quad + \int_0^\pi B(\pi, t)\sin\left(\lambda_n^0 + \varepsilon_n\right) t\, dt = 0.
\end{aligned} \tag{50}$$

On the other hand, since

$$\begin{aligned}
&\Delta_0(\lambda) \\
&\quad = \alpha^+\cos\left(\lambda\pi - \beta^+(\pi)\right) + \alpha^-\cos\left(\lambda(2a - \pi) + \beta^-(\pi)\right), \\
&\quad \Delta_0\left(\lambda_n^0 + \varepsilon_n\right) = \dot{\Delta}_o\left(\lambda_n^0\right)\varepsilon_n + o(\varepsilon_n), \quad n \longrightarrow \infty,
\end{aligned} \tag{51}$$

(50) takes the form of

$$\begin{aligned}
&\dot{\Delta}_0\left(\lambda_n^0\right)\varepsilon_n + \int_0^\pi A(\pi, t)\cos\left(\lambda_n^0 + \varepsilon_n\right) t\, dt \\
&\quad + \int_0^\pi B(\pi, t)\sin\left(\lambda_n^0 + \varepsilon_n\right) t\, dt + o(\varepsilon_n) = 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{52}$$

It is easy to see that the function $\Delta_0(\lambda)$ is type of "Sine" [39], so there exists $\gamma_\delta > 0$ such that $|\dot{\Delta}_o(\lambda_n^0)| \geq \gamma_\delta > 0$ is satisfied for all n. We also have

$$\lambda_n^0 = n + \frac{1}{\pi}\beta^+(\pi) + h_n, \tag{53}$$

where $\sup_n|h_n| \leq M$ for some constant $M > 0$ [40] (see also [41]). Further, substituting (53) into (52) after certain transformations [1, page 67], we reach $\varepsilon_n \in l_2$. We can obtain more precisely

$$\begin{aligned}
\varepsilon_n = {} & \frac{1}{2\dot{\Delta}_o\left(\lambda_n^0\right)\lambda_n^0}\left\{\left[\alpha^+\sin\left(\lambda_n^0\pi - \beta_1(\pi)\right)\right.\right. \\
& \left. + \alpha^-\sin\left(\lambda_n^0(2a - \pi) + \beta_2(\pi)\right)\right] \\
& \times \int_0^\pi \left(q(x) + p^2(x)\right) dx \\
& - \left[\alpha^+\cos\left(\lambda_n^0\pi - \beta_1(\pi)\right)\right. \\
& \left. + \alpha^-\cos\left(\lambda_n^0(2a - \pi) + \beta_2(\pi)\right)\right] \\
& \left. \times \left(p(\pi) - p(0)\right)\right\} \\
& + \frac{1}{\dot{\Delta}_o\left(\lambda_n^0\right)\lambda_n^0}\left[\int_0^\pi B_t'(\pi, t)\cos\lambda_n^0 t\, dt\right. \\
& \left. - \int_0^\pi A_t'(\pi, t)\sin\lambda_n^0 t\, dt\right] \\
& + \frac{o(\varepsilon_n)}{\lambda_n^0}, \quad n \longrightarrow \infty.
\end{aligned} \tag{54}$$

Since $\int_0^\pi B_t'(\pi,t)\cos\lambda_n^0 t\,dt \in l_2$, $\int_0^\pi A_t'(\pi,t)\sin\lambda_n^0 t\,dt \in l_2$, we have

$$\begin{aligned}\varepsilon_n = \frac{1}{2\dot{\Delta}_o(\lambda_n^0)\lambda_n^0}\Big\{&\Big[\alpha^+\sin\left(\lambda_n^0\pi-\beta_1(\pi)\right)\\&+\alpha^-\sin\left(\lambda_n^0(2a-\pi)+\beta_2(\pi)\right)\Big]\\&\times\int_0^\pi\left(q(x)+p^2(x)\right)dx\\&-\Big[\alpha^+\cos\left(\lambda_n^0\pi-\beta_1(\pi)\right)\\&+\alpha^-\cos\left(\lambda_n^0(2a-\pi)+\beta_2(\pi)\right)\Big]\\&\times\left(p(\pi)-p(0)\right)\Big\}+\frac{\delta_n}{\lambda_n^0},\end{aligned}\tag{55}$$

where $\delta_n \in l_2$. Hence we obtain

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{\delta_n}{\lambda_n^0},\tag{56}$$

where

$$\begin{aligned}d_n = \frac{1}{2\dot{\Delta}_o(\lambda_n^0)}\Big\{&\Big[\alpha^+\sin\left(\lambda_n^0\pi-\beta^+(\pi)\right)\\&+\alpha^-\sin\left(\lambda_n^0(2a-\pi)+\beta^-(\pi)\right)\Big]\\&\times\int_0^\pi\left(q(x)+p^2(x)\right)dx\\&-\Big[\alpha^+\cos\left(\lambda_n^0\pi-\beta^+(\pi)\right)\\&+\alpha^-\cos\left(\lambda_n^0(2a-\pi)+\beta^-(\pi)\right)\Big]\\&\times\left(p(\pi)-p(0)\right)\Big\}\end{aligned}\tag{57}$$

is a bounded sequence. The proof is completed. □

Lemma 13. *Normalizing numbers α_n of the problem $L(\alpha,a)$ are positive and the formula*

$$\alpha_n = \frac{\pi}{2}\left[(\alpha^+)^2+(\alpha^-)^2\right]+\frac{d_{11}}{\lambda_n^0}+\frac{d_{1n}}{n}\tag{58}$$

holds, where $d_{11} = -(\alpha\pi/2)p(0)$, $d_{1n} \in l_2$.

Proof. The formula (58) can be easily obtained from the equalities

$$\begin{aligned}&A(x,x)\sin\lambda_n x - B(x,x)\cos\lambda_n x\\&\quad=\frac{\alpha^+}{2}\Big\{\left(p(x)-p(0)\right)\cos\left(\lambda_n x-\beta^+(x)\right)\\&\qquad+\int_0^x\left(q(t)+p^2(t)\right)dt\sin\left(\lambda_n x-\beta^+(x)\right)\Big\},\\&\left[A(x,2a-x+0)-A(x,2a-x-0)\right]\sin\lambda_n(2a-x)\\&\quad-\left[B(x,2a-x+0)-B(x,2a-x-0)\right]\cos\lambda_n(2a-x)\\&\quad=\frac{\alpha^-}{2}\Big\{\int_0^x\left(q(t)+p^2(t)\right)dt\sin\left(\lambda_n(2a-x)+\beta^-(x)\right)\\&\qquad-\left(p(x)-p(0)\right)\cos\left(\lambda_n(2a-x)+\beta^-(x)\right)\Big\}\end{aligned}\tag{59}$$

by using the asymptotic formula (49) for λ_n. □

4. Inverse Problems

Together with $L(\alpha,a)$, we consider the boundary value problem $\widetilde{L}(\alpha,a)$ of the same form but with different coefficients $(\widetilde{q},\widetilde{p},\widetilde{\alpha},\widetilde{a})$. It is assumed in what follows that if a certain symbol γ denotes an object related to the problem $L(\alpha,a)$, then $\widetilde{\gamma}$ will denote the corresponding object related to the problem $\widetilde{L}(\alpha,a)$.

In the present section, we investigate some inverse spectral problem of the reconstruction of a boundary value problem $L(\alpha,a)$ of type (1)–(4) from its spectral characteristics. Namely, we consider the inverse problems of reconstruction of the boundary value problem $L(\alpha,a)$ from the Weyl function, from the spectral data $\{\lambda_n,\alpha_n\}_{n\geq 0}$, and from two spectra $\{\lambda_n,\mu_n\}_{n\geq 0}$ and prove that the following two lemmas can be easily obtained from asymptotic behavior (49) of the eigenvalues λ_n.

Lemma 14. *If $\lambda_n = \widetilde{\lambda}_n$, $n = 0,\pm1,\pm2,\ldots$, then $\beta^+(\pi) = \widetilde{\beta}^+(\pi)$, $\beta^-(\pi) = \widetilde{\beta}^-(\pi)$, that is, the sequence $\{\lambda_n\}$ uniquely determines $\beta^\pm(\pi)$.*

Lemma 15. *If $\lambda_n = \widetilde{\lambda}_n$, $n = 0,\pm1,\pm2,\ldots$, then $a = \widetilde{a}$, $\alpha = \widetilde{\alpha}$, that is, the sequence $\{\lambda_n\}$ uniquely determines numbers a and α.*

Let $\Phi(x,\lambda)$ be the solution of (1) under the conditions $U(\Phi) = 1, V(\Phi) = 0$ and under the jump conditions (3). One sets $M(\lambda) := \Phi(0,\lambda)$. The functions $\Phi(x,\lambda)$ and $M(\lambda)$ are called the Weyl solution and Weyl function for the boundary value problem $L(\alpha,a)$, respectively. Using the solution $\varphi(x,\lambda)$ defined in the previous sections one has

$$\begin{aligned}\Phi(x,\lambda) = -\frac{\psi(x,\lambda)}{\Delta(\lambda)} = S(x,\lambda)+M(\lambda)\varphi(x,\lambda),\\ M(\lambda) = -\frac{\psi(0,\lambda)}{\Delta(\lambda)},\end{aligned}\tag{60}$$

where $\psi(x,\lambda)$ is a solution of (1) satisfying the conditions $\psi(\pi,\lambda) = 0$, $\psi'(\pi,\lambda) = -1$, and the jump conditions (3), and $S(x,\lambda)$ is defined from the equality

$$\psi(x,\lambda) = \psi(0,\lambda)\varphi(x,\lambda) - \Delta(\lambda)S(x,\lambda).\tag{61}$$

Note that, by virtue of equalities $\langle\varphi(x,\lambda), S(x,\lambda)\rangle \equiv 1$ and (60), one has

$$\begin{gathered}\langle\Phi(x,\lambda),\varphi(x,\lambda)\rangle \equiv 1,\\ \langle\varphi(x,\lambda),\psi(x,\lambda)\rangle \equiv -\Delta(\lambda) \quad \text{for } x\neq a.\end{gathered} \tag{62}$$

The following theorem shows that the Weyl function uniquely determines the potentials and the coefficients of the boundary value problem $L(\alpha,a)$.

Theorem 16. *If $M(\lambda)=\widetilde{M}(\lambda)$, then $L(\alpha,a)=\widetilde{L}(\alpha,a)$. Thus, the boundary value problem $L(\alpha,a)$ is uniquely defined by the Weyl function.*

Proof. Since

$$\psi^{(\nu)}(x,\lambda)=O\left(|\lambda|^{\nu-1}\exp\left(|\operatorname{Im}\lambda|(\pi-x)\right)\right),\quad \lambda\in\widetilde{G}_\delta, \tag{63}$$

$$|\Delta(\lambda)|\geq C_\delta\exp\left(|\operatorname{Im}\lambda|\pi\right),\quad \lambda\in\widetilde{G}_\delta,\ C_\delta<0,\ \nu=0,1, \tag{64}$$

it is easy to observe that

$$\left|\Phi^{(\nu)}(x,\lambda)\right|\leq C_\delta|\lambda|^{\nu-1}\exp\left(-|\operatorname{Im}\lambda|x\right),\quad \lambda\in G_\delta. \tag{65}$$

Let us define the matrix $P(x,\lambda)=[P_{jk}(x,\lambda)]_{j,k=1,2}$, where

$$\begin{aligned}P_{j1}(x,\lambda)&=\varphi^{(j-1)}(x,\lambda)\widetilde{\Phi}'(x,\lambda)-\Phi^{(j-1)}(x,\lambda)\widetilde{\varphi}'(x,\lambda),\\ P_{j2}(x,\lambda)&=\Phi^{(j-1)}(x,\lambda)\widetilde{\varphi}'(x,\lambda)-\varphi^{(j-1)}(x,\lambda)\widetilde{\Phi}(x,\lambda).\end{aligned} \tag{66}$$

Then we have

$$\begin{aligned}\varphi(x,\lambda)&=P_{11}(x,\lambda)\widetilde{\varphi}(x,\lambda)+P_{12}(x,\lambda)\widetilde{\varphi}'(x,\lambda),\\ \Phi(x,\lambda)&=P_{11}(x,\lambda)\widetilde{\Phi}(x,\lambda)+P_{12}(x,\lambda)\widetilde{\Phi}'(x,\lambda).\end{aligned} \tag{67}$$

According to (60) and (65), for each fixed x, the functions $P_{jk}(x,\lambda)$ are meromorphic in λ with poles at points λ_n and $\widetilde{\lambda}_n$. Denote $G_\delta^\circ=G_\delta\cap\widetilde{G}_\delta$. By virtue of (65), (66), and

$$\varphi^{(\nu)}(x,\lambda)=O\left(|\lambda|^{\nu}\exp\left(|\operatorname{Im}\lambda|x\right)\right),\quad \lambda\in G_\delta^\circ, \tag{68}$$

we get

$$\left|P_{12}(x,\lambda)\right|\leq C_\delta|\lambda|^{-1},\quad \left|P_{11}(x,\lambda)\right|\leq C_\delta,\quad \lambda\in G_\delta^\circ. \tag{69}$$

It follows from (60) and (66) that if $M(\lambda)\equiv\widetilde{M}(\lambda)$, then for each fixed x the functions $P_{1k}(x,\lambda)$ are entire in k. Together with (69) this yields $P_{12}(x,\lambda)\equiv 0$, $P_{12}(x,\lambda)\equiv A(x)$. Now using (67), we obtain

$$\varphi(x,\lambda)\equiv A(x)\widetilde{\varphi}(x,\lambda),\qquad \Phi(x,\lambda)\equiv A(x)\widetilde{\Phi}(x,\lambda). \tag{70}$$

Therefore, for $|\lambda|\to\infty$, $\arg\lambda\in[\varepsilon,\pi-\varepsilon]$ $(\varepsilon>0)$, we have

$$\varphi(x,\lambda)=\frac{b}{2}\exp\left(-i\left(\lambda x-\beta_1(x)\right)\right)\left(1+O\left(\frac{1}{\lambda}\right)\right), \tag{71}$$

where $b=1$ for $x<a$ and $b=\alpha^+$ for $x>a$. Similarly, one can calculate

$$\begin{gathered}\Phi(x,\lambda)=(i\lambda b)^{-1}\exp\left(i\left(\lambda x-\beta_1(x)\right)\right)\left(1+O\left(\frac{1}{\lambda}\right)\right),\\ |\lambda|\longrightarrow\infty,\ \arg\lambda\in[\varepsilon,\pi-\varepsilon].\end{gathered} \tag{72}$$

Finally, taking into account the relations $\langle\Phi(x,\lambda),\varphi(x,\lambda)\rangle\equiv 1$ and (65), we have $\alpha^+=\widetilde{\alpha}^+$, $A(x)\equiv 1$, that is, $\varphi(x,\lambda)\equiv\widetilde{\varphi}(x,\lambda)$, $\Phi(x,\lambda)\equiv\widetilde{\Phi}(x,\lambda)$ for all x and λ. Consequently, $L(\alpha,a)=\widetilde{L}(\alpha,a)$. The theorem is proved. □

The following two theorems show that two spectra and spectral data also uniquely determine the potentials and the coefficients of the boundary value problem $L(\alpha,a)$.

Theorem 17. *If $\lambda_n=\widetilde{\lambda}_n$, $\mu_n=\widetilde{\mu}_n$, $n=0,\pm1,\pm2,\ldots$, then $L(\alpha,a)=\widetilde{L}(\alpha,a)$.*

Proof. It is obvious that characteristic functions $\Delta(\lambda)$ and $\psi(0,\lambda)$ are uniquely determined by the sequences $\{\lambda_n^2\}$ and $\{\mu_n^2\}$ $(n=0,\pm1,\pm2,\ldots)$, respectively. If $\lambda_n=\widetilde{\lambda}_n$, $\mu_n=\widetilde{\mu}_n$, $n=0,\pm1,\pm2,\ldots$, then $\Delta(\lambda)\equiv\widetilde{\Delta}(\lambda)$, $\psi(0,\lambda)\equiv\widetilde{\psi}(0,\lambda)$. It follows from (60) that $M(\lambda)\equiv\widetilde{M}(\lambda)$. Therefore, applying Theorem 16, we conclude that $L(\alpha,a)=\widetilde{L}(\alpha,a)$. The proof is completed. □

Theorem 18. *If $\lambda_n=\widetilde{\lambda}_n$, $\alpha_n=\widetilde{\alpha}_n$, $n=0,\pm1,\pm2,\ldots$, then $L(\alpha,a)=\widetilde{L}(\alpha,a)$, that is, spectral data $\{\lambda_n,\alpha_n\}$ uniquely determines the problem $L(\alpha,a)$.*

Proof. It is obvious that the Weyl function $M(\lambda)$ is meromorphic with simple poles at points λ_n^2. Using the expression

$$\Delta(\lambda)=\Delta_0(\lambda)+\int_0^\pi A(\pi,\lambda)\cos\lambda t\,dt+\int_0^\pi B(\pi,\lambda)\sin\lambda t\,dt \tag{73}$$

and equalities $2\lambda_n\beta_n\alpha_n=-\dot{\Delta}(\lambda_n)$, $\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n)$, we have

$$\operatorname*{Re}_{\lambda=\lambda_n} sM(\lambda)=-\frac{\psi(0,\lambda_n)}{\dot{\Delta}(\lambda_n)}=-\frac{\beta_n}{\dot{\Delta}(\lambda_n)}=\frac{1}{2\lambda_n\alpha_n}. \tag{74}$$

Since the Weyl function $M(\lambda)$ is regular for $\lambda\in\Gamma_n$, applying the Rouche theorem, we conclude that

$$M(\lambda)=\frac{1}{2\pi i}\int_{\Gamma_n}\frac{M(\mu)}{\mu-\lambda}d\mu,\quad \lambda\in\operatorname{int}\Gamma_n. \tag{75}$$

Taking (60) and (63) into account, we arrive at $|M(\lambda)|\leq C_\delta|\lambda|^{-1}$, $\lambda\in G_\delta$. Therefore

$$M(\lambda)=\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{n1}}\frac{M(\mu)}{\mu-\lambda}d\mu, \tag{76}$$

where $\Gamma_{n1}=\{\lambda:|\lambda|=|\lambda_n^0|\}$, $n=0,\pm1,\pm2,\ldots$. Hence, by the residue theorem, we have

$$M(\lambda)=\sum_{n=-\infty}^{\infty}\frac{1}{2\lambda_n\alpha_n(\lambda-\lambda_n)}. \tag{77}$$

Finally, from the equality $M(\lambda) \equiv \widetilde{M}(\lambda)$, applying Theorem 16, we conclude that $L(\alpha, a) = \widetilde{L}(\alpha, a)$. The theorem is proved. □

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