Research Article

The Existence of Positive Solutions for a New Coupled System of Multiterm Singular Fractional Integrodifferential Boundary Value Problems

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1. Introduction

During the last decade, there were a lot of manuscripts on fractional differential equations (see, e.g., [1–19] and the references therein). Fractional equations have been discussed extensively as valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in fluid mechanics, viscoelasticity, edge detection, porous media, and electromagnetism, as well as in various other areas. For more examples and details, see [3,13–15] and references therein. On the other hand, there are many works about the existence of positive solutions of fractional differential equations (see, e.g., [1,2,6–8,10,17,19] and the references therein).

2. The Problem

In this paper, we investigate the following coupled system of multiterm singular fractional integrodifferential boundary value problem:

\[ D_0^\alpha u(t) + f_1(t, u(t), v(t), (\psi_1 u)(t), D_0^p u(t), D_0^\mu_1 v(t), \ldots, D_0^\mu_m v(t)) = 0, \]
\[ D_0^\beta v(t) + f_2(t, u(t), v(t), (\phi_1 u)(t), (\psi_2 v)(t), D_0^q v(t), D_0^\mu_1 u(t), D_0^\mu_2 u(t), \ldots, D_0^\mu_m u(t)) = 0, \]

\[ u^{(i)}(0) = 0 \quad \text{and} \quad v^{(i)}(0) = 0 \]
for all \( 0 \leq i \leq n - 2 \), where \( n \geq 4 \), \( n - 1 < \alpha, \beta < n \), \( 0 < p, q < 1 \), \( 1 < \mu_i < 2 \) \( (i = 1, 2, \ldots, m) \), \( y_1, \lambda : [0, 1] \times [0, 1] \rightarrow (0, \infty) \) are continuous functions \( (j = 1, 2) \) and \( (\phi_1 u)(t) = \int_0^t \lambda_1(t, s) u(s) ds, (\psi_2 v)(t) = \int_0^t \lambda_2(t, s) v(s) ds \). Here \( D \) is the standard Riemann-Liouville fractional derivative, \( f_j \ (j = 1, 2) \) is a Carathéodory function, and \( f_j(t, x, y, z, w, u_1, u_2, \ldots, u_m) \) is singular at the value 0 of its variables.
\begin{align}
\mathcal{V}_2(v)(t), D^\nu_0 v(t), D^\nu_1 u(t), \\
D^\nu_2 u(t), \ldots, D^\nu_m u(t) &= 0, \\
\mathcal{V}_1(v)(t), D^\nu_0 v(t), D^\nu_1 u(t), \\
D^\nu_2 u(t), \ldots, D^\nu_m u(t) &= 0,
\end{align}
(1)
\begin{align}
u(i)(0) = 0 \quad \forall 0 \leq i \leq n - 2, \\
[D^\nu_0 u(t)]_{i=1} = 0 \quad \text{for } 2 < \delta_1 < n - 1, \alpha - \delta_1 \geq 1, \\
\nu(i)(0) = 0 \quad \forall 0 \leq i \leq n - 2, \\
[D^\nu_0 v(t)]_{i=1} = 0 \quad \text{for } 2 < \delta_2 < n - 1, \beta - \delta_2 \geq 1,
\end{align}
(2)
where \( n \geq 4, n - 1 < \alpha, \beta < n, 0 < p, q < 1, 1 < \mu_i, \nu_i < 2 \) (\( i = 1, 2, \ldots, m \)), \( \gamma_i \) and \( \lambda_i \) are positive-valued continuous functions on \([0, 1] \times [0, 1] \) \((i = 1, 2, \ldots, m)\), \( y_i \) and \( \lambda_i \) are positive-valued continuous functions on \([0, 1] \times [0, 1] \) \((i = 1, 2, \ldots, m)\). Here, \( D \) is the standard Riemann-Liouville fractional derivative. In fact, a function \( f \) satisfies the local Carathéodory condition on \([0, 1] \times \mathcal{D} \) whenever
\begin{enumerate}
\item[(i)] \( f(t; x, y, z, w, v, u_1, u_2, \ldots, u_m) : [0, 1] \to \mathbb{R} \) is measurable for all \((x, y, z, w, v, u_1, u_2, \ldots, u_m) \in \mathcal{D}\).
\item[(ii)] \( f(t; \cdot, \cdot, \ldots, \cdot) : \mathcal{D} \to \mathbb{R} \) is continuous for almost all \( t \in [0, 1]\).
\item[(iii)] for each compact subset \( \kappa \subset \mathcal{D} \), there is a function \( \varphi_k \in L^1[0, 1] \) such that
\begin{equation}
\left| f(t; x, y, z, w, v, u_1, u_2, \ldots, u_m) \right| \leq \varphi_k(t),
\end{equation}
(5)
for almost all \( t \in [0, 1] \) and all \((x, y, z, w, v, u_1, u_2, \ldots, u_m) \in \kappa\). The functions \( u, v \in C^2[0, 1] \) are called positive solutions of the problem (1), (2), and (3) whenever \( u > 0 \) and \( v > 0 \) on \((0, 1], D^\nu_0 u, D^\nu_1 u \in L^1[0, 1], u, v \) satisfy boundary conditions (2), (3), and (1) holds for almost all \( t \in [0, 1] \). In 2010, Agarwal et al. reviewed positive solutions of the singular Dirichlet problem
\begin{equation}
D^\nu_0 u(t) + f(t, u(t), D^\nu_1 u(t)) = 0,
\end{equation}
(6)
\begin{equation}
u(0) = u(0) = u'(0) = 0, \quad \forall 0 < \alpha < 2, 0 < \mu \leq \alpha - 1 \) and \( f \) satisfies the local Carathéodory condition on \([0, 1] \times (0, \infty) \times \mathbb{R} \) \([1]\). In 2011, Staněk reviewed the singular problem
\begin{equation}
D^\nu_0 u(t) + f(t, u(t), u'(t), D^\nu_0 u(t)) = 0,
\end{equation}
(7)
\begin{equation}
u(0) = u(0) = u'(0) = 0, \quad 2 < \alpha < 3, 0 < \mu < 1 \) and \( f \) satisfies the local Carathéodory condition on \([0, 1] \times \mathcal{D} \) \([19]\).
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\[
\int_0^1 P_j\left(M_1^\alpha, M_2^\alpha, \ldots, M_n^\alpha\right) ds < \infty,
\]

for almost all \( t \in [0, 1] \) and \( x, y, z, \ldots, u, v, u_1, u_2, \ldots, u_m \) \( \in \mathfrak{D} \).

Since we suppose that the problem (1) is singular, we use regularization and sequential techniques for the existence of positive solutions of the problem. In this way, for each natural number \( n \) define the function \( f_n^j \) \( (j = 1, 2) \) by

\[ f_n^j(t, x, y, z, w, u_1, u_2, \ldots, u_m) \leq P_j(t, x, y, z, w, u_1, u_2, \ldots, u_m) \]

\[ + k_j(t) h_j(1 + x, 1 + y, 1 + z, 1 + w, 1 + v, 1 + u_1, 1 + u_2, \ldots, 1 + u_m) \]

for almost all \( t \in [0, 1] \) and \( x, y, z, w, u_1, u_2, \ldots, u_m \) \( \in \mathbb{R}^m \), where \( \forall n \in [\alpha] + 1 \) and \( \forall \epsilon \in \mathbb{R} \) for \( \forall k = 1, 2, \ldots, n \). Suppose that \( v \in (0, 1) \), \( \mu \in (1, 2) \), \( x \in C^2[0, 1] \), and \( x(0) = x'(0) = 0 \). Then, \( D_0^\alpha, x(t) \in C[0, 1] \), \( D_0^\alpha, x(t) = (1/\Gamma(2 - \mu)) \int_0^t (t - s)^{1-\mu} x''(s) ds, \) and \( D_0^\alpha, x(t) = (1/\Gamma(2 - \mu)) \int_0^t (t - s)^{1-\mu} x''(s) ds \). 

\[ D_0^\alpha, u(t) + f_n^j(t, u(t), v(t), \phi(u)(t), \psi(u)(t)) = 0, \]

\[ D_0^\alpha, v(t) + f_n^j(t, u(t), v(t), \phi(u)(t), \psi(u)(t)) = 0, \]

\[ D_0^\alpha, u(t) = 0, \]

\[ D_0^\alpha, v(t) = 0, \]

via the boundary conditions (2) and (3) and by using solutions of this system, we will obtain solution of the system (1). (2), and (3), it has been proved that the fractional integral \( I_0^\alpha \) maps \( L^1([0, 1]) \) into \( L^1([0, 1]) \) whenever \( \alpha \in (0, 1) \) and \( L^1([0, 1]) \) into \( AC([0, 1]) \) whenever \( \alpha = 1 \) [19]. Here, \( [\alpha] \) means the integral part of \( \alpha \) and \( AC([0, 1]) = AC([0, 1]) \) Supposes that \( \alpha > 0 \) and \( \alpha \) is not a natural number. If \( x \in C([0, 1]) \) and \( D_0^\alpha, x \in L^1([0, 1]) \), then \( x(t) = I_0^\alpha D_0^\alpha, x(t) + \sum_{k=1}^n \eta_k t^{\alpha - k} \) for all \( t \in (0, 1) \), where \( n = [\alpha] + 1 \) and \( \eta_k \in \mathbb{R} \) for \( k = 1, 2, \ldots, n \). Suppose that \( v \in (0, 1) \), \( \mu \in (1, 2) \), \( x \in C^2([0, 1]) \), and \( x(0) = x'(0) = 0 \). Then, \( D_0^\alpha, x(t) \in C[0, 1] \), \( D_0^\alpha, x(t) = (1/\Gamma(2 - \mu)) \int_0^t (t - s)^{1-\mu} x''(s) ds, \) and \( D_0^\alpha, x(t) = (1/\Gamma(2 - \mu)) \int_0^t (t - s)^{1-\mu} x''(s) ds \). 

3. Main Results

Now, we are ready to state and prove our main results. One can find main idea of next result in [17].

Lemma 1. Let \( n = 4 \) and \( n - 1 < \alpha < n \). For each \( g \in L^1([0, 1]), \) \( u(t) = \int_0^t G_{\alpha}(t, s) g(s) ds \) is the unique solution of the equation \( D_0^\alpha, u(t) + g(t) = 0 \) in \( C^2([0, 1]) \) which satisfies the boundary condition (2), where

\[ G_{\alpha}(t, s) = \begin{cases} \frac{\Gamma(\alpha)}{\Gamma(\alpha - s - 1)} \left[ (t - s)^{\alpha - 1} - (t - s)^{\alpha - 1} \right] & 0 \leq s \leq t \leq 1, \\ \frac{\Gamma(\alpha)}{\Gamma(\alpha - s - 1)} \left[ t^{\alpha - 1} - (t - s)^{\alpha - 1} \right] & 0 \leq t \leq s \leq 1. \end{cases} \]
Proof. It is easy to see that the functions \( u(t) = -I_0^\alpha g(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n} \) are solutions of \( D_0^\alpha u(t) + g(t) = 0 \) in \( C(0, 1] \) for all \( c_i \in \mathbb{R} \). Since \( |\alpha| \geq 3, I_0^\alpha g \in AC^{[\alpha-1]}[0, 1] \) and so \( u(t) = -I_0^\alpha g(t) + c_1 t^{\alpha-1} \) are solutions of \( D_0^\alpha u(t) + g(t) = 0 \) in \( C^2[0, 1] \), where \( c_1 \in \mathbb{R} \). The boundary condition (2) implies that \( c_2 = \cdots = c_{n-1} = 0 \) and
\[
0 = c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha - \delta_1)} (1)^{\alpha - \delta_1} - \frac{1}{\Gamma(\alpha - \delta_1)} \int_0^1 (1-s)^{\alpha - \delta_1} g(s) \, ds.
\]

Thus, \( u(t) = (t^{\alpha-1}/\Gamma(\alpha)) \int_0^1 (1-s)^{\alpha - \delta_1} g(s) \, ds (1/\Gamma(\alpha)) \int_0^1 (t-s)^{\alpha - \delta_1} g(s) \, ds = \int_0^1 G_\alpha(t, s) g(s) \, ds \) is the unique solution of the problem in \( C^2[0, 1] \).

Note that the Green function \( G_\alpha \) in Lemma 1 has some properties. For example, \( G_\alpha \), \( (\partial/\partial t)G_\alpha \) and \( (\partial^2/\partial t^2)G_\alpha \) are continuous functions on \( [0, 1] \times [0, 1] \), \( G_\alpha(t, s) > 0 \) on \( (0, 1) \times (0, 1) \), \( G_\alpha(t, s) \leq 1/\Gamma(\alpha) \) for all \( (t, s) \in [0, 1] \times [0, 1] \),
\[
\int_0^1 G_\alpha(t, s) \, ds \geq r^{\alpha-1}/(\alpha - \delta_1) \Gamma(\alpha + 1) \text{ for all } t \in [0, 1],
\]
\[
(\partial/\partial t)G_\alpha(t, s) > 0 \text{ on } (0, 1) \times (0, 1), \quad (\partial/\partial s)G_\alpha(t, s) \leq 1/\Gamma(\alpha) \text{ for all } t \in [0, 1] \times [0, 1], \quad \text{and}
\]
\[
(\partial^2/\partial t^2)G_\alpha(t, s) \leq 1/\Gamma(\alpha - 2) \text{ for all } t \in [0, 1] \times [0, 1], \quad \text{and}
\]
\[
(\partial^2/\partial t^2)G_\alpha(t, s) ds \geq (\delta_1 - 2) r^{\alpha-3}/(\alpha - \delta_1) \Gamma(\alpha - 1) \text{ for all } t \in [0, 1].
\]

Let \( X = C^2[0, 1] \). Define the cone \( P \) on \( X \times X \) by
\[
P = \bigl\{ (x, y) \in X \times X : \begin{align*}
x(0) &= x'(0) = y(0) = y'(0) = 0, \\
x(t), &x'(t), x''(t), y(t), y'(t), y''(t) \geq 0, \\
\forall t &\in [0, 1]igr\}.
\]

It is easy to see that for each \( (x, y) \in P, i = 1, 2, \ldots, m \) and \( t \in [0, 1] \) we have \( D_0^\mu x(t), D_0^\nu y(t), D_0^\mu y(t), y(t), y'(t), y''(t) \geq 0 \) now for each natural number \( n \), define the operator \( T_n \) on \( P \) by
\[
T_n(x, y)(t) = \begin{cases}
T_n^1(x, y)(t) & (t \in [0, 1]), \\
T_n^2(x, y)(t) & (t \in [0, 1]).
\end{cases}
\]

Lemma 2. For each natural number \( n, T_n \) is a completely continuous operator on \( P \).

Proof. Let \( n \) be a natural number, \( (x, y) \in P \),
\[
\rho^1(t) = f_n^1 \bigl( x(t), y(t), (\phi_1 x)(t), (\psi_1 y)(t), D_0^\mu x(t), D_0^\nu y(t), D_0^\mu y(t), y(t) \bigr)
\]

and
\[
\rho^2(t) = f_n^2 \bigl( x(t), y(t), (\phi_2 x)(t), (\psi_2 y)(t), D_0^\mu x(t), D_0^\nu y(t), D_0^\mu y(t), y(t) \bigr)
\]

Then, \( \rho^1, \rho^2 \in L^1[0, 1] \) and there exist positive constants \( m_1 \) and \( m_2 \) such that \( \rho^1(t) \geq m_1 \) and \( \rho^2(t) \geq m_2 \) for almost all \( t \in [0, 1] \).

Since \( G_\alpha, G_{\alpha,\beta}, (\partial/\partial t)G_\alpha, (\partial^2/\partial t^2)G_{\alpha,\beta} \) are nonnegative and continuous functions on \( [0, 1] \times [0, 1] \), \( G_\alpha(0, s) = 0, \quad G_{\alpha,\beta}(0, s) = 0 \) for all \( s \in [0, 1] \), we get
\[
T_n^1(x, y)(t), T_n^2(x, y)(t) \in C^2[0, 1], \quad T_n^1(x, y)(0) = T_n^2(x, y)(0) = (T_n^1(x, y)'(t))(0) = 0, \quad T_n^1(x, y)(0) \geq 0, \quad T_n^1(x, y)'(t) \geq 0, \quad (D_0^\mu x(t), y(t))' \geq 0, \quad (D_0^\nu y(t))' \geq 0, \quad (T_n^1(x, y))''(t) \geq 0, \quad (T_n^2(x, y))''(t) \geq 0, \quad (D_0^\mu x(t), y(t))'' \geq 0.
\]

and \( T_n^1(x, y)(t), T_n^2(x, y)(t) \geq 0 \) on \( [0, 1] \). Thus, \( T_n \) maps \( P \) into \( P \). Suppose that \( \{(x_k, y_k)\}_{k=1} \) is a convergent sequence in \( P \) and \( \lim_{k \to \infty} x_k(t) = x(t) \). Then, \( \lim_{k \to \infty} y_k(t) = y(t) \) and \( \lim_{k \to \infty} y_k^{(j)}(t) = y^{(j)}(t) \) uniformly on \( [0, 1] \) for \( j = 0, 1, 2 \).

Since for \( p, q \in (0, 1), \mu, \nu \in (1, 2) \),
\[
\begin{align*}
(D_0^\mu x_k(t) - D_0^\mu x(t)) &\leq \frac{\|x_k'' - x''\|}{\Gamma(2 - p)} \int_0^t (t-s)^{1-p} \, ds \leq \frac{\|x_k'' - x''\|}{\Gamma(3 - p)}, \\
(D_0^\nu y_k(t) - D_0^\nu y(t)) &\leq \frac{\|y_k'' - y''\|}{\Gamma(2 - q)} \int_0^t (t-s)^{1-q} \, ds \leq \frac{\|y_k'' - y''\|}{\Gamma(3 - q)}, \\
(D_0^\mu x_k(t) - D_0^\mu x(t)) &\leq \frac{\|x_k'' - x''\|}{\Gamma(2 - \nu_1)} \int_0^t (t-s)^{1-\nu_1} \, ds \leq \frac{\|x_k'' - x''\|}{\Gamma(3 - \nu_1)}.
\end{align*}
\]
\[ \left| D^\gamma_0 y_k(t) - D^\gamma_0 y(t) \right| \leq \frac{\| y'' - y'' \|}{\Gamma (2 - \mu)} \int_0^t (t - s)^{1-\mu} ds \leq \frac{\| y'' - y'' \|}{\Gamma (3 - \mu)}, \]
\[ \left| (\phi, x_k(t)) - (\phi, x(t)) \right| \leq \sup_{t \in [0,1]} \left| \int_0^t y_j(t, s) ds \right| \| x_k - x \| = y_0 \| x_k - x \|, \]
\[ \left| (\psi, y_k(t)) - (\psi, y(t)) \right| \leq \sup_{t \in [0,1]} \left| \int_0^t \lambda_j(t, s) ds \right| \| y - y \| = \lambda_0 \| y - y \|, \]
\[
(21)
\]
for \( j = 1, 2 \) and \( k \geq 1 \), we have \( \lim_{k \to \infty} D^\gamma_0 x_k(t) = D^\gamma_0 x(t), \lim_{k \to \infty} D^\gamma_0 y_k(t) = D^\gamma_0 y(t), \lim_{k \to \infty} D^\gamma_0 x_k(t) = D^\gamma_0 x(t), \) and \( \lim_{k \to \infty} D^\gamma_0 y_k(t) = D^\gamma_0 y(t) \) for \( i = 1, 2, \ldots, m \), and \( \lim_{k \to \infty} (\phi, x_k(t)) = (\phi, x(t)), \lim_{k \to \infty} (\psi, y_k(t)) = (\psi, y(t)) \) uniformly on \([0, 1]\) for \( j = 1, 2 \). Also, we have
\[
\begin{align*}
\| D^\gamma_0 x_k \| &\leq \frac{\| y'' \|}{\Gamma (2 - p)} \int_0^t (t - s)^{1-p} ds \leq \frac{\| y'' \|}{\Gamma (3 - p)}, \\
\| D^\gamma_0 y_k \| &\leq \frac{\| y'' \|}{\Gamma (2 - q)} \int_0^t (t - s)^{1-q} ds \leq \frac{\| y'' \|}{\Gamma (3 - q)}, \\
\| D^\gamma_0 x_k \| &\leq \frac{\| y'' \|}{\Gamma (2 - \nu)} \int_0^t (t - s)^{1-\nu} ds \leq \frac{\| y'' \|}{\Gamma (3 - \nu)}, \\
\| D^\gamma_0 y_k \| &\leq \frac{\| y'' \|}{\Gamma (2 - \mu)} \int_0^t (t - s)^{1-\mu} ds \leq \frac{\| y'' \|}{\Gamma (3 - \mu)},
\end{align*}
\]
and also \( \| \phi, x_k \| \leq y_0^\| x_k \| \) and \( \| \psi, y_k \| \leq \lambda_0^\| y_k \| \) for \( j = 1, 2 \) and \( k \geq 1 \). Now, put
\[
\rho_k^j (t) = f_j^1 (t, x_k(t), y_k(t), (\phi, x_k(t)), (\psi, y_k(t)), (\psi_1, y_k(t)), (\psi_2, y_k(t)), (\psi_3, y_k(t)), (\psi_4, y_k(t)), (\psi_5, y_k(t)), (\psi_6, y_k(t)), \) \]
\[ D^\gamma_0 x_k(t), \ldots, D^\gamma_0 y_k(t), \quad (23) \]
and \( \rho_2^j (t) = f_2^1 (t, x_k(t), y_k(t), (\phi, x_k(t)), (\psi, y_k(t)), (\psi_1, y_k(t)), (\psi_2, y_k(t)), (\psi_3, y_k(t)), (\psi_4, y_k(t)), (\psi_5, y_k(t)), (\psi_6, y_k(t)), (\psi_7, y_k(t)), \)
\[ D^\gamma_0 x_k(t), \ldots, D^\gamma_0 y_k(t), \]
and\( \rho_k^j (t) = f_j^1 (t, x_k(t), y_k(t), (\phi, x_k(t)), (\psi, y_k(t)), (\psi_1, y_k(t)), (\psi_2, y_k(t)), (\psi_3, y_k(t)), (\psi_4, y_k(t)), (\psi_5, y_k(t)), (\psi_6, y_k(t)), (\psi_7, y_k(t)), \)
\[ D^\gamma_0 x_k(t), \ldots, D^\gamma_0 y_k(t), \]
and \( \rho_k^j (t) = f_j^1 (t, x_k(t), y_k(t), (\phi, x_k(t)), (\psi, y_k(t)), (\psi_1, y_k(t)), (\psi_2, y_k(t)), (\psi_3, y_k(t)), (\psi_4, y_k(t)), (\psi_5, y_k(t)), (\psi_6, y_k(t)), (\psi_7, y_k(t)), \)
\[ D^\gamma_0 x_k(t), \ldots, D^\gamma_0 y_k(t), \]
and \( \rho_k^j (t) = f_j^1 (t, x_k(t), y_k(t), (\phi, x_k(t)), (\psi, y_k(t)), (\psi_1, y_k(t)), (\psi_2, y_k(t)), (\psi_3, y_k(t)), (\psi_4, y_k(t)), (\psi_5, y_k(t)), (\psi_6, y_k(t)), (\psi_7, y_k(t)), \)
\[ D^\gamma_0 x_k(t), \ldots, D^\gamma_0 y_k(t), \]
and \( \rho_k^j (t) = f_j^1 (t, x_k(t), y_k(t), (\phi, x_k(t)), (\psi, y_k(t)), (\psi_1, y_k(t)), (\psi_2, y_k(t)), (\psi_3, y_k(t)), (\psi_4, y_k(t)), (\psi_5, y_k(t)), (\psi_6, y_k(t)), (\psi_7, y_k(t)), \)
\[ D^\gamma_0 x_k(t), \ldots, D^\gamma_0 y_k(t), \]
and \( \rho_k^j (t) = f_j^1 (t, x_k(t), y_k(t), (\phi, x_k(t)), (\psi, y_k(t)), (\psi_1, y_k(t)), (\psi_2, y_k(t)), (\psi_3, y_k(t)), (\psi_4, y_k(t)), (\psi_5, y_k(t)), (\psi_6, y_k(t)), (\psi_7, y_k(t)), \)
\[ D^\gamma_0 x_k(t), \ldots, D^\gamma_0 y_k(t), \]
and \( \rho_k^j (t) = f_j^1 (t, x_k(t), y_k(t), (\phi, x_k(t)), (\psi, y_k(t)), (\psi_1, y_k(t)), (\psi_2, y_k(t)), (\psi_3, y_k(t)), (\psi_4, y_k(t)), (\psi_5, y_k(t)), (\psi_6, y_k(t)), (\psi_7, y_k(t)), \)
\[ D^\gamma_0 x_k(t), \ldots, D^\gamma_0 y_k(t), \]
and \( \rho_k^j (t) = f_j^1 (t, x_k(t), y_k(t), (\phi, x_k(t)), (\psi, y_k(t)), (\psi_1, y_k(t)), (\psi_2, y_k(t)), (\psi_3, y_k(t)), (\psi_4, y_k(t)), (\psi_5, y_k(t)), (\psi_6, y_k(t)), (\psi_7, y_k(t)), \)
\[ D^\gamma_0 x_k(t), \ldots, D^\gamma_0 y_k(t), \]
and \( \rho_k^j (t) = f_j^1 (t, x_k(t), y_k(t), (\phi, x_k(t)), (\psi, y_k(t)), (\psi_1, y_k(t)), (\psi_2, y_k(t)), (\psi_3, y_k(t)), (\psi_4, y_k(t)), (\psi_5, y_k(t)), (\psi_6, y_k(t)), (\psi_7, y_k(t)), \)
\[ D^\gamma_0 x_k(t), \ldots, D^\gamma_0 y_k(t), \]
and \( \rho_k^j (t) = f_j^1 (t, x_k(t), y_k(t), (\phi, x_k(t)), (\psi, y_k(t)), (\psi_1, y_k(t)), (\psi_2, y_k(t)), (\psi_3, y_k(t)), (\psi_4, y_k(t)), (\psi_5, y_k(t)), (\psi_6, y_k(t)), (\psi_7, y_k(t)), \)
\[ D^\gamma_0 x_k(t), \ldots, D^\gamma_0 y_k(t), \]
and \( \rho_k^j (t) = f_j^1 (t, x_k(t), y_k(t), (\phi, x_k(t)), (\psi, y_k(t)), (\psi_1, y_k(t)), (\psi_2, y_k(t)), (\psi_3, y_k(t)), (\psi_4, y_k(t)), (\psi_5, y_k(t)), (\psi_6, y_k(t)), (\psi_7, y_k(t)), \)
\[ D^\gamma_0 x_k(t), \ldots, D^\gamma_0 y_k(t), \]
and \( \rho_k^j (t) = f_j^1 (t, x_k(t), y_k(t), (\phi, x_k(t)), (\psi, y_k(t)), (\psi_1, y_k(t)), (\psi_2, y_k(t)), (\psi_3, y_k(t)), (\psi_4, y_k(t)), (\psi_5, y_k(t)), (\psi_6, y_k(t)), (\psi_7, y_k(t)), \)
\[ D^\gamma_0 x_k(t), \ldots, D^\gamma_0 y_k(t), \]
and \( \rho_k^j (t) = f_j^1 (t, x_k(t), y_k(t), (\phi, x_k(t)), (\psi, y_k(t)), (\psi_1, y_k(t)), (\psi_2, y_k(t)), (\psi_3, y_k(t)), (\psi_4, y_k(t)), (\psi_5, y_k(t)), (\psi_6, y_k(t)), (\psi_7, y_k(t)), \)
\[ D^\gamma_0 x_k(t), \ldots, D^\gamma_0 y_k(t), \]
\begin{align*}
0 \leq & \left( T_n^1 (x_k, y_k) \right)' (t) = \int_0^1 \frac{\partial}{\partial t} G_n(t, s) \rho_k^1 (s) \, ds \\
& \leq \frac{1}{\Gamma (\alpha - 1)} \int_0^1 g_1 (s) \, ds = \frac{\| g_1 \|}{\Gamma (\alpha - 1)}, \\
0 \leq & \left( T_n^1 (x_k, y_k) \right)'' (t) = \int_0^1 \frac{\partial^2}{\partial t^2} G_n(t, s) \rho_k^1 (s) \, ds \\
& \leq \frac{1}{\Gamma (\alpha - 2)} \int_0^1 g_1 (s) \, ds = \frac{\| g_1 \|}{\Gamma (\alpha - 2)}. \\
\| T_n^1 (x_k, y_k) \|_* & \leq \frac{\| g_1 \|}{\Gamma (\alpha - 2)} = \| g_2 \|_{\Gamma (\beta - 2)}. \\
\text{Similarly, we can verify that} & \| T_n^2 (x_k, y_k) \|_* \leq \frac{\| g_2 \|}{\Gamma (\beta - 2)}. \\
\text{for all} k. \text{ Hence,} & \| T_n (x_k, y_k) \|_{\Omega_1} \text{ is bounded in} X \times X. \text{ Also, for} \\
0 \leq & t_1 < t_2 \leq 1 \text{ we have} \\
\left| \left( T_n^1 (x_k, y_k) \right)'' (t_2) - \left( T_n^1 (x_k, y_k) \right)'' (t_1) \right| & \leq \frac{\Gamma (\alpha - 3)}{\Gamma (\alpha - 2)} \int_{t_1}^{t_2} (t_1 - s)^{\alpha - 3 - 1} \rho_k^1 (s) \, ds + \frac{1}{\Gamma (\alpha - 2)} \\
& \times \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 3} \rho_k^1 (s) \, ds - \int_{t_0}^{t_1} (t_1 - s)^{\alpha - 3} \rho_k^1 (s) \, ds \\
& \leq \frac{\| \rho_k \|}{\Gamma (\alpha - 2)} \left( \frac{\Gamma (\alpha - 3)}{\Gamma (\alpha - 2)} + \frac{1}{\Gamma (\alpha - 2)} \right) \\
& \times \left[ \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 3} \rho_k^1 (s) \, ds + \int_{t_0}^{t_1} (t_1 - s)^{\alpha - 3} \rho_k^1 (s) \, ds \right] \\
& \leq \frac{\| g_1 \|}{\Gamma (\alpha - 2)} \left( \frac{\Gamma (\alpha - 3)}{\Gamma (\alpha - 2)} + \frac{1}{\Gamma (\alpha - 2)} \right) \\
& \times \left[ (t_2 - t_1)^{\alpha - 3} \| g_1 \| + \int_0^{t_1} (t_1 - s)^{\alpha - 3} \rho_1 (s) \, ds \right].
\end{align*}
\[
0 \leq T^1_n(x, y)(t) = \int_0^t (T^1_n(x, y))''(s) \, ds
\]

\[
0 \leq T^1_n(x, y)(t) = \int_0^t (T^1_n(x, y))''(s) \, ds
\]

Similarly, we have

\[
T^2_n(x, y) \leq \frac{1}{\Gamma(\beta - 2)} \left( W^2_n + h_1 \left( 1 + \|x\|, 1 + \|y\|, 1 + \|\phi_1 x\|, 1 + \|\psi_1 y\|^2, 1 + \|D_{\phi}^0 x\|, 1 + \|D_{\psi}^0 y\|, 1 + \|\phi_1 x\|, 1 + \|\psi_1 y\|, 1 + \|D_{\phi}^0 x\|, 1 + \|D_{\psi}^0 y\|, \|k_1\|_1 \right)
\]

(32)

for all \((x, y) \in P\) and \(t \in [0, 1]\). Hence,

\[
T^1_n(x, y) \leq \frac{1}{\Gamma(\alpha - 2)} \left( W^1_n + h_1 \left( 1 + \|x\|, 1 + \|y\|, 1 + \|\phi_1 x\|, 1 + \|\psi_1 y\|, 1 + \|D_{\phi}^0 x\|, 1 + \|D_{\psi}^0 y\|, \|k_1\|_1 \right)
\]

(31)

for all \((x, y) \in P\). Since \(\lim_{x \to \infty} h_j(x, x, \ldots, x)/x = 0\) for \(j = 1, 2\), there exists \(S > 0\) such that

\[
\frac{1}{\Gamma(\alpha - 2)} \left( W^1_n + h_1 \left( 1 + \|x\|, 1 + \|\phi_1 x\|, 1 + \|\psi_1 y\|, 1 + \|D_{\phi}^0 x\|, 1 + \|D_{\psi}^0 y\|, \|k_1\|_1 \right)
\]

\[
1 + S, 1 + \gamma_0^1 S, 1 + \lambda_1^1 S, \frac{S}{\Gamma(3 - \mu_1)}, \frac{S}{\Gamma(3 - \mu_2)}, \ldots, \frac{S}{\Gamma(3 - \mu_m)}
\]

(33)
\[
\frac {1} {\Gamma (\beta - 2)} \left( W_n^2 + h_2 \right) \left( 1 + S, 1 + S, 1 + S, 1 + \lambda S \right),
\]
\[
\frac {1} {\Gamma (3 - \eta)} \left( 1 + \frac {S} {\Gamma (3 - \eta)} \right),
\]
\[
\frac {1} {\Gamma (3 - \nu_2)}, \ldots,
\]
\[
\frac {S} {\Gamma (3 - \nu_2)} \left\| h_2 \right\| .
\]

Put \( \Omega_2 = \{(x, y) \in X \times X \text{ s.t. } \|(x, y)\|_{**} < S\} \). Then, the above inequalities imply that \( \| T_n (x, y) \|_{**} \leq \| (x, y) \|_{**} \) for all \((x, y) \in P \cap \partial \Omega_2\). Now by using Lemma 3, we conclude that the operator \( T_n \) has a fixed point \((u_n, v_n)\) in \( P \cap (\Omega_2 \setminus \Omega_1) \). It is easy to see that \((u_n, v_n)\) is a desired solution of the system, that is, \(u_n \geq M_1t^{\beta-1}\) and \(v_n(t) \geq M_2t^{\beta-1}\) for all \(t \in [0,1]\).

We need the following lemma.

**Lemma 5.** The set of solutions \(\{(u_n, v_n)\}_{n \geq 1}\) of the system (15), (2), and (3) is a relatively compact subset of \(X \times X\).

**Proof.** It is easy to check that

\[
u_n(t) = \int_0^1 G_\beta (t, s) f_n^2 (s, u_n (s), v_n (s), (\phi_2 u_n) (s), (\psi_2 v_n) (s), D^0_0 u_n (s), D^\mu_0 v_n (s), D^0_0 u_n (s), \ldots, D^\nu_0 u_n (s), D^\nu_0 u_n (s)) ds,
\]

for \(t \in [0,1]\) and \(n \geq 1\), satisfy in Theorem 4. Also, \(u_n'(t) \geq m_1 \int_0^1 (\partial / \partial t) G_\alpha (t, s) ds \geq m_1 t^{\alpha-2}/\Gamma (\alpha) (\alpha - \delta_1), u_n''(t) \geq m_1 \int_0^1 (\partial^2 / \partial t^2) G_\alpha (t, s) ds \geq m_1 (\delta_1 - 2) t^{\alpha-2}/\Gamma (\alpha) (\alpha - \delta_1), v_n'(t) \geq m_2 \int_0^1 (\partial / \partial t) G_\beta (t, s) ds \geq m_2 t^{\beta-2}/\Gamma (\beta) (\beta - \delta_2),\) and

\[
v_n''(t) \geq m_2 \int_0^1 (\partial^2 / \partial t^2) G_\beta (t, s) ds \geq m_2 t^{\beta-2}/\Gamma (\beta) (\beta - \delta_2)\) for \(t \in [0,1]\) and \(n \geq 1\). Moreover,

\[
D^p_0 u_n (t) = \frac {1} {\Gamma (2 - p)} \int_0^t (t - s)^{1-p} u_n'' (s) ds
\]
\[
\geq \frac {m_1 (\delta_1 - 2)} {\Gamma (2 - q) \Gamma (\alpha - 1) (\alpha - \delta_1)} \int_0^t (t - s)^{1-q} s^{\alpha-3} ds,
\]
\[
D^q_0 v_n (t) = \frac {1} {\Gamma (2 - q)} \int_0^t (t - s)^{1-q} v_n'' (s) ds
\]
\[
\geq \frac {m_2 (\delta_2 - 2)} {\Gamma (2 - q) \Gamma (\beta - 1) (\beta - \delta_2)} \int_0^t (t - s)^{1-q} s^{\beta-3} ds,
\]
\[
D^{\mu}_0 u_n (t)
\]
\[
\geq \frac {m_1 (\delta_1 - 2)} {\Gamma (2 - q) \Gamma (\alpha - 1) (\alpha - \delta_1)} \int_0^t (t - s)^{1-q} s^{\alpha-3} ds,
\]
\[
D^{\nu}_0 v_n (t)
\]
\[
\geq \frac {m_2 (\delta_2 - 2)} {\Gamma (2 - q) \Gamma (\beta - 1) (\beta - \delta_2)} \int_0^t (t - s)^{1-q} s^{\beta-3} ds,
\]
\[
\begin{align*}
\int_0^t (t-s)^{-\gamma} s^{\beta-3} ds & \geq \frac{\Gamma(2-q) \Gamma(\beta-2)}{\Gamma(\beta-q)} t^{\beta-\gamma}, \\
\int_0^t (t-s)^{-\nu} s^{\alpha-3} ds & \geq \frac{\Gamma(2-q) \Gamma(\alpha-2)}{\Gamma(\alpha-\nu)} t^{\alpha-\nu}, \\
\int_0^t (t-s)^{-\mu} s^{\beta-3} ds & \geq \frac{\Gamma(2-q) \Gamma(\beta-2)}{\Gamma(\beta-\mu)} t^{\beta-\mu},
\end{align*}
\]

(37)

for \( i = 1, 2, \ldots, m \), we get \( D^p_{0, t} u_n(t) \geq (m_1(\delta_1-2)/(\alpha-2)(\alpha-\delta_1) \Gamma(\alpha-\beta)) t^{\alpha-\gamma}, D^p_{0, t} v_n(t) \geq (m_2(\delta_2-2)/(\beta-2)(\beta-\delta_2) \Gamma(\beta-\gamma)) t^{\beta-\gamma} \), and \( D^\gamma_{0, t} u_n(t) \geq (m_1(\delta_1-2)/\alpha-2)(\alpha-\delta_1) \Gamma(\alpha-\nu_i)) t^{\alpha-\nu_i} \), and \( D^\gamma_{0, t} v_n(t) \geq (m_2(\delta_2-2)/(\beta-2)(\beta-\delta_2) \Gamma(\beta-\nu_i)) t^{\beta-\nu_i} \) for all \( t \in [0,1] \) and \( n \geq 1 \). Since

\[
m_1 \cdot \min \left\{ \frac{1}{(\alpha-\delta_1) \Gamma(\alpha+1)}, \frac{1}{(\alpha-\delta_1) \Gamma(\alpha+1)} \right\} = M_1
\]

(38)

and \( m_2 \cdot \min \{1/(\beta-\delta_2) \Gamma(\beta+1), 1/(\beta-\delta_2) \Gamma(\beta+1) \} = M_2 \), for each \( t \in [0,1] \) and \( n \geq 1 \), we get \( u_n(t) \geq M_1 t^{\alpha-\gamma}, v_n(t) \geq M_2 t^{\beta-\gamma}, D^\gamma_{0, t} u_n(t) \geq (m_1(\delta_1-2)/\alpha-2)(\alpha-\delta_1) \Gamma(\alpha-\nu_i)) t^{\alpha-\nu_i} \), and \( D^\gamma_{0, t} v_n(t) \geq (m_2(\delta_2-2)/(\beta-2)(\beta-\delta_2) \Gamma(\beta-\nu_i)) t^{\beta-\nu_i} \) for \( i = 1, 2, \ldots, m \) and \( (\phi_i u_n)(t) \geq M_1 \int_0^t \gamma_i(t,s) s^{\alpha-1} ds \) and \( (\psi_j v_n)(t) \geq M_2 \int_0^t \lambda_j(t,s) s^{\beta-1} ds \) for \( j = 1, 2 \). Thus,

\[
P_1 \left( u_n(t), v_n(t), (\phi_i u_n)(t), (\psi_j v_n)(t) \right),
\]

\[
D^p_{0, t} u_n(t), D^p_{0, t} v_n(t), D^\gamma_{0, t} u_n(t), \ldots, D^\gamma_{0, t} v_n(t)
\]

\[
\leq P_1 \left( M_1 t^{\alpha-1}, M_2 t^{\beta-1}, M_1 \int_0^t \gamma_i(t,s) s^{\alpha-1} ds, \right.
\]

\[
M_2 \int_0^t \lambda_j(t,s) s^{\beta-1} ds, \quad \gamma_i(t,s) = \frac{\Gamma(\beta-\mu_i)}{\Gamma(\beta-\mu_i)} t^{\beta-\gamma},
\]

\[
= \frac{\Gamma(\beta-\mu_i)}{\Gamma(\beta-\mu_i)}, \quad (\delta_2-2) M_2 t^{\beta-\mu_2}, \ldots,
\]

\[
(\beta-2) M_2 t^{\beta-\mu_m} \bigg) ds
\]

\[
+ h_1 \left( 1 + \|u_n\|_s, 1 + \|v_n\|_s, \right.
\]

\[
1 + \|\phi_i u_n\|_s, 1 + \|\psi_j v_n\|_s,
\]

for all \( t \in [0,1] \) and \( n \geq 1 \) and so

\[
0 \leq u_n''(t)
\]

\[
= \int_0^1 \frac{\partial^2}{\partial t^2} G_{a}(t,s) f_n(s, u_n(s), v_n(s), \phi_i u_n(s), \psi_j v_n(s), D^p_{0, t} u_n(s), D^p_{0, t} v_n(s), M_1 \int_0^s \gamma_i(s, \tau) \tau^{\alpha-1} d\tau, M_2 \int_0^s \lambda_j(s, \tau) \tau^{\beta-1} d\tau, (\delta_2-2) M_2 s^{\beta-\mu_2}, \ldots, (\beta-2) M_2 s^{\beta-\mu_m} \bigg) ds
\]

\[
\leq \frac{1}{\Gamma(\alpha-2)} \left[ \int_0^1 \frac{\partial^2}{\partial t^2} G_{a}(t,s) f_n(s, u_n(s), v_n(s), \phi_i u_n(s), \psi_j v_n(s), D^p_{0, t} u_n(s), D^p_{0, t} v_n(s), M_1 \int_0^s \gamma_i(s, \tau) \tau^{\alpha-1} d\tau, M_2 \int_0^s \lambda_j(s, \tau) \tau^{\beta-1} d\tau, (\delta_2-2) M_2 s^{\beta-\mu_2}, \ldots, (\beta-2) M_2 s^{\beta-\mu_m} \bigg) ds
\]

\[
+ h_1 \left( 1 + \|u_n\|_s, 1 + \|v_n\|_s, \right.
\]

\[
1 + \|\phi_i u_n\|_s, 1 + \|\psi_j v_n\|_s,
\]
1 + \frac{\|u_n\|_*}{\Gamma(3-p)}, 1 + \frac{\|v_n\|_*}{\Gamma(3-\mu_1)}, \\
1 + \frac{\|v_n\|_*}{\Gamma(3-\mu_2)}, \ldots, 1 + \frac{\|v_n\|_*}{\Gamma(3-\mu_m)}, \\
\times \int_0^1 k_1(s) \, ds.
\]

for all $t \in [0,1]$ and $n \geq 1$ and also $\Lambda_1 < \infty$, where

$$\Lambda_1 = \int_0^1 P_1 \left(M_1 s^{\alpha-1}, M_2 s^{\beta-1}, \\
M_1 \int_0^t \gamma_1(s, \tau) \tau^{\alpha-1} \, d\tau, \\
M_2 \int_0^t \lambda_1(s, \tau) \tau^{\beta-1} \, d\tau, \\
(\delta_1 - 2) M_1 s^{\alpha-p}, (\delta_2 - 2) M_2 s^{\beta-\mu_1}, \\
\Gamma(\alpha-p), (\delta_2 - 2) M_2 s^{\beta-\mu}, \ldots, \\
\Gamma(\beta-\mu_m) \right) \, ds.$$
\[
\frac{1}{\Gamma(\alpha - 2)} \left( A_1 + h_1 \left( 1 + \| (u_n, v_n) \|_{*}, 1 + \lambda_1^0 \| (u_n, v_n) \|_{**}, \right.ight.
\]
\[
1 + \lambda_2^0 \| (u_n, v_n) \|_{**}, 1 + \| (u_n, v_n) \|_{**}, \Gamma(3 - p), 1 + \| (u_n, v_n) \|_{**}, \Gamma(3 - \mu_2), \ldots, 1 + \| (u_n, v_n) \|_{**}, \Gamma(3 - \mu_m)) \| k_1 \|_1 \right) = G_n^3
\]
(42)

for all \( n \geq 1 \). Similarly, we have
\[
\| v_n \|_* \leq \frac{1}{\Gamma(\beta - 2)} \left( A_2 + h_2 \left( 1 + \| (u_n, v_n) \|_{*}, 1 + \| (u_n, v_n) \|_{**}, \right.ight.
\]
\[
1 + \lambda_2^0 \| (u_n, v_n) \|_{**}, 1 + \| (u_n, v_n) \|_{**}, \Gamma(3 - q), 1 + \| (u_n, v_n) \|_{**}, \Gamma(3 - \nu_2), \ldots, 1 + \| (u_n, v_n) \|_{**}, \Gamma(3 - \nu_m)) \| k_2 \|_1 \right) \leq G_n^2
\]
(43)

for all \( n \geq 1 \), where
\[
A_2 = \int_0^1 P_2 \left( M_1 s^{\alpha - 1}, M_2 s^{\beta - 1}, \right.
\]
\[
M_1 \int_0^t \gamma_2 (s, \tau) \tau^{\alpha - 1} d\tau,\right.
\]
\[
M_2 \int_0^t \lambda_2 (s, \tau) \tau^{\beta - 1} d\tau, \right.
\]
\[
(\delta_2 - 2) M_2 s^{\beta - q}, \left. \Gamma(\beta - q) \right)\]

Thus, \( \| (u_n, v_n) \|_{**} \leq \max(G_n^1, G_n^2) \) for all \( n \geq 1 \). Since \( \lim_{x \to \infty} f_j(x, x, \ldots, x) / x = 0 \) for \( j = 1, 2 \), there exists \( L > 0 \) such that
\[
\frac{1}{\Gamma(\alpha - 2)} \left( A_1 + h_1 \left( 1 + v, 1 + v, 1 + \gamma_0^1 v, \right. \right.
\]
\[
1 + \lambda_1^0 v, 1 + \frac{v}{\Gamma(3 - p)}, \right.
\]
\[
1 + \frac{v}{\Gamma(3 - \mu_1)}, \right. \right.
\]
\[
1 + \frac{v}{\Gamma(3 - \mu_2)}, \ldots, \right.
\]
\[
1 + \frac{v}{\Gamma(3 - \mu_m))} \| k_1 \|_1 < v,
\]
(44)

for all \( v \geq L \). Consequently, \( \| (u_n, v_n) \|_{**} < L \) for all \( n \geq 1 \) and so \( \{(u_n, v_n)_{n \geq 1} \) is a bounded sequence in \( X \times X \). It remains to prove that \( \{(u_n, v_n)_{n} \) is equicontinuous on \([0, 1]\). Put
\[
V_1 = h_1 \left( 1 + L, 1 + L, 1 + \gamma_0^1 L, \right.
\]
\[
1 + \lambda_1^0 L, 1 + \frac{L}{\Gamma(3 - p)}, 1 + \frac{L}{\Gamma(3 - \mu_1)}, \right.
\]
\[
1 + \frac{L}{\Gamma(3 - \mu_2)}, \ldots, 1 + \frac{L}{\Gamma(3 - \mu_m)} \right) \right).
\]
\[ V_2 = h_2 \left( 1 + L, 1 + L, 1 + \gamma_0^2 L \right), \]
\[ 1 + \lambda_0^2 L, 1 + \frac{L}{\Gamma (3 - q)}, 1 + \frac{L}{\Gamma (3 - \nu_1)} \],
\[ 1 + \frac{L}{\Gamma (3 - \nu_2)}, \ldots, 1 + \frac{L}{\Gamma (3 - \nu_m)} \),
\[ \Phi_1 (t) = P_1 \left( M_1 t^{\alpha-1}, M_2 t^{\beta-1}, \right) \]
\[ M_1 \int_0^t \gamma_1 (t, s) s^{\alpha-1} ds, \]
\[ M_2 \int_0^t \lambda_1 (t, s) s^{\beta-1} ds \]
\[ \left( \frac{\delta_2 - 2}{\Gamma (\beta - q)} M_2 t^{\beta-q} \right) M_1 t^{\alpha-\gamma}, \]
\[ \left( \frac{\delta_2 - 2}{\Gamma (\beta - q)} M_2 t^{\beta-q} \right), \ldots, \left( \frac{\delta_2 - 2}{\Gamma (\beta - q)} M_2 t^{\beta-q} \right) \]
\[ \phi_1 \left( \frac{\delta_2 - 2}{\Gamma (\beta - q)} M_2 t^{\beta-q} \right) \]
\[ (\psi_1, v_n (t)), (\phi_1 u_n (t)), (\psi_1, D_0^\mu u_n (t)) \]
\[ D_0^\mu v_n (t), \ldots, D_0^{\mu_n} v_n (t) \]
\[ \leq \Phi_1 (t) + V_1 k_1 (t) \]
\[ f_2 (t, u_n (t), v_n (t), \phi_2 u_n (t)), \]
\[ (\psi_2, v_n (t)), (\phi_2 u_n (t)), (\psi_2, D_0^\mu u_n (t)) \]
\[ D_0^\mu v_n (t), \ldots, D_0^{\mu_n} u_n (t) \]
\[ \leq \Phi_2 (t) + V_2 k_2 (t) \]

for all \( t \in (0, 1] \). Note that, \( \Lambda_j = \int_0^1 \Phi_j (t) dt \) for \( j = 1, 2 \) and
\[ f_1 (t, u_n (t), v_n (t), (\phi_1 u_n (t)), \]
\[ (\psi_1, v_n (t)), (\phi_1 u_n (t)), \]
\[ (\psi_1, D_0^\mu u_n (t)), (\psi_1, v_n (t)), (\phi_1 u_n (t)), \]
\[ D_0^\mu v_n (t), \ldots, D_0^{\mu_n} u_n (t) \]
\[ \leq \Phi_1 (t) + V_1 k_1 (t) \]
\[ f_2 (t, u_n (t), v_n (t), (\phi_2 u_n (t)), \]
\[ (\psi_2, v_n (t)), (\phi_2 u_n (t)), (\psi_2, D_0^\mu u_n (t)) \]
\[ D_0^\mu v_n (t), \ldots, D_0^{\mu_n} u_n (t) \]
\[ \leq \Phi_2 (t) + V_2 k_2 (t) \]
Now, we give our main result.

**Theorem 6.** The system (1), (2), and (3) has a positive solution \((u, v)\) such that \(u(t) \geq M_1 t^{\beta - 1}, v(t) \geq M_2 t^{\beta - 1}, D^\rho_0 u(t) \geq ((\delta_1 - 2)M_1 / \Gamma(\alpha - p)) t^{\alpha - p}, D^\rho_0 v(t) \geq ((\delta_2 - 2)M_2 / \Gamma(\beta - q)) t^{\beta - q}, D^\gamma_0 u(t) \geq ((\delta_1 - 2)M_1 / \Gamma(\alpha - \mu_1)) t^{\alpha - \mu_1},\) and \(D^\gamma_0 v(t) \geq ((\delta_2 - 2)M_2 / \Gamma(\beta - \mu_2)) t^{\beta - \mu_2} for \(i = 1, 2, \ldots, m\) and \(t \in [0, 1]\) and \((\phi, \psi)(t) \geq M_1 \int_0^t \lambda_j(t, s) s^{\alpha - 1} \, ds\) and

\[
(\psi_j v)(t) \geq M_2 \int_0^t \lambda_j(t, s) s^{\beta - 1} \, ds \tag{51}
\]

for \(j = 1, 2\) and \(t \in [0, 1].\)

**Proof.** By using Theorem 4, for each natural number \(n\), the system (15), (2), and (3) has a solution \((u_n, v_n)\) in \(P\). Also by using Lemma 5, the set \(\{(u_n, v_n)\}_{n \geq 1}\) is a relatively compact subset of \(X \times X\). By using the Arzela-Ascoli theorem, without loss of generality we can assume that \(\{(u_n, v_n)\}_{n \geq 1}\) is convergent in \(X \times X\) to some element \((u, v)\) of \(P\). It is easy to check that \((u, v)\) satisfy the boundary conditions (2) and (3) and also \(\lim_{n \to \infty} D^\rho_0 u_n = D^\rho_0 u, \lim_{n \to \infty} D^\rho_0 v_n = D^\rho_0 v, \lim_{n \to \infty} D^\rho_0 u_n = D^\rho_0 u,\) and \(\lim_{n \to \infty} D^\rho_0 v_n = D^\rho_0 v\) for \(i = 1, 2, \ldots, m\) and \(\lim_{n \to \infty} \phi_j u_n = \phi_j u\) and \(\lim_{n \to \infty} \psi_j v_n = \psi_j v\) for \(j = 1, 2\). Thus, it is easy to see that \((u, v)\) satisfy the desired conditions.

Also,

\[
\lim_{n \to \infty} f^1_n \left( t, u_n(t), v_n(t), \phi_j u_n(t) \right)
\]

\[
(\psi_1 v_n(t), D^\rho_0 u_n(t), D^{\rho_0} v_n(t), D^{\mu_1} u_n(t), D^{\mu_1} v_n(t))
\]

\[
= f_1 \left( t, u(t), v(t), \phi_j u(t), \psi_1 v(t) \right),
\]

\[
D^\rho_0 u(t), D^{\rho_0} v(t), D^{\mu_1} u(t), D^{\mu_1} v(t),
\]

\[
D^{\gamma_2} u_n(t), D^{\gamma_2} v_n(t), \ldots, D^{\gamma_2} v_n(t)) \tag{52}
\]

\[
\lim_{n \to \infty} f^2_n \left( t, u_n(t), v_n(t), \phi_j u_n(t) \right)
\]

\[
(\psi_2 v_n(t), D^\rho_0 u_n(t), D^{\rho_0} v_n(t), D^{\mu_2} u_n(t), D^{\mu_2} v_n(t)),
\]

\[
D^\rho_0 u_n(t), \ldots, D^{\mu_2} u_n(t),
\]

\[
D^{\gamma_2} u_n(t), \ldots, D^{\gamma_2} u_n(t)
\]

\[
= f_2 \left( t, u(t), v(t), \phi_j u(t), \psi_2 v(t) \right),
\]

\[
(\psi_2 v(t), D^\rho_0 u(t), D^{\rho_0} v(t), D^{\mu_2} u(t), D^{\mu_2} v(t)),
\]

\[
D^\rho_0 u(t), \ldots, D^{\mu_2} u(t),
\]

\[
D^{\gamma_2} u(t), \ldots, D^{\gamma_2} u(t) \tag{53}
\]

for almost all \((t, s) \in [0, 1] \times [0, 1]\) and \(n \geq 1\). Hence, by using the Lebesgue dominated convergence theorem we get

\[
u(t) = \int_0^1 G_a(t, s) f_1 \left( (s, u(s), v(s), \phi_j u(s), \psi_1 v(s)) \right)
\]

\[
D^\rho_0 u(s), D^{\rho_0} v(s), \ldots,
\]

\[
D^{\mu_2} v(s) \, ds,
\]

\[n \geq 1 \|\phi_j u_n\| \leq \xi \gamma_j and \|\psi_j v_n\| \leq \xi \lambda_j for j = 1, 2 and n \geq 1.\]

\[
0 \leq G_a(t, s) f^1_n \left( (s, u_n(s), v_n(s), \phi_j u_n(s), \psi_1 v_n(s)) \right)
\]

\[
D^\rho_0 u_n(s), D^{\rho_0} v_n(s), \ldots,
\]

\[
D^{\mu_2} v_n(s) \right) \right)
\]

\[
\leq \frac{1}{\Gamma(\alpha)} \left( \Phi_1(s) + h_1 \left( 1 + \xi, \right.ight.
\]

\[
1 + \xi, 1 + \gamma_0 \xi, 1 + \lambda_0 \xi,
\]

\[
1 + \frac{\xi}{\Gamma(3 - p)}, 1 + \frac{\xi}{\Gamma(3 - \mu_1)}
\]

\[
1 + \frac{\xi}{\Gamma(3 - \mu_2)}, \ldots,
\]

\[
1 + \frac{\xi}{\Gamma(3 - \mu_m)} \right) k_1(s), \tag{53}
\]

\[
0 \leq G_a(t, s) f^2_n \left( (s, u_n(s), v_n(s), \phi_j u_n(s), \psi_2 v_n(s)) \right)
\]

\[
D^\rho_0 u_n(s), D^{\rho_0} v_n(s), \ldots,
\]

\[
D^{\mu_2} u_n(s) \right) \right)
\]

\[
= \frac{1}{\Gamma(\beta)} \left( \Phi_2(s) + h_2 \left( 1 + \xi, \right.ight.
\]

\[
1 + \xi, 1 + \gamma_0 \xi, 1 + \lambda_0 \xi,
\]

\[
1 + \frac{\xi}{\Gamma(3 - q)}, 1 + \frac{\xi}{\Gamma(3 - v_1)}
\]

\[
1 + \frac{\xi}{\Gamma(3 - v_2)}, \ldots,
\]

\[
1 + \frac{\xi}{\Gamma(3 - v_m)} \right) k_2(s) \tag{53}\)

for almost all \((t, s) \in [0, 1] \times [0, 1]\) and \(n \geq 1\). Hence, by using the Lebesgue dominated convergence theorem we get

\[
u(t) = \int_0^1 G_a(t, s) f_1 \left( (s, u(s), v(s), \phi_j u(s), \psi_1 v(s)) \right)
\]

\[
D^\rho_0 u(s), D^{\rho_0} v(s), \ldots,
\]

\[
D^{\mu_2} v(s) \, ds,
\]
for all $t \in [0, 1]$. Therefore, $(u, v)$ is a positive solution of the system (1), (2), and (3).

4. Example

Here, we give an example to illustrate our last result.

Example 1. Let $m_1 > 0$, $m_2 > 0$, $n \geq 4$, $n - 1 < \alpha, \beta < n$, $0 < p, q < 1$, $1 < \mu, \gamma < 2$ for $i = 1, 2, \ldots, m$, $\rho_1, \rho_2, \sigma_1, \sigma_2 \in L^1[0, 1]$, $\sigma_1(t) \geq m_1$ and $\sigma_2(t) \geq m_2$ for almost all $t \in [0, 1]$. Suppose that $a_i, c_i \in (0, 1/(\alpha - 1)), a_2, c_2 \in (0, 1/(\beta - 1)), a_3 \in (0, 1/(\alpha - p)), c_3 \in (0, 1/(\beta - q)), d_i \in (0, 1/(\alpha - \beta)), \gamma \in (0, 1/(\alpha - \gamma))$ for $i = 1, 2, \ldots, m$ and also $b_1, b_2, b_3, b_4, b_i, b'_i, b''_i, \ldots, b'_m, d_i, d_2, d_3, d_4, d_5, d'_i, d'_2, \ldots, d'_m \in (0, 1)$. Assume that $a_3, a_4, c_3$ and $c_4$ are positive real numbers such that $\int_0^1 \left(\int_0^1 y_1(s, \tau) t^{\alpha - 1} d\tau\right)^{-a_3} ds < \infty$, $\int_0^1 \left(\int_0^1 y_2(s, \tau) t^{\alpha - 1} d\tau\right)^{-a_3} ds < \infty$, $\int_0^1 \left(\int_0^1 y_3(s, \tau) t^{\alpha - 1} d\tau\right)^{-a_3} ds < \infty$, and $\int_0^1 \left(\int_0^1 y_4(s, \tau) t^{\alpha - 1} d\tau\right)^{-a_3} ds < \infty$. Define the functions $f_1$ and $f_2$ on $[0, 1] \times \mathbb{D}$ by

$$f_1(t, x, y, z, w, u_1, u_2, \ldots, u_m) = \frac{1}{x^{\gamma_1}} + \frac{1}{y^{\gamma_2}} + \frac{1}{z^{\gamma_3}} + \frac{1}{w^{\gamma_4}} + \frac{1}{u_1^{\gamma_5}} + \frac{1}{u_2^{\gamma_6}} + \cdots + \frac{1}{u_m^{\gamma_m}} + |\rho_1(t)| \left(x^{\beta_1} + y^{\beta_2} + z^{\beta_2} + w^{\beta_3} + u_1^{\beta_4} + u_2^{\beta_5} + \cdots + u_m^{\beta_m}\right) + \sigma_1(t),$$

and

$$f_2(t, x, y, z, w, u_1, u_2, \ldots, u_m) = \frac{1}{x^{\gamma_1}} + \frac{1}{y^{\gamma_2}} + \frac{1}{z^{\gamma_3}} + \frac{1}{w^{\gamma_4}} + \frac{1}{u_1^{\gamma_5}} + \frac{1}{u_2^{\gamma_6}} + \cdots + \frac{1}{u_m^{\gamma_m}} + |\rho_2(t)| \left(x^{\beta_1} + y^{\beta_2} + z^{\beta_2} + w^{\beta_3} + u_1^{\beta_4} + u_2^{\beta_5} + \cdots + u_m^{\beta_m}\right) + \sigma_2(t).$$

Note that the functions $f_1$ and $f_2$ satisfy the conditions (H_1) and (H_2), where

$$P_1(x, y, z, w, u_1, u_2, \ldots, u_m) = \frac{1}{x^{\gamma_1}} + \frac{1}{y^{\gamma_2}} + \frac{1}{z^{\gamma_3}} + \frac{1}{w^{\gamma_4}} + \frac{1}{u_1^{\gamma_5}} + \frac{1}{u_2^{\gamma_6}} + \cdots + \frac{1}{u_m^{\gamma_m}},$$

$$P_2(x, y, z, w, u_1, u_2, \ldots, u_m) = \frac{1}{x^{\gamma_1}} + \frac{1}{y^{\gamma_2}} + \frac{1}{z^{\gamma_3}} + \frac{1}{w^{\gamma_4}} + \frac{1}{u_1^{\gamma_5}} + \frac{1}{u_2^{\gamma_6}} + \cdots + \frac{1}{u_m^{\gamma_m}},$$

$$h_1(x, y, z, w, u_1, u_2, \ldots, u_m) = 1 + x^{\delta_1} + y^{\delta_2} + z^{\delta_3} + w^{\delta_4} + u_1^{\delta_5} + u_2^{\delta_6} + \cdots + u_m^{\delta_m},$$

and

$$h_2(x, y, z, w, u_1, u_2, \ldots, u_m) = 1 + x^{\delta_1} + y^{\delta_2} + z^{\delta_3} + w^{\delta_4} + u_1^{\delta_5} + u_2^{\delta_6} + \cdots + u_m^{\delta_m},$$

where $k_1(t) = |\rho_1(t)| + \sigma_1(t)$ and $k_2(t) = |\rho_2(t)| + \sigma_2(t)$. Theorem 6 guarantees that the system (1), (2), and (3) via these functions has a positive solution $(u, v)$ satisfying the desired inequalities in Theorem 6 whenever $M_1 = m_1/(\alpha - \delta_1)\Gamma(\alpha + 1)$ and $M_2 = m_2/(\beta - \delta_2)\Gamma(\beta + 1)$.

5. Conclusions

One of the most interesting branches is obtaining solutions of singular fractional differential equations via boundary value problems. Having these thought in mind we discuss the existence of positive solutions for a coupled system of multiterm singular fractional integrodifferential boundary value problems. An illustrative example illustrates the applicability of the proposed method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.
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