Research Article

The Intersection of Upper and Lower Semi-Browder Spectrum of Upper-Triangular Operator Matrices

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Received 22 January 2013; Accepted 3 August 2013

Academic Editor: Alberto Fiorenza

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When $A \in B(H)$ and $B \in B(K)$ are given, we denote by M_C the operator acting on the infinite-dimensional separable Hilbert space $H \oplus K$ of the form $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. In this paper, it is proved that there exists some operator $C \in B(K, H)$ such that M_C is upper semi-Browder if and only if there exists some left invertible operator $C \in B(K, H)$ such that M_C is upper semi-Browder. Moreover, a necessary and sufficient condition for M_C to be upper semi-Browder for some $C \in G(K, H)$ is given, where $G(K, H)$ denotes the subset of all of the invertible operators of $B(K, H)$.

1. Introduction

It is well known that if H is a Hilbert space, T is a bounded linear operator defined on H , and H_1 is an invariant closed subspace of T , then T can be represented in the following form:

$$T = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : H_1 \oplus H_1^\perp \longrightarrow H_1 \oplus H_1^\perp, \quad (1)$$

which motivated the interest in 2×2 upper-triangular operator matrices. For recent investigations on this subject, see references [1–23].

Throughout this paper, let H and K be separable infinite-dimensional complex Hilbert spaces, and let $B(H, K)$ be the set of all bounded linear operators from H into K ; when $H = K$, we write $B(H, H)$ as $B(H)$. For $A \in B(H)$, $B \in B(K)$, and $C \in B(K, H)$, we have $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in B(H \oplus K)$. For $T \in B(H, K)$, let $R(T)$ and $N(T)$ denote the range and the kernel of T , respectively, and denote that $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim K/R(T)$. If $T \in B(H)$, the ascent $\text{asc}(T)$ of T is defined to be the smallest nonnegative integer k which satisfies $N(T^k) = N(T^{k+1})$. If such k does not exist, then the ascent of T is defined as infinity. Similarly, the descent $\text{des}(T)$ of T is defined as the smallest nonnegative integer k for which $R(T^k) = R(T^{k+1})$ holds. If such k does not exist,

then $\text{des}(T)$ is defined as infinity, too. If the ascent and the descent of T are finite, then they are equal (see [6]). For $T \in B(H)$, if $R(T)$ is closed and $\alpha(T) < \infty$, then T is said to be an upper semi-Fredholm operator; if $\beta(T) < \infty$, which implies that $R(T)$ is closed, then T is said to be a lower semi-Fredholm operator. If $T \in B(H)$ is either upper or lower semi-Fredholm operator, then T is said to be a semi-Fredholm operator. If both $\alpha(T) < \infty$ and $\beta(T) < \infty$, then T is said to be a Fredholm operator. For a semi-Fredholm operator T , its index $\text{ind}(T)$ is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$.

For a semi-Fredholm operator $T \in B(H)$, its shift Samuel multiplicity $s_mul(T)$ and backward shift Samuel multiplicity $b.s._mul(T)$ are defined, respectively, by the following (see [24]):

$$\begin{aligned} s_mul(T) &= \lim_{k \rightarrow \infty} \frac{\beta(T^k)}{k}, \\ b.s._mul(T) &= \lim_{k \rightarrow \infty} \frac{\alpha(T^k)}{k}. \end{aligned} \quad (2)$$

Moreover, it has been proved that $s_mul(T), b.s._mul(T) \in \{0, 1, 2, \dots, \infty\}$ and that $\text{ind}(T) = b.s._mul(T) - s_mul(T)$. These two invariants refine the Fredholm index and can be regarded as the stabilized dimensions of the kernel and the cokernel (see [24]).

In this paper, the sets of invertible operators and left invertible operators from H into K are denoted by $G(H, K)$ and $G_l(H, K)$, respectively; the sets of all Fredholm operators, upper semi-Fredholm operators, and lower semi-Fredholm operators from H into K are denoted by $\Phi(H, K)$, $\Phi_+(H, K)$, and $\Phi_-(H, K)$, respectively; the sets of all Browder operators, upper semi-Browder operators, and lower semi-Browder operators, on H are defined, respectively, by the following:

$$\begin{aligned}\Phi_b(H) &:= \{T \in \Phi(H) : \text{asc}(T) = \text{des}(T) < \infty\}, \\ \Phi_{ab}(H) &:= \{T \in \Phi_+(H) : \text{asc}(T) < \infty\}, \\ \Phi_{sb}(H) &:= \{T \in \Phi_-(H) : \text{des}(T) < \infty\}.\end{aligned}\quad (3)$$

Moreover, for $T \in B(H)$, we introduce its corresponding spectra as follows.

The spectrum is given as $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin G(H)\}$.

The left spectrum is given as $\sigma_l(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin G_l(H)\}$.

The essential spectrum is defined as $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi(H)\}$.

The upper semi-Fredholm spectrum is defined as $\sigma_{SF+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_+(X)\}$.

The lower semi-Fredholm spectrum is presented as $\sigma_{SF-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_-(X)\}$.

The Browder spectrum is presented as $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_b(H)\}$.

The upper semi-Browder spectrum is defined as $\sigma_{ab}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_{ab}(X)\}$.

The lower semi-Browder spectrum is presented as $\sigma_{sb}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_{sb}(X)\}$.

Using the Samuel multiplicities, Zhang and Wu (see [20]) gave a necessary and sufficient condition for which $M_C \in \Phi_{ab}(H \oplus K)$ for some $C \in B(K, H)$ and characterized the set of $\cap_{C \in B(K, H)} \sigma_{ab}(M_C)$. In this paper, our main goal is to characterize the intersection of $\cap_{C \in G_l(K, H)} \sigma_{ab}(M_C)$ and $\cap_{C \in G(K, H)} \sigma_{ab}(M_C)$. This paper is organized as follows. In Section 2, we give a necessary and sufficient condition for which $M_C \in \Phi_{ab}(H \oplus K)$ for some $C \in G_l(K, H)$ and get

$$\bigcap_{C \in G_l(K, H)} \sigma_{ab}(M_C) = \bigcap_{C \in B(K, H)} \sigma_{ab}(M_C). \quad (4)$$

In Section 3, we give a necessary and sufficient condition for which $M_C \in \Phi_{ab}(H \oplus K)$ for some $C \in G(K, H)$ and get

$$\begin{aligned}\bigcap_{C \in G(K, H)} \sigma_{ab}(M_C) &= \bigcap_{C \in \Phi(K, H)} \sigma_{ab}(M_C) \\ &= \left(\bigcap_{C \in B(K, H)} \sigma_{ab}(M_C) \right) \\ &\cup \{\lambda \in \mathbb{C} : B - \lambda \text{ is compact}\}.\end{aligned}\quad (5)$$

For the sake of convenience, we now present some lemmas which will be used in the sequel.

Lemma 1 (see [20, 24]). *An operator $T \in B(H)$ is semi-Fredholm if and only if T can be decomposed into the following form with respect to some orthogonal decomposition $H = H_1 \oplus H_2 \oplus H_3$:*

$$T = \begin{pmatrix} T_1 & T_{12} & T_{13} \\ 0 & T_2 & T_{23} \\ 0 & 0 & T_3 \end{pmatrix} : H_1 \oplus H_2 \oplus H_3 \longrightarrow H_1 \oplus H_2 \oplus H_3, \quad (6)$$

where $\dim(H_3) < \infty$, T_1 is a right invertible operator, T_3 is a finite nilpotent operator, T_2 is a left invertible operator, and $\min\{\text{ind}(T_1), -\text{ind}(T_2)\} < \infty$. Moreover, $\text{ind}(T_1) = \alpha(T_1) = b.s._-\text{mul}(T)$, $\text{ind}(T_2) = -\beta(T_2) = -s._-\text{mul}(T)$, and $\text{ind}(T) = \alpha(T_1) - \beta(T_2)$.

Lemma 2 (see [18]). *Let $A \in B(H)$, $B \in B(K)$, and $C \in B(K, H)$.*

- (1) *If $A \in \Phi_b(H)$, then $B \in \Phi_{ab}(K)$ if $M_C \in \Phi_{ab}(H \oplus K)$ for some $C \in B(K, H)$.*
- (2) *If $M_C \in \Phi_{ab}(H \oplus K)$ for some $C \in B(K, H)$, then $A \in \Phi_{ab}(H)$.*
- (3) *If $A \in \Phi_{ab}(H)$ and $B \in \Phi_{ab}(K)$, then $M_C \in \Phi_{ab}(H \oplus K)$ for any $C \in B(K, H)$.*
- (4) *If $B \in \Phi_b(K)$, then $A \in \Phi_{ab}(H)$ if $M_C \in \Phi_{ab}(H \oplus K)$ for some $C \in B(K, H)$; $A \in \Phi_{sb}(H)$ if $M_C \in \Phi_{sb}(H \oplus K)$ for some $C \in B(K, H)$.*
- (5) *If $M_C \in \Phi_b(H \oplus K)$ for some $C \in B(K, H)$, then $A \in \Phi_{ab}(H)$ and $B \in \Phi_{sb}(K)$.*
- (6) *If two of A , B , and M_C are Browder, then so is the third.*

Lemma 3 (see [20]). *Let $T \in B(H)$. Then, T is upper semi-Browder if T can be decomposed into the following form with respect to some orthogonal decomposition $H = H_1 \oplus H_2$:*

$$T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix}, \quad (7)$$

where $\dim(H_1) < \infty$, T_1 is nilpotent, T_2 is left invertible, and $\beta(T_2) = s._-\text{mul}(T) = -\text{ind}(T)$.

Lemma 4 (see [20]). *Let $T \in B(H)$. Then, T is lower semi-Browder if T can be decomposed into the following form with respect to some orthogonal decomposition $H = H_1 \oplus H_2$:*

$$T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix}, \quad (8)$$

where $\dim(H_2) < \infty$, T_1 is right invertible, T_2 is nilpotent, and $\alpha(T_1) = b.s._-\text{mul}(T) = \text{ind}(T)$.

Lemma 5 (see [20]). *For any given $A \in B(H)$ and $B \in B(K)$, $M_C \in \Phi_{ab}(H \oplus K)$ for some $C \in B(K, H)$ if $A \in \Phi_{ab}(H)$ and*

$$\begin{aligned}s._-\text{mul}(A) &= \infty \quad \text{if } B \notin \Phi_+(K), \\ b.s._-\text{mul}(B) &\leq s._-\text{mul}(A) \quad \text{if } B \in \Phi_+(K).\end{aligned}\quad (9)$$

Lemma 6 (see [9]). For any given $A \in B(H)$ and $B \in B(K)$, M_C is left invertible for some $C \in B(K, H)$ if A is left invertible and

$$\begin{aligned} \alpha(B) &\leq \beta(A) \quad \text{if } R(B) \text{ is closed,} \\ \beta(A) &= \infty \quad \text{if } R(B) \text{ is not closed.} \end{aligned} \quad (10)$$

Lemma 7 (see [25]). Let V be a linear subspace of H . Then, the following statements are equivalent.

- (1) Any bounded operator $A \in B(H)$ with $R(A) \subseteq V$ is compact.
- (2) V contains no closed infinite-dimensional subspace.

2. $\bigcap_{C \in B(K, H)} \sigma_{ab}(M_C)$ and $\bigcap_{C \in G_l(K, H)} \sigma_{ab}(M_C)$

In [1, 20], the authors have proved that

$$\begin{aligned} &\bigcap_{C \in B(K, H)} \sigma_b(M_C) \\ &= \sigma_{ab}(A) \cup \sigma_{sb}(B) \cup \{\lambda \in \mathbb{C} : \alpha(A - \lambda) \\ &\quad + \alpha(B - \lambda) \neq \beta(A - \lambda) \\ &\quad + \beta(B - \lambda)\}. \end{aligned} \quad (11)$$

They, moreover, proved that

$$\begin{aligned} &\bigcap_{C \in B(K, H)} \sigma_{ab}(M_C) \\ &= \sigma_{ab}(A) \cup \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SF+}(B), \\ &\quad s_mul(A - \lambda) < \infty\} \\ &\cup \{\lambda \in \Phi(A) \cap \Phi_+(B) : b.s_mul(B - \lambda) \\ &\quad > s_mul(A - \lambda)\}. \end{aligned} \quad (12)$$

Comparing the above two kinds of spectra with the upper semi-Weyl spectrum and Weyl spectrum, one may expect that the following equality holds:

$$\begin{aligned} &\bigcap_{C \in B(K, H)} \sigma_{ab}(M_C) \\ &= \sigma_{ab}(A) \cup \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SF+}(B), \beta(A - \lambda) < \infty\} \\ &\cup \{\lambda \in \mathbb{C} : \alpha(A - \lambda) + \alpha(B - \lambda) \\ &\quad > \beta(A - \lambda) + \beta(B - \lambda)\}. \end{aligned} \quad (13)$$

However, it is not that case, as the following example shows.

Example 8. Let A be the unilateral shift on ℓ^2 , that is,

$$V: \ell^2 \longrightarrow \ell^2, \quad \{z_1, z_2, \dots\} \longmapsto \{0, z_1, z_2, \dots\}, \quad (14)$$

and let the operators A and B be defined by

$$A = V, \quad B = \begin{pmatrix} (V^*)^2 & 0 \\ 0 & V^5 \end{pmatrix}: \ell^2 \oplus \ell^2 \longrightarrow \ell^2 \oplus \ell^2. \quad (15)$$

Then, we have $b.s_mul(B) = 2 > s_mul(A) = 1$, while $\alpha(A) + \alpha(B) = 2 < \beta(A) + \beta(B) = 6$. Moreover, $0 \in \bigcap_{C \in B(K, H)} \sigma_{ab}(M_C)$, while $0 \notin \sigma_{ab}(A) \cup \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SF+}(B) \text{ and } \beta(A - \lambda) < \infty\} \cup \{\lambda \in \mathbb{C} : \alpha(A - \lambda) + \alpha(B - \lambda) > \beta(A - \lambda) + \beta(B - \lambda)\}$. Thus, (13) does not hold.

In spite of the above counter example, we have the following.

Proposition 9. For any given $A \in B(H)$ and $B \in B(K)$, one has

$$\begin{aligned} &\bigcap_{C \in B(K, H)} \sigma_{ab}(M_C) \supseteq \sigma_{ab}(A) \\ &\cup \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SF+}(B), \beta(A - \lambda) < \infty\} \\ &\cup \{\lambda \in \mathbb{C} : \alpha(A - \lambda) + \alpha(B - \lambda) \\ &\quad > \beta(A - \lambda) + \beta(B - \lambda)\}. \end{aligned} \quad (16)$$

Proof. From the proof of Theorem 2.3 in [20], we know that when $A \in \Phi_{ab}(H)$, $s_mul(A) < \infty$ if and only if $\beta(A) < \infty$. Combining this fact with Corollary 2.5 of [20], it is easy to see that

$$\begin{aligned} &\bigcap_{C \in B(K, H)} \sigma_{ab}(M_C) \supseteq \sigma_{ab}(A) \cup \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SF+}(B), \\ &\quad \beta(A - \lambda) < \infty\}. \end{aligned} \quad (17)$$

Noting that $\beta(B - \lambda) < \infty$ implies that $R(B - \lambda)$ is closed, it follows from corollary 2.5 of [2] that

$$\begin{aligned} &\bigcap_{C \in B(K, H)} \sigma_{ab}(M_C) \supseteq \bigcap_{C \in B(K, H)} \sigma_{aw}(M_C) \\ &\supseteq \{\lambda \in \mathbb{C} : \alpha(A - \lambda) + \alpha(B - \lambda) \\ &\quad > \beta(A - \lambda) + \beta(B - \lambda)\}, \end{aligned} \quad (18)$$

where $\sigma_{aw}(M_C) = \{\lambda \in \mathbb{C} : M_C - \lambda \text{ is not uppersemi-Fredholm operator with index less than or equal to } 0\}$. $\square$

Now, we are ready to present the main result of this section.

Theorem 10. For any given $A \in B(H)$ and $B \in B(K)$, one has

$$\bigcap_{C \in G_l(K, H)} \sigma_{ab}(M_C) = \bigcap_{C \in B(K, H)} \sigma_{ab}(M_C). \quad (19)$$

Proof. Since $\bigcap_{C \in G_l(K, H)} \sigma_{ab}(M_C) \supseteq \bigcap_{C \in B(K, H)} \sigma_{ab}(M_C)$ is obvious, it is sufficient to prove that if $M_C \in \Phi_{ab}(H \oplus K)$, then there exists some left invertible operator $Q \in B(K, H)$ such that $M_Q \in \Phi_{ab}(H \oplus K)$.

Suppose that $M_C \in \Phi_{ab}(H \oplus K)$. It follows from Lemma 5 that $A \in \Phi_{ab}(H)$ and

$$\begin{aligned} s_mul(A) &= \infty \quad \text{if } B \notin \Phi_+(K), \\ b.s_mul(B) &\leq s_mul(A) \quad \text{if } B \in \Phi_+(K). \end{aligned} \quad (20)$$

There are two cases to consider.

Case 1. Assume that $A \in \Phi_{ab}(H)$, $s_mul(A) = \infty$, and $B \notin \Phi_+(K)$. Then, it follows from Lemma 3 that A can be decomposed into the following form:

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} : H_1 \oplus H_2 \longrightarrow H_1 \oplus H_2, \quad (21)$$

where $\dim(H_1) < \infty$, A_1 is nilpotent, A_2 is a left invertible operator, and $\beta(A_2) = s_mul(A) = \infty$. So, we can let

$$Q = \begin{pmatrix} 0 \\ 0 \\ V \end{pmatrix} : K \longrightarrow H_1 \oplus R(A_2) \oplus (H_2 \ominus R(A_2)), \quad (22)$$

where $V \in B(K, (H_2 \ominus R(A_2)))$ is unitary. Obviously, Q is left invertible. Now, M_Q can be rewritten as

$$M_Q = \begin{pmatrix} A_1 & A_{12} & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & V \\ 0 & 0 & B \end{pmatrix} : H_1 \oplus H_2 \oplus K \longrightarrow H_1 \oplus R(A_2) \oplus (H_2 \ominus R(A_2)) \oplus K. \quad (23)$$

Since A_2 is left invertible and V is invertible, then there exist unique A'_2 and V' such that $A'_2 A_2 = I_{H_2}$ and $V' V = I_K$, and

$$\begin{pmatrix} A'_2 & 0 & 0 \\ 0 & V' & 0 \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ 0 & V \\ 0 & B \end{pmatrix} = I_{H_2} \oplus I_K. \quad (24)$$

This implies that $\begin{pmatrix} A_2 & 0 \\ 0 & V \\ 0 & B \end{pmatrix}$ is left invertible. And, hence, Lemma 2 leads to $M_Q \in \Phi_{ab}(H \oplus K)$.

Case 2. Assume that $A \in \Phi_{ab}(H)$, $b.s_mul(B) \leq s_mul(A)$, and $B \in \Phi_+(K)$. Then, it follows from Lemma 3 that A can be decomposed into the following form:

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} : H_1 \oplus H_2 \longrightarrow H_1 \oplus H_2, \quad (25)$$

where $\dim(H_1) < \infty$, A_1 is nilpotent, A_2 is a left invertible operator, and $\beta(A_2) = s_mul(A)$. By the assumption that $B \in \Phi_+(K)$ and Lemma 1, we know that B can be decomposed into the following form with respect to some orthogonal decomposition $K = K_1 \oplus K_2 \oplus K_3$:

$$B = \begin{pmatrix} B_1 & * & * \\ 0 & B_2 & * \\ 0 & 0 & B_3 \end{pmatrix}, \quad (26)$$

where $\dim(K_3) < \infty$, B_1 is a right invertible operator, B_2 is a left invertible operator, B_3 is a finite nilpotent operator, and the parts marked by $*$ can be any operators. Moreover, $\infty > \alpha(B_1) = b.s_mul(B)$. Thus, $\beta(A_2) \geq \alpha(B_1)$, and then there exists some left invertible $C_1 \in B(N(B_1), H_2 \ominus R(A_2))$. Noting that $\dim((K_1 \ominus N(B_1)) \oplus K_2 \oplus K_3) = \dim(H_1 \oplus R(A_2)) = \infty$, we can let $C_2 \in G((K_1 \ominus N(B_1)) \oplus K_2 \oplus K_3, H_1 \oplus R(A))$. Consider

$$Q = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} : N(B_1) \oplus [K_1 \ominus N(B_1) \oplus K_2 \oplus K_3] \longrightarrow (H_2 \ominus R(A_2)) \oplus [H_1 \oplus R(A_2)]. \quad (27)$$

Obviously, Q is left invertible, and M_Q can be rewritten as

$$M_Q = \begin{pmatrix} A_1 & A_{12} & C_{11} & 0 & C_{12} & C_{13} \\ 0 & A_{21} & C_{21} & 0 & C_{22} & C_{23} \\ 0 & 0 & 0 & C_1 & 0 & 0 \\ 0 & 0 & B_{11} & 0 & * & * \\ 0 & 0 & 0 & 0 & B_2 & * \\ 0 & 0 & 0 & 0 & 0 & B_3 \end{pmatrix} : H_1 \oplus H_2 \oplus (K_1 \ominus N(B_1) \oplus N(B_1)) \oplus K_2 \oplus K_3 \longrightarrow H_1 \oplus R(A_2) \oplus (H_2 \ominus R(A_2)) \oplus K_1 \oplus K_2 \oplus K_3, \quad (28)$$

where A_{21} and B_{11} are invertible and C_1 and B_2 are left invertible. Similar to the proof of Case 1, through direct calculation we can show that $\begin{pmatrix} A_{21} & C_{21} & 0 \\ 0 & 0 & C_1 \\ 0 & B_{11} & 0 \end{pmatrix}$ is left invertible. Also since $\dim(H_1) < \infty$ and $\dim(K_3) < \infty$, we have $A_1 \in \Phi_b(H_1)$ and $B_3 \in \Phi_b(K_3)$. Thus, it follows from Lemma 2 that $M_C \in \Phi_{ab}(H \oplus K)$. $\square$

By duality, we have the following.

Theorem 11. For any given $A \in B(H)$ and $B \in B(K)$, one has

$$\bigcap_{C \in G_r(K, H)} \sigma_{sb}(M_C) = \bigcap_{C \in B(K, H)} \sigma_{sb}(M_C). \quad (29)$$

3. $\bigcap_{C \in G(K, H)} \sigma_{ab}(M_C)$ and $\bigcap_{C \in \Phi(K, H)} \sigma_{ab}(M_C)$

In this section, we give the characterization of invertible and Fredholm perturbations of upper semi-Browder spectra of 2×2 upper-triangular matrices. We begin with some lemmas.

Lemma 12 (see [19]). For a given pair $(A, B) \in B(H) \times B(K)$, if either A or B is a compact operator, then, for each $C \in \Phi(K, H)$, M_C is not a semi-Fredholm operator.

In particular, if B is not compact, then M_C is not semi-Browder for any invertible operator C .

Lemma 13. The following statements are equivalent.

- (i) B is not compact.
- (ii) For each given $A \in \Phi_{ab}(H)$, if $\beta(A) = \infty$, then there exists an operator $C \in G(K, H)$ such that M_C is an upper semi-Browder operator.
- (iii) For each given $A \in \Phi_{ab}(H)$, if $\beta(A) = \infty$, then there exists an operator $C \in \Phi(K, H)$ such that M_C is an upper semi-Browder operator.

Proof. Obviously, we only need to prove the implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (i). If B is compact, then it follows from Lemma 12 that M_C is not a semi-Fredholm operator for each $C \in \Phi(K, H)$, which contradicts with (iii). Thus, B is not compact.

(i) $\Rightarrow$ (ii). Suppose that B is not compact. Then, we consider the following two cases.

Case 1. Assume that $R(B)$ is closed. It follows from Lemma 3 that A can be decomposed into the following form with respect to some orthogonal decomposition $H = H_1 \oplus H_2$:

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} : H_1 \oplus H_2 \longrightarrow H_1 \oplus H_2, \quad (30)$$

where $\dim(H_1) < \infty$, A_1 is nilpotent, and A_2 is a left invertible operator. Noting that $\beta(A) = \infty$, we have $\beta(A_2) = \infty$. Since the assumption that B is not compact, we have that $\dim N(B)^\perp = \infty$. Also since $\beta(A_2) = \infty$, let $R(A_2)^\perp = H_3 \oplus H_4$ with $\dim(H_3) = \dim N(B)$ and $\dim(H_4) = \infty$. Define an operator $C : K \rightarrow H$ by

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} : N(B) \oplus N(B)^\perp \longrightarrow H_3 \oplus (H_1 \oplus R(A_2) \oplus H_4), \quad (31)$$

where $C_1 \in B(N(B), H_3)$ and $C_2 \in B(N(B)^\perp, H_1 \oplus R(A_2) \oplus H_4)$ are invertible operators. Obviously, $C \in B(K, H)$ is invertible. Next, we claim that M_C is an upper semi-Browder operator. To see this, M_C can be rewritten as

$$M_C = \begin{pmatrix} A_1 & A_{12} & C_{11} & 0 \\ 0 & A_{22} & C_{21} & 0 \\ 0 & 0 & 0 & C_1 \\ 0 & 0 & C_{41} & 0 \\ 0 & 0 & B_1 & 0 \end{pmatrix} : H_1 \oplus H_2 \oplus N(B)^\perp \oplus N(B) \longrightarrow H_1 \oplus R(A_2) \oplus H_3 \oplus H_4 \oplus K, \quad (32)$$

where $A_{22} \in B(H_2, R(A_2))$ is invertible and $B_1 \in B(N(B)^\perp, R(B))$ is left invertible. By Lemma 2 and the fact that $A_1 \in \Phi_b(H_1)$, it is sufficient to prove that

$$M_1 =: \begin{pmatrix} A_{22} & C_{21} & 0 \\ 0 & 0 & C_1 \\ 0 & C_{41} & 0 \\ 0 & B_1 & 0 \end{pmatrix} \quad (33)$$

is semi-Browder. For this, we only need to show that M_1 is left invertible. In fact, since A_{22} is invertible and B_1 and C_1 are left invertible, we can set A'_{22} , B'_1 , and C'_1 such that

$$A'_{22}A_{22} = I_{H_2}, \quad B'_1B_1 = I_{N(B)^\perp}, \quad C'_1C_1 = I_{N(B)}. \quad (34)$$

Direct calculation shows that

$$\begin{pmatrix} A'_{22} & 0 & 0 & -A'_{22}C_{21}B'_1 \\ 0 & 0 & 0 & B'_1 \\ 0 & C'_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_{22} & C_{21} & 0 \\ 0 & 0 & C_1 \\ 0 & C_{41} & 0 \\ 0 & B_1 & 0 \end{pmatrix} = \begin{pmatrix} I_{H_2} & 0 & 0 \\ 0 & I_{N(B)^\perp} & 0 \\ 0 & 0 & I_{N(B)} \end{pmatrix}, \quad (35)$$

which implies that M_1 is left invertible. Noting that $A_1 \in \Phi(H_1)$, by Lemma 2 we have that M_C is upper semi-Browder.

Case 2. Assume that $R(B)$ is not closed. If B is not compact, then by Lemma 7, $R(B)$ contains a closed infinite-dimensional subspace. Without loss of generality, suppose that $\widetilde{K}_1$ is a closed subspace of $R(B)$ with $\dim \widetilde{K}_1 = \infty$ and $\dim \widetilde{K}_1^\perp = \infty$. Let $K_1 = \{x \in N(B)^\perp : Bx \in \widetilde{K}_1\}$. Thus, K_1 is a closed subspace of $N(B)^\perp$, and $\dim(K_1) = \infty$. Denote $K_2 = N(B)^\perp \ominus K_1$. Without loss of generality, we may assume that $\dim(K_2) = \infty$ (otherwise, suppose that $\{e_n\}_{n=1}^\infty$ is an orthonormal basis of K_1 . Denote $K'_1 = \text{span}\{e_n : n = 2, 4, 6, \dots\}$ and $\widetilde{K}'_1 = \{Bx : x \in K'_1\}$, then K_1 and $\widetilde{K}_1$ can be replaced by K'_1 and $\widetilde{K}'_1$, resp.). Since $\beta(A_2) = \infty$, let $R(A_2)^\perp = H_3 \oplus H_4$ with $\dim(H_3) = \dim N(B)$ and $\dim H_4 = \infty$. Define an operator $C : K \rightarrow H$ by

$$C = \begin{pmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{pmatrix} : K_1 \oplus K_2 \oplus N(B) \longrightarrow (H_1 \oplus R(A_2)) \oplus H_4 \oplus H_3, \quad (36)$$

where C_1 , C_2 , and C_3 are unitary operators. Obviously, C is invertible. M_C can be rewritten as

$$M_C = \begin{pmatrix} A_1 & A_{12} & C_{11} & 0 & 0 \\ 0 & A_{22} & C_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_3 \\ 0 & 0 & 0 & C_2 & 0 \\ 0 & 0 & B_{11} & B_{12} & 0 \\ 0 & 0 & 0 & B_{22} & 0 \end{pmatrix} : H_1 \oplus H_2 \oplus K_1 \oplus K_2 \oplus N(B) \longrightarrow H_1 \oplus R(A_2) \oplus H_3 \oplus H_4 \oplus \widetilde{K}_1 \oplus \widetilde{K}_1^\perp, \quad (37)$$

where A_{22} and B_{11} are invertible and $C_1 = \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix}$.

Next, we prove that $M_C \in \Phi_{ab}(H \oplus K)$. Noting that $\dim(H_1) < \infty$, then, by Lemma 2, it is sufficient to prove that

$$M_1 =: \begin{pmatrix} A_{22} & C_{21} & 0 & 0 \\ 0 & 0 & 0 & C_3 \\ 0 & 0 & C_2 & 0 \\ 0 & B_{11} & B_{12} & 0 \\ 0 & 0 & B_{22} & 0 \end{pmatrix} \quad (38)$$

is left invertible. For this, let A'_{22} , B'_{11} , C'_1 , and C'_2 be operators satisfying

$$\begin{aligned} A'_{22}A_{22} &= I_{H_2}, & B'_{11}B_{11} &= I_{K_1}, \\ C'_2C_2 &= I_{K_2}, & C'_3C_3 &= I_{N(B)}. \end{aligned} \quad (39)$$

Direct calculation shows that

$$\begin{aligned}
 & \begin{pmatrix} A'_{22} & 0 & A'_{22}C_{21}B'_{11}B_{12}C'_2 & -A'_{22}C_{21}B'_{11} & 0 \\ 0 & 0 & B'_{11}B_{12}C'_2 & B'_{11} & 0 \\ 0 & 0 & C'_2 & 0 & 0 \\ 0 & C'_3 & 0 & 0 & 0 \end{pmatrix} \\
 & \times \begin{pmatrix} A_{22} & C_{21} & 0 & 0 \\ 0 & 0 & 0 & C_3 \\ 0 & 0 & C_2 & 0 \\ 0 & B_{11} & B_{12} & 0 \\ 0 & 0 & B_{22} & 0 \end{pmatrix} \\
 & = \begin{pmatrix} I_{H_2} & 0 & 0 & 0 \\ 0 & I_{K_1} & 0 & 0 \\ 0 & 0 & I_{K_2} & 0 \\ 0 & 0 & 0 & I_{N(B)} \end{pmatrix},
 \end{aligned} \tag{40}$$

which implies that M_1 is left invertible.

Combining Case 1 with Case 2, the lemma is proved. $\square$

Similarly, we have the following.

Lemma 14. *The following statements are equivalent:*

- (i) A is not compact.
- (ii) For each given $B \in \Phi_{sb}(H)$, if $\alpha(B) = \infty$, then there exists an operator $C \in G(K, H)$ such that M_C is a lower semi-Browder operator.
- (iii) For each given $B \in \Phi_{sb}(H)$, if $\alpha(B) = \infty$, then there exists an operator $C \in \Phi(K, H)$ such that M_C is a lower semi-Browder operator.

One is now ready to prove the main result of this section.

Theorem 15. *For a given pair $(A, B) \in B(H) \times B(K)$, one has*

$$\begin{aligned}
 \bigcap_{C \in G(K, H)} \sigma_{ab}(M_C) &= \bigcap_{C \in \Phi(K, H)} \sigma_{ab}(M_C) \\
 &= \left(\bigcap_{C \in B(K, H)} \sigma_{ab}(M_C) \right) \\
 &\quad \cup \{ \lambda \in \mathbb{C} : B - \lambda \text{ is compact} \}.
 \end{aligned} \tag{41}$$

Proof. According to Lemma 12, it is clear that

$$\begin{aligned}
 \bigcap_{C \in G(K, H)} \sigma_{ab}(M_C) &\supseteq \bigcap_{C \in \Phi(K, H)} \sigma_{ab}(M_C) \\
 &\supseteq \left(\bigcap_{C \in B(K, H)} \sigma_{ab}(M_C) \right) \\
 &\quad \cup \{ \lambda \in \mathbb{C} : B - \lambda \text{ is compact} \}.
 \end{aligned} \tag{42}$$

For the conversion, without loss of generality, suppose that

$$0 \notin \left(\bigcap_{C \in B(K, H)} \sigma_{ab}(M_C) \right) \cup \{ \lambda \in \mathbb{C} : B - \lambda \text{ is compact} \}. \tag{43}$$

Then, B is not compact, and there exists some $C \in B(K, H)$ such that $M_C \in \Phi_{ab}(H \oplus K)$, and, hence, $A \in \Phi_{ab}(H)$.

Case 1. $\beta(A) = \infty$. It follows from Lemma 13 that there exists some $C \in G(K, H)$ such that M_C is an upper semi-Browder operator. This implies that $\lambda \notin \bigcap_{C \in G(K, H)} \sigma_{ab}(M_C)$. In this case, we have proved. Consider that

$$\bigcap_{C \in G(K, H)} \sigma_{ab}(M_C) \subseteq \left(\bigcap_{C \in B(K, H)} \sigma_{ab}(M_C) \right) \cup \{ \lambda \in \mathbb{C} : B - \lambda \text{ is compact} \}. \tag{44}$$

Case 2. Consider $\beta(A) < \infty$. This implies that $A \in \Phi(H)$, and, thus, $B \in \Phi_+(K)$ since $M_C \in \Phi_{ab}(H \oplus K)$. It follows from Lemma 5 that $b.s.\text{-mul}(B) \leq s.\text{-mul}(A)$. Moreover, using Lemmas 1 and 3, we have

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} : H_1 \oplus H_2 \mapsto H_1 \oplus H_2,$$

$$B = \begin{pmatrix} B_1 & * & * \\ 0 & B_2 & * \\ 0 & 0 & B_3 \end{pmatrix} : K_1 \oplus K_2 \oplus K_3 \mapsto K_1 \oplus K_2 \oplus K_3, \tag{45}$$

where $\dim(H_1) < \infty$, A_1 is nilpotent, A_2 is a left invertible operator, $\dim(K_3) < \infty$, B_1 is a right invertible operator, B_2 is a left invertible operator, B_3 is a finite nilpotent operator, and the parts marked by $*$ can be any operators. Moreover, $\beta(A_2) = s.\text{-mul}(A)$, and $\alpha(B_1) = b.s.\text{-mul}(B)$. Hence, $\alpha(B_1) \leq \beta(A_2)$. Now, put $H_2 = R(A_2) \oplus H_3 \oplus H_4$, where $\dim(H_3) = \alpha(B_1) < \infty$. Noting that $\dim H_1 \oplus R(A_2) \oplus H_4 = N(B_1)^\perp \oplus K_2 \oplus K_3 = \infty$, there exist unitaries $C_{33} \in B(N(B_1), H_3)$ and $C' \in B(H_1 \oplus R(A_2) \oplus H_4, N(B_1)^\perp \oplus K_2 \oplus K_3)$. Let $C = \begin{pmatrix} C_{33} & 0 \\ 0 & C' \end{pmatrix}$. Obviously, $C = \begin{pmatrix} C_{33} & 0 \\ 0 & C' \end{pmatrix} \in G(H \oplus K)$.

Consider that operator

$$\begin{aligned}
 M_C &= \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : H \oplus K \rightarrow H \oplus K \\
 &= \begin{pmatrix} A_1 & A_{12} & C_{11} & 0 & C_{13} & C_{14} \\ 0 & A_2 & C_{21} & 0 & C_{23} & C_{24} \\ 0 & 0 & 0 & C_{33} & 0 & 0 \\ 0 & 0 & C_{41} & 0 & C_{43} & C_{44} \\ 0 & 0 & B_{11} & 0 & * & * \\ 0 & 0 & 0 & 0 & B_2 & * \\ 0 & 0 & 0 & 0 & 0 & B_3 \end{pmatrix} : H_1 \oplus H_2 \\
 &\quad \oplus N(B_1)^\perp \oplus N(B_1) \oplus K_2 \oplus K_3 \\
 &\rightarrow H_1 \oplus R(A_2) \oplus H_3 \oplus H_4 \oplus K_1 \oplus K_2 \oplus K_3,
 \end{aligned} \tag{46}$$

where

$$\begin{aligned}
 C' &= \begin{pmatrix} C_{11} & C_{13} & C_{14} \\ C_{21} & C_{23} & C_{24} \\ C_{41} & C_{43} & C_{44} \end{pmatrix} : H_1 \oplus H_2 \oplus N(B_1)^\perp \\
 &\rightarrow H_1 \oplus R(A_2) \oplus H_4, \quad B_1 = \begin{pmatrix} B_{11} \\ 0 \end{pmatrix}.
 \end{aligned} \tag{47}$$

We claim that $M_C \in \Phi_{ab}(H \oplus K)$. In fact, since A_1 and B_3 are Browder operators, then, by Lemma 2, it is sufficient to show that

$$M =: \begin{pmatrix} A_2 & C_{21} & 0 & C_{23} \\ 0 & 0 & C_{33} & 0 \\ 0 & C_{41} & 0 & C_{43} \\ 0 & B_{11} & 0 & * \\ 0 & 0 & 0 & B_2 \end{pmatrix} : H_2 \oplus N(B_1)^\perp \oplus N(B_1) \oplus K_2 \longrightarrow R(A_2) \oplus H_3 \oplus H_4 \oplus K_1 \oplus K_2 \quad (48)$$

is upper semi-Browder. Observe that A_2 and B_2 are left invertible; C_{33} and B_{11} are invertible. Direct calculation shows that M is injective. Since $A \in \Phi(H)$ and $B \in \Phi_+(K)$, we have $M_C \in \Phi_{ab}(H \oplus K)$, and, hence, M is an upper semi-Fredholm operator. Thus, M is left invertible. Combining this with Lemma 2 yields $M_C \in \Phi_{ab}(H \oplus K)$, which means that $\lambda \notin \bigcap_{C \in G(K, H)} \sigma_{ab}(M_C)$. Thus,

$$\bigcap_{C \in G(K, H)} \sigma_{ab}(M_C) \subseteq \bigcap_{C \in B(K, H)} \sigma_{ab}(M_C) \cup \{\lambda \in \mathbb{C} : B - \lambda \text{ is compact}\}. \quad (49)$$

Combining Case 1 with Case 2 leads to

$$\begin{aligned} \bigcap_{C \in G(K, H)} \sigma_{ab}(M_C) &= \bigcap_{C \in \phi(K, H)} \sigma_{ab}(M_C) \\ &= \left(\bigcap_{C \in B(K, H)} \sigma_{ab}(M_C) \right) \cup \{\lambda \in \mathbb{C} : B - \lambda \text{ is compact}\}. \end{aligned} \quad (50)$$

This completes the proof. $\square$

By duality, we have

Theorem 16. For a given pair $(A, B) \in B(H) \times B(K)$, one has

$$\begin{aligned} \bigcap_{C \in G(K, H)} \sigma_{sb}(M_C) &= \bigcap_{C \in \Phi(K, H)} \sigma_{sb}(M_C) \\ &= \left(\bigcap_{C \in B(K, H)} \sigma_{sb}(M_C) \right) \cup \{\lambda \in \mathbb{C} : A - \lambda \text{ is compact}\}. \end{aligned} \quad (51)$$

Acknowledgments

Long Long research is supported by the Freedom Explore Program of Central South University (Grant no. 2012QNZT040) and the Post Doctoral Program Foundation of Central South University. This work is also supported by the National Natural Science Foundation of China (11226113, 11301077, 11171066, and 10771191), the Foundation of the Education Department of Fujian Province (JA12074), and the Natural Science Foundation of Fujian Province (2012J05003).

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