Research Article

Some Curvature Properties of $(LCS)_n$ -Manifolds

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The object of the present paper is to study $(LCS)_n$ -manifolds with vanishing quasi-conformal curvature tensor. $(LCS)_n$ -manifolds satisfying Ricci-symmetric condition are also characterized.

1. Introduction

Recently, in [1], Shaikh introduced and studied Lorentzian concircular structure manifolds (briefly (LCS) -manifold) which generalizes the notion of LP-Sasakian manifolds, introduced by Matsumoto [2].

Generalizing the notion of LP-Sasakian manifold in 2003 [1], Shaikh introduced the notion of $(LCS)_n$ -manifolds along with their existence and applications to the general theory of relativity and cosmology. Also, Shaikh and his coauthors studied various types of $(LCS)_n$ -manifolds by imposing the curvature restrictions (see [3–6]). In [7, 8], the authors also studied $(LCS)_{2n+1}$ -manifolds.

The submanifold of an $(LCS)_n$ -manifold is studied by Ateken and Hui [9, 10] and Shukla et al. [11]. In [12], Yano and Sawaki introduced the quasi-conformal curvature tensor, and later it was studied by many authors with curvature restrictions on various structures [13].

After then, the same author studied weakly symmetric $(LCS)_n$ -manifolds by several examples and obtain various results in such manifolds. In [7], authors shown that a pseudo projectively flat and pseudo projectively recurrent $(LCS)_n$ manifolds are η -Einstein manifold.

On the other hand, in [5], authors proved the existence of ϕ -recurrent $(LCS)_3$ manifold which is neither locally symmetric nor locally ϕ -symmetric by nontrivial examples. Furthermore, they also give the necessary and sufficient conditions for a $(LCS)_n$ -manifold to be locally ϕ -recurrent.

In this study, we have investigated the quasi-conformal flat $(LCS)_n$ -manifolds satisfying properties such as Ricci-symmetric, locally symmetric, and η -Einstein. Finally, we give an example for η -Einstein manifolds.

2. Preliminaries

An n -dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric tensor g , that is, M admits a smooth symmetric tensor field g of the type $(2, 0)$ such that, for each $p \in M$,

$$g_p : T_M(p) \times T_M(p) \longrightarrow \mathbb{R} \quad (1)$$

is a nondegenerate inner product of signature $(-, +, +, \dots, +)$. In such a manifold, a nonzero vector $X_p \in T_M(p)$ is said to be timelike (resp., nonspacelike, null, and spacelike) if it satisfies the condition $g_p(X_p, X_p) < 0$ (resp., $\leq 0, = 0, > 0$). These cases are called casual character of the vectors.

Definition 1. In a Lorentzian manifold (M, g) , a vector field P defined by

$$g(X, P) = A(X) \quad (2)$$

for any $X \in \Gamma(TM)$ is said to be a concircular vector field if

$$(\nabla_X A)Y = \alpha \{g(X, Y) + w(X)A(Y)\} \quad (3)$$

for $Y \in \Gamma(TM)$, where α is a nonzero scalar function, A is a 1-form, w is also closed 1-form, and ∇ denotes the Levi-Civita connection on M [7].

Let M be a Lorentzian manifold admitting a unit timelike concircular vector field ξ , called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \quad (4)$$

Since ξ is a unit concircular unit vector field, there exists a nonzero 1-form η such that

$$g(X, \xi) = \eta(X). \quad (5)$$

The equation of the following form holds:

$$(\nabla_X \eta)Y = \alpha \{g(X, Y) + \eta(X)\eta(Y)\}, \quad \alpha \neq 0 \quad (6)$$

for all $X, Y \in \Gamma(TM)$, where α is a nonzero scalar function satisfying

$$\nabla_X \alpha = X(\alpha) = d\alpha(X) = \rho\eta(X), \quad (7)$$

ρ being a certain scalar function given by $\rho = -\xi(\alpha)$.

Let us put

$$\nabla_X \xi = \alpha\phi X, \quad (8)$$

then from (6) and (8), we can derive

$$\phi X = X + \eta(X)\xi, \quad (9)$$

which tell us that ϕ is a symmetric $(1, 1)$ -tensor. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ , its associated 1-form η , and $(1, 1)$ -type tensor field ϕ is said to be a Lorentzian concircular structure manifold.

A differentiable manifold M of dimension n is called (LCS)-manifold if it admits a $(1, 1)$ -type tensor field ϕ , a covariant vector field η , and a Lorentzian metric g which satisfy

$$\eta(\xi) = g(\xi, \xi) = -1, \quad (10)$$

$$\phi^2 X = X + \eta(X)\xi, \quad (11)$$

$$g(X, \xi) = \eta(X), \quad (12)$$

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \quad (13)$$

for all $X \in \Gamma(TM)$. Particularly, if we take $\alpha = 1$, then we can obtain the LP -Sasakian structure of Matsumoto [2].

Also, in an $(LCS)_n$ -manifold M , the following relations are satisfied (see [3–6]):

$$\eta(R(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \quad (14)$$

$$R(\xi, X)Y = (\alpha^2 - \rho)[g(X, Y)\xi - \eta(Y)X], \quad (15)$$

$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \quad (16)$$

$$(\nabla_X \phi)Y = \alpha[g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \quad (17)$$

$$S(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X), \quad (18)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y) \quad (19)$$

for all vector fields X, Y, Z on M , where R and S denote the Riemannian curvature tensor and Ricci curvature, respectively, Q is also the Ricci operator given by $S(X, Y) = g(QX, Y)$.

Now let (M, g) be an n -dimensional Riemannian manifold; then the concircular curvature tensor $\widetilde{C}$, the Weyl conformal curvature tensor C , and the pseudo projective curvature tensor $\widetilde{P}$ are, respectively, defined by

$$\begin{aligned} \widetilde{C}(X, Y)Z &= R(X, Y)Z \\ &\quad - \frac{\tau}{n(n-1)}[g(Y, Z)X - g(X, Z)Y], \end{aligned} \quad (20)$$

$$\begin{aligned} C(X, Y)Z &= R(X, Y)Z - \frac{1}{n-2} \\ &\quad \times [S(Y, Z)X - S(X, Z)Y \\ &\quad + g(Y, Z)QX - g(X, Z)QY] \\ &\quad + \frac{\tau}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y], \end{aligned} \quad (21)$$

$$\begin{aligned} \widetilde{P}(X, Y)Z &= aR(X, Y)Z \\ &\quad + b[S(Y, Z)X - S(X, Z)Y] \\ &\quad - \frac{\tau}{n}\left[\frac{a}{n-1} + b\right][g(Y, Z)X - g(X, Z)Y], \end{aligned} \quad (22)$$

where a and b are constants such that $a, b \neq 0$, and τ is also the scalar curvature of M [7].

For an n -dimensional $(LCS)_n$ -manifold the quasi-conformal curvature tensor $\widetilde{\mathcal{C}}$ is given by

$$\begin{aligned} \widetilde{\mathcal{C}}(X, Y)Z &= aR(X, Y)Z \\ &\quad + b[S(Y, Z)X - S(X, Z)Y \\ &\quad + g(Y, Z)QX - g(X, Z)QY] \\ &\quad - \frac{\tau}{n}\left[\frac{a}{n-1} + 2b\right][g(Y, Z)X - g(X, Z)Y] \end{aligned} \quad (23)$$

for all $X, Y, Z \in \Gamma(TM)$ [14].

The notion of quasi-conformal curvature tensor was defined by Yano and Swaki [12]. If $a = 1$ and $b = -1/(n-1)$, then quasi-conformal curvature tensor reduces to conformal curvature tensor.

3. Quasi-Conformally Flat $(LCS)_n$ -Manifolds and Some of Their Properties

For an n -dimensional quasi-conformally flat $(LCS)_n$ -manifold, we know for $Z = \xi$ from (23),

$$\begin{aligned} & aR(X, Y)\xi + b[S(Y, \xi)X - S(X, \xi)Y \\ & \quad + g(Y, \xi)QX - g(X, \xi)QY] \\ & - \frac{\tau}{n} \left[\frac{a}{n-1} + 2b \right] [g(Y, \xi)X - g(X, \xi)Y] = 0. \end{aligned} \quad (24)$$

Here, taking into account of (16), we have

$$\begin{aligned} & [\eta(Y)X - \eta(X)Y] \left[a(\alpha^2 - \rho) + b(n-1)(\alpha^2 - \rho) \right. \\ & \quad \left. - \frac{\tau}{n} \left(\frac{a}{n-1} + 2b \right) \right] \\ & + b[\eta(Y)QX - \eta(X)QY] = 0. \end{aligned} \quad (25)$$

Let $Y = \xi$ be in (25); then also by using (18) we obtain

$$\begin{aligned} & [-X - \eta(X)\xi] \left[a(\alpha^2 - \rho) - \frac{\tau}{n} \left(\frac{a}{n-1} + 2b \right) \right. \\ & \quad \left. + b(n-1)(\alpha^2 - \rho) \right] \\ & + b[-QX - \eta(X)(n-1)(\alpha^2 - \rho)\xi] = 0. \end{aligned} \quad (26)$$

Taking the inner product on both sides of the last equation by Y , we obtain

$$\begin{aligned} & [g(X, Y) + \eta(X)\eta(Y)] \left[a(\alpha^2 - \rho) + b(n-1) \right. \\ & \quad \left. \times (\alpha^2 - \rho) - \frac{\tau}{n} \left(\frac{a}{n-1} + 2b \right) \right] \\ & + b[S(X, Y) + \eta(X)\eta(Y)(\alpha^2 - \rho)(n-1)] = 0, \end{aligned} \quad (27)$$

that is,

$$\begin{aligned} & S(X, Y) = g(X, Y) \\ & \quad \times \left[\frac{\tau}{nb} \left(\frac{a}{n-1} + 2b \right) - (\alpha^2 - \rho) \left(\frac{a}{b} + (n-1) \right) \right] \\ & \quad + \eta(X)\eta(Y) \left[\frac{\tau}{nb} \left(\frac{a}{n-1} + 2b \right) \right. \\ & \quad \left. - (\alpha^2 - \rho) \left(\frac{a}{b} + 2(n-1) \right) \right]. \end{aligned} \quad (28)$$

Now we are in a proposition to state the following.

Theorem 2. *If an n -dimensional $(LCS)_n$ -manifold M is quasi-conformally flat, then M is an η -Einstein manifold.*

Now, let $\{e_1, e_2, \dots, e_{n-1}, \xi\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$, ξ in (28), and taking summation for $1 \leq i \leq n-1$, we have

$$\tau = n(n-1)(\alpha^2 - \rho) \quad \text{if } a + (n-2)b \neq 0. \quad (29)$$

In view of (28) and (29), we obtain

$$S(X, Y) = (n-1)(\alpha^2 - \rho)g(X, Y), \quad (30)$$

which is equivalent to

$$QX = (n-1)(\alpha^2 - \rho)X \quad (31)$$

for any $X \in \Gamma(TM)$.

By using (29) and (31) in (23) for a quasi-conformally flat $(LCS)_n$ -manifold M , we get

$$R(X, Y)Z = (\alpha^2 - \rho)\{g(Y, Z)X - g(X, Z)Y\}, \quad (32)$$

for all $X, Y, Z \in \Gamma(TM)$. If we consider Schur's Theorem, we can give the following the theorem.

Theorem 3. *A quasi-conformally flat $(LCS)_n$ -manifold M ($n > 1$) is a manifold of constant curvature $(\alpha^2 - \rho)$ provided that $a + b(n-2) \neq 0$.*

Now let us consider an $(LCS)_n$ -manifold M which is conformally flat. Thus we have from (21) that

$$\begin{aligned} R(X, Y)Z &= \frac{1}{n-2} \{S(Y, Z)X - S(X, Z)Y \\ & \quad + g(Y, Z)QX - g(X, Z)QY\} \\ & - \frac{\tau}{(n-1)(n-2)} \{g(Y, Z)X - g(X, Z)Y\}, \end{aligned} \quad (33)$$

for all vector fields X, Y, Z tangent to M . Setting $Z = \xi$ in (33) and using (16), (18) we have

$$\begin{aligned} & \left[\frac{\tau}{n-1} - (\alpha^2 - \rho) \right] [\eta(Y)X - \eta(X)Y] \\ & = [\eta(Y)QX - \eta(X)QY]. \end{aligned} \quad (34)$$

If we put $Y = \xi$ in (34) and also using (18), we obtain

$$QX = \left[\frac{\tau}{n-1} - (\alpha^2 - \rho) \right] X + \left[\frac{\tau}{n-1} - n(\alpha^2 - \rho) \right] \eta(X)\xi. \quad (35)$$

Corollary 4. *A conformally flat $(LCS)_n$ -manifold is an η -Einstein manifold.*

Generalizing the notion of a manifold of constant curvature, Chen and Yano [15] introduced the notion of a manifold of quasi-constant curvature which can be defined as follows:

Definition 5. A Riemannian manifold is said to be a manifold of quasi-constant curvature if it is conformally flat and its curvature tensor $\bar{R}$ of type $(0, 4)$ is of the form

$$\begin{aligned} & \bar{R}(X, Y, Z, W) \\ & = a\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\ & \quad + b\{g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W) \\ & \quad + g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)\}, \end{aligned} \quad (36)$$

for all $X, Y, Z, W \in \Gamma(TM)$, where a, b are scalars of which $b \neq 0$ and A is a nonzero 1-form (for more details, we refer to [13, 16]).

Thus we have the following theorem for $(LCS)_n$ -conformally flat manifolds.

Theorem 6. *A conformally flat $(LCS)_n$ -manifold is a manifold of quasi-constant curvature.*

Proof. From (33) and (35), we obtain

$$\begin{aligned} \bar{R}(X, Y, Z, W) &= \left(\frac{\tau - 2(n-1)(\alpha^2 - \rho)}{(n-1)(n-2)} \right) \\ &\times \{g(X, W)g(Y, Z) - g(Y, W)g(X, Z)\} \\ &+ \left(\frac{\tau - n(n-1)(\alpha^2 - \rho)}{(n-1)(n-2)} \right) \\ &\times \{g(X, W)\eta(Y)\eta(Z) - g(Y, W)\eta(X)\eta(Z) \\ &\quad + g(Y, Z)\eta(X)\eta(W) - g(X, Z)\eta(Y)\eta(W)\}. \end{aligned} \quad (37)$$

This implies (36) for

$$\begin{aligned} a &= \frac{\tau - 2(n-1)(\alpha^2 - \rho)}{(n-1)(n-2)}, \\ b &= \frac{\tau - n(n-1)(\alpha^2 - \rho)}{(n-1)(n-2)}, \quad A = \eta. \end{aligned} \quad (38)$$

This proves our assertion. $\square$

Next, differentiating the (19) covariantly with respect to W , we get

$$\begin{aligned} \nabla_W S(\phi X, \phi Y) &= \nabla_W S(X, Y) + (n-1)W(\alpha^2 - \rho) \\ &\quad + (n-1)(\alpha^2 - \rho)W[\eta(X)\eta(Y)], \end{aligned} \quad (39)$$

for any $X, Y \in \Gamma(TM)$. Making use of the definition of ∇S and (8), we have

$$\begin{aligned} (\nabla_W S)(\phi X, \phi Y) &+ S(\nabla_W \phi X, \phi Y) + S(\phi X, \nabla_W \phi Y) \\ &= (\nabla_W S)(X, Y) + S(\nabla_W X, Y) + S(X, \nabla_W Y) \\ &\quad + (n-1)W(\alpha^2 - \rho)\eta(X)\eta(Y) \\ &\quad + (n-1)(\alpha^2 - \rho)\eta(Y)\{\eta(\nabla_W X) + \alpha g(X, \phi W)\} \\ &\quad + (n-1)(\alpha^2 - \rho)\eta(X)\{\eta(\nabla_W Y) + \alpha g(Y, \phi W)\}. \end{aligned} \quad (40)$$

Thus we have

$$\begin{aligned} &(\nabla_W S)(\phi X, \phi Y) - (\nabla_W S)(X, Y) \\ &= -S((\nabla_W \phi)X + \phi \nabla_W X, \phi Y) \\ &\quad - S(\phi X, (\nabla_W \phi)Y + \phi \nabla_W Y) + S(\nabla_W X, Y) \\ &\quad + S(X, \nabla_W Y) + (n-1)W(\alpha^2 - \rho)\eta(X)\eta(Y) \\ &\quad + (n-1)(\alpha^2 - \rho)\eta(Y)\{\eta(\nabla_W X) + \alpha g(X, \phi W)\} \\ &\quad + (n-1)(\alpha^2 - \rho)\eta(X)\{\eta(\nabla_W Y) + \alpha g(Y, \phi W)\}. \end{aligned} \quad (41)$$

Here taking account of (17), we arrive at

$$\begin{aligned} &(\nabla_W S)(\phi X, \phi Y) - (\nabla_W S)(X, Y) \\ &= -S(\alpha\{g(X, W)\xi + 2\eta(X)\eta(W)\xi + \eta(X)W\}, \phi Y) \\ &\quad - S(\phi X, \alpha\{g(Y, W)\xi + 2\eta(Y)\eta(W)\xi + \eta(Y)W\}) \\ &\quad - S(\phi X, \phi \nabla_W Y) + S(\nabla_W X, Y) \\ &\quad + S(X, \nabla_W Y) + (n-1)W(\alpha^2 - \rho)\eta(X)\eta(Y) \\ &\quad - S(\phi \nabla_W X, \phi Y) + (n-1)(\alpha^2 - \rho)\eta(Y) \\ &\quad \times \{\eta(\nabla_W X) + \alpha g(X, \phi W)\} + (n-1)(\alpha^2 - \rho)\eta(X) \\ &\quad \times \{\eta(\nabla_W Y) + \alpha g(Y, \phi W)\} \\ &= -\alpha\{g(X, W)S(\xi, \phi Y) + 2\eta(X)\eta(W)S(\xi, \phi Y) \\ &\quad + \eta(X)S(W, \phi Y)\} \\ &\quad - \alpha\{g(Y, W)S(\phi X, \xi) + 2\eta(Y)\eta(W)S(\phi X, \xi) \\ &\quad + \eta(Y)S(\phi X, W)\} \\ &\quad - S(\phi X, \phi \nabla_W Y) + S(\nabla_W X, Y) + S(X, \nabla_W Y) \\ &\quad - S(\phi \nabla_W X, \phi Y) + (n-1)W(\alpha^2 - \rho)\eta(X)\eta(Y) \\ &\quad + (n-1)(\alpha^2 - \rho)\eta(Y)\{\eta(\nabla_W X) + \alpha g(X, \phi W)\} \\ &\quad + (n-1)(\alpha^2 - \rho)\eta(X)\{\eta(\nabla_W Y) + \alpha g(Y, \phi W)\}. \end{aligned} \quad (42)$$

Again, by using (13), (18), and (19), we reach

$$\begin{aligned} &(\nabla_W S)(\phi X, \phi Y) - (\nabla_W S)(X, Y) \\ &= -\alpha\eta(X)S(W, \phi Y) - \alpha\eta(Y)S(\phi X, W) \\ &\quad - (n-1)(\alpha^2 - \rho)\eta(X)\eta(\nabla_W X) \\ &\quad - (n-1)(\alpha^2 - \rho)\eta(Y)\eta(\nabla_W X) \end{aligned}$$

$$\begin{aligned}
 & + (n-1) W (\alpha^2 - \rho) \eta(X) \eta(Y) \\
 & + (n-1) (\alpha^2 - \rho) \\
 & \times \{ \eta(\nabla_W X) \eta(Y) + \alpha \eta(Y) g(X, \phi W) \\
 & \quad + \eta(\nabla_W Y) \eta(X) + \alpha \eta(X) g(Y, \phi W) \} \\
 & = -\alpha \eta(X) S(W, \phi Y) - \alpha \eta(Y) S(\phi X, W) \\
 & \quad + \alpha (n-1) (\alpha^2 - \rho) \\
 & \times \{ \eta(Y) g(X, \phi W) + \eta(X) g(Y, \phi W) \} \\
 & \quad + (n-1) W (\alpha^2 - \rho) \eta(X) \eta(Y).
 \end{aligned} \tag{43}$$

Thus we have the following theorem.

Theorem 7. *If an $(LCS)_n$ -manifold M is Ricci-symmetric; then $\alpha^2 - \rho$ is constant.*

Proof. If $n > 1$ -dimensional $(LCS)_n$ -manifold M is Ricci-symmetric, then from (43) we conclude that

$$\begin{aligned}
 & \alpha (n-1) (\alpha^2 - \rho) \{ \eta(Y) g(X, \phi W) + \eta(X) g(Y, \phi W) \} \\
 & \quad + (n-1) W (\alpha^2 - \rho) \eta(X) \eta(Y) \\
 & \quad - \alpha \eta(X) S(W, \phi Y) - \alpha \eta(Y) S(\phi X, W) = 0.
 \end{aligned} \tag{44}$$

It follows that

$$\begin{aligned}
 & \alpha (n-1) (\alpha^2 - \rho) \{ g(X, \phi W) \xi - \eta(X) \phi W \} \\
 & \quad + (n-1) W (\alpha^2 - \rho) \eta(X) \xi \\
 & \quad - \alpha \eta(X) \phi QW - \alpha S(\phi X, W) \xi = 0,
 \end{aligned} \tag{45}$$

from which

$$\begin{aligned}
 & -\alpha (n-1) (\alpha^2 - \rho) g(X, \phi W) \\
 & \quad - (n-1) W (\alpha^2 - \rho) \eta(X) + S(\phi X, W) = 0,
 \end{aligned} \tag{46}$$

which is equivalent to

$$\begin{aligned}
 & -\alpha (n-1) (\alpha^2 - \rho) \phi W - (n-1) W (\alpha^2 - \rho) \xi \\
 & \quad + \alpha \phi QW = 0,
 \end{aligned} \tag{47}$$

that is,

$$W (\alpha^2 - \rho) = 0, \tag{48}$$

which proves our assertion. $\square$

Since $\nabla R = 0$ implies that $\nabla S = 0$, we can give the following corollary.

Corollary 8. *If an n -dimensional $(LCS)_n$ -manifold M is locally symmetric, then $\alpha^2 - \rho$ is constant.*

Now, taking the covariant derivation of the both sides of (18) with respect to Y , we have

$$YS(X, \xi) = (n-1) W [(\alpha^2 - \rho) \eta(X)]. \tag{49}$$

From the definition of the covariant derivation of Ricci-tensor, we have

$$\begin{aligned}
 (\nabla_Y S)(X, \xi) &= \nabla_Y S(X, \xi) - S(\nabla_Y X, \xi) - S(X, \nabla_Y \xi) \\
 &= (n-1) \{ Y (\alpha^2 - \rho) \eta(X) + (\alpha^2 - \rho) \\
 & \quad \times [\eta(\nabla_Y X) + \alpha g(X, \phi Y)] \} \\
 & \quad - (n-1) (\alpha^2 - \rho) \eta(\nabla_Y X) - \alpha S(X, \phi Y) \\
 &= (n-1) Y (\alpha^2 - \rho) \eta(X) \\
 & \quad + \alpha (n-1) (\alpha^2 - \rho) g(X, \phi Y) - \alpha S(X, \phi Y).
 \end{aligned} \tag{50}$$

If an $(LCS)_n$ -manifold M Ricci symmetric, then Theorem 7 and (43) imply that

$$S(X, \phi Y) = (n-1) (\alpha^2 - \rho) g(\phi Y, X). \tag{51}$$

This leads us to state the following.

Theorem 9. *If an $(LCS)_n$ -manifold M is Ricci symmetric, then it is an Einstein manifold.*

Corollary 10. *If an $(LCS)_n$ -manifold M is locally symmetric, then it is an Einstein manifold.*

In this section, an example is used to demonstrate that the method presented in this paper is effective. But this example is a special case of Example 6.1 of [6].

Example 11. Now, we consider the 3-dimensional manifold

$$M = \{ (x, y, z) \in \mathbb{R}^3, z \neq 0 \}, \tag{52}$$

where (x, y, z) denote the standard coordinates in $\mathbb{R}^3$. The vector fields

$$\begin{aligned}
 e_1 &= e^z \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right), & e_2 &= e^z \frac{\partial}{\partial y}, \\
 e_3 &= \frac{\partial}{\partial z}
 \end{aligned} \tag{53}$$

are linearly independent of each point of M . Let g be the Lorentzian metric tensor defined by

$$\begin{aligned}
 g(e_1, e_1) &= g(e_2, e_2) = -g(e_3, e_3) = 1, \\
 g(e_i, e_j) &= 0, \quad i \neq j,
 \end{aligned} \tag{54}$$

for $i, j = 1, 2, 3$. Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \Gamma(TM)$. Let ϕ be the (1,1)-tensor field defined by

$$\phi e_1 = e_1, \quad \phi e_2 = e_2, \quad \phi e_3 = 0. \quad (55)$$

Then using the linearity of ϕ and g , we have $\eta(e_3) = -1$,

$$\phi^2 Z = Z + \eta(Z) e_3, \quad (56)$$

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W),$$

for all $Z, W \in \Gamma(TM)$. Thus for $\xi = e_3$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M .

Now, let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g , and let R be the Riemannian curvature tensor of g . Then we have

$$[e_1, e_2] = -e^z e_2, \quad [e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2. \quad (57)$$

Making use of the Koszul formulae for the Lorentzian metric tensor g , we can easily calculate the covariant derivations as follows:

$$\begin{aligned} \nabla_{e_1} e_1 &= -e_3, & \nabla_{e_2} e_1 &= e^z e_2, & \nabla_{e_1} e_3 &= -e_1, \\ \nabla_{e_2} e_3 &= -e_2, & \nabla_{e_2} e_2 &= -e^z e_1 - e_3, & \\ \nabla_{e_1} e_2 &= \nabla_{e_3} e_1 = \nabla_{e_3} e_2 = \nabla_{e_3} e_3 = 0. \end{aligned} \quad (58)$$

From the previously mentioned, it can be easily seen that (ϕ, ξ, η, g) is an $(LCS)_3$ -structure on M , that is, M is an $(LCS)_3$ -manifold with $\alpha = -1$ and $\rho = 0$. Using the previous relations, we can easily calculate the components of the Riemannian curvature tensor as follows:

$$\begin{aligned} R(e_1, e_2)e_1 &= (e^{2z} - 1)e_2, & R(e_1, e_2)e_2 &= (1 - e^{2z})e_1, \\ R(e_1, e_3)e_1 &= -e_3, & R(e_1, e_3)e_3 &= -e_1, \\ R(e_2, e_3)e_2 &= -e_3, & R(e_2, e_3)e_3 &= -e_2, \\ R(e_1, e_2)e_3 &= R(e_1, e_3)e_2 = R(e_2, e_3)e_1 = 0. \end{aligned} \quad (59)$$

By using the properties of R and definition of the Ricci tensor, we obtain

$$\begin{aligned} S(e_1, e_1) &= S(e_2, e_2) = -e^{2z}, & S(e_3, e_3) &= -2, \\ S(e_1, e_2) &= S(e_1, e_3) = S(e_2, e_3) = 0. \end{aligned} \quad (60)$$

Thus the scalar curvature τ of M is given by

$$\tau = \sum_{i=1}^3 g(e_i, e_i) S(e_i, e_i) = 2(1 - e^{2z}). \quad (61)$$

On the other hand, for any $Z, W \in \Gamma(TM)$, Z and W can be written as $Z = \sum_{i=1}^3 f_i e_i$ and $W = \sum_{j=1}^3 g_j e_j$, where f_i and g_i are smooth functions on M . By direct calculations, we have

$$\begin{aligned} S(Z, W) &= -e^{2z}(f_1 g_1 + f_2 g_2) - 2f_3 g_3 \\ &= -e^{2z}(f_1 g_1 + f_2 g_2 - f_3 g_3) - f_3 g_3(e^{2z} + 2). \end{aligned} \quad (62)$$

Since $\eta(Z) = -f_3$ and $\eta(W) = -g_3$ and $g(Z, W) = f_1 g_1 + f_2 g_2 - f_3 g_3$, we have

$$S(Z, W) = -e^{2z}g(Z, W) - (e^{2z} + 2)\eta(Z)\eta(W). \quad (63)$$

This tells us that M is an η -Einstein manifold.

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