Research Article

Persistence Property and Estimate on Momentum Support for the Integrable Degasperis-Procesi Equation

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It is shown that a strong solution of the Degasperis-Procesi equation possesses persistence property in the sense that the solution with algebraically decaying initial data and its spatial derivative must retain this property. Moreover, we give estimates of measure for the momentum support.

1. Introduction

Recently, Degasperis and Procesi [1] consider the following family of third order dispersive conservation laws:

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = (c_1 u^2 + c_2 u_x^2 + c_3 uu_{xx})_x, \quad (1)$$

where α , γ , c_0 , c_1 , c_2 , and c_3 are real constants. Within this family, only three equations that satisfy asymptotic integrability condition up to third order are singled out, namely, the KdV equation

$$u_t + u_x + uu_x + u_{xxx} = 0, \quad (2)$$

the Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad (3)$$

and a new equation (the Degasperis-Procesi equation, the DP equation, for simplicity) which can be written as (after rescaling) the dispersionless form [1]

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \quad (4)$$

It is worth noting that in [2] both the Camassa-Holm and DP equations are derived as members of a one-parameter family of asymptotic shallow water approximations to the Euler equations: this is important because it shows that (after

the addition of linear dispersion terms) both the Camassa-Holm and DP equations are physically relevant; otherwise the DP equation would be of purely theoretical interest.

When $c_1 = -3c_3/2\alpha^2$ and $c_2 = c_3/2$ in (1), we recover the Camassa-Holm equation derived physically by Camassa and Holm in [3] by approximating directly the Hamiltonian for Euler's equations in the shallow water regime, where $u(x, t)$ represents the free surface above a flat bottom. There is also a geometric approach which is used to prove the least action principle holding for the Camassa-Holm equation, compared with [4]. It is worth pointing out that a fundamental aspect of the Camassa-Holm equation, the fact that it is a completely integrable system, was shown in [5, 6]. Some satisfactory results have been obtained for this shallow water equation recently, we refer the readers to see [7–19].

Although, the DP equation (4) has a similar form to the Camassa-Holm equation and admits exact peakon solutions analogous to the Camassa-Holm peakons [20], these two equations are pretty different. The isospectral problem for equation (4) is

$$\Psi_x - \Psi_{xxx} - \lambda y \Psi = 0, \quad (5)$$

while for Camassa-Holm equation it is

$$\Psi_{xx} - \frac{1}{4}\Psi - \lambda y \Psi = 0, \quad (6)$$

where $y = u - u_{xx}$ for both cases. This implies that the inside structures of the DP equation (4) and the Camassa-Holm equation are truly different. However, we not only have some similar results [21–23], but also have considerable differences in the scattering/inverse scattering approach, compared with the discussion in [5, 6] and in the paper [24].

Analogous to the Camassa-Holm equation, (4) can be written in Hamiltonian form and has infinitely many conservation laws. Here we list some of the simplest conserved quantities [20]:

$$\begin{aligned} H_{-1} &= \int_{\mathbb{R}} u^3 dx, & H_0 &= \int_{\mathbb{R}} y dx, & H_1 &= \int_{\mathbb{R}} yv dx, \\ H_5 &= \int_{\mathbb{R}} y^{1/3} dx, & H_7 &= \int_{\mathbb{R}} (y_x^2 y^{-7/3} + 9y^{-1/3}) dx, \end{aligned} \quad (7)$$

where $v = (4 - \partial_x^2)^{-1}u$. So they are different from the invariants of the Camassa-Holm equation

$$E(u) = \int_{\mathbb{R}} (u^2 + u_x^2) dx, \quad F(u) = \int_{\mathbb{R}} (u^3 + uu_x^2) dx. \quad (8)$$

Set $Q = (1 - \partial_x^2)$; then the operator Q^{-1} in $\mathbb{R}$ can be expressed by

$$Q^{-1}f = G * f = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} f(y) dy. \quad (9)$$

Equation (4) can be written as

$$u_t + uu_x + \partial_x G * \left(\frac{3}{2}u^2\right) = 0, \quad (10)$$

while the Camassa-Holm equation can be written as

$$u_t + uu_x + \partial_x G * \left(u^2 + \frac{1}{2}u_x^2\right) = 0. \quad (11)$$

On the other hand, the DP equation can also be expressed in the following momentum form:

$$\begin{aligned} y_t + y_x u &= -3yu_x \\ y &= (1 - \partial_x^2)u. \end{aligned} \quad (12)$$

This formulation is important to motivate us to consider the measure of momentum support which is the second object of this paper, since we found that (12) is similar to the vorticity equation of the three-dimensional Euler equation for incompressible perfect fluids (U is the speed, and ω is its vorticity)

$$\begin{aligned} \omega_t + (U \cdot \nabla)\omega &= (\omega \cdot \nabla)U, \\ \operatorname{div} U &= 0, \\ \operatorname{curl} U &= \omega. \end{aligned} \quad (13)$$

The stretching term $(\omega \cdot \nabla)U$ in (13) is similar to the term $-3yu_x$ in (12).

One can follow the argument for the Camassa-Holm equation [8] to establish the following well posedness theorem for the Degasperis-Procesi equation.

Theorem 1 (see [23]). *Given $u(x, t = 0) = u_0 \in H^s(\mathbb{R})$, $s > 3/2$, then there exist a T and a unique solution u to (4) (also (10)) such that*

$$u(x, t) \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R})). \quad (14)$$

It should be mentioned that due to the form of (10) (no derivative appears in the convolution term), Coclite and Karlsen [25] established global existence and uniqueness result for entropy weak solutions belonging to the class $L^1(\mathbb{R}) \cap BV(\mathbb{R})$.

2. Unique Continuation

The purpose of this section is to show that the solution to (10) and its first-order spatial derivative retain algebraic decay at infinity as their initial values do. Precisely, we prove.

Theorem 2. *Assume that for some $T > 0$ and $s > 3/2$, $u \in C([0, T]; H^s(\mathbb{R}))$ is a strong solution of the initial value problem associated with (10), and that $u_0(x) = u(x, 0)$ satisfies that for some $\theta > 1$*

$$|u_0(x)|, \quad |\partial_x u_0(x)| = O(x^{-\theta}) \quad \text{as } x \uparrow \infty. \quad (15)$$

Then

$$|u(x, t)|, \quad |\partial_x u(x, t)| = O(x^{-\theta}) \quad \text{as } x \uparrow \infty, \quad (16)$$

uniformly in the time interval $[0, T]$.

Notation. We will say that

$$|f(x)| = O(x^{-\theta}) \quad \text{as } x \uparrow \infty \quad \text{if } \lim_{x \rightarrow \infty} \frac{|f(x)|}{x^{-\theta}} = L, \quad (17)$$

where L is a nonnegative constant.

Proof. We introduce the following notations:

$$F(u) = \frac{3}{2}u^2, \quad (18)$$

$$M = \sup_{t \in [0, T]} \|u(t)\|_{H^s}. \quad (19)$$

Multiplying (10) by u^{2p-1} with $p \in \mathbb{Z}^+$ and integrating the result in the x -variable, one gets

$$\int_{-\infty}^{\infty} u^{2p-1} (u_t + uu_x + \partial_x G * F(u)) dx = 0. \quad (20)$$

The first term in (20) is

$$\begin{aligned} \int_{-\infty}^{\infty} u^{2p-1} u_t dx &= \int_{-\infty}^{\infty} \frac{1}{2p} \frac{du^{2p}}{dt} dx \\ &= \frac{1}{2p} \frac{d}{dt} \int_{-\infty}^{\infty} u^{2p} dx = \|u(t)\|_{2p}^{2p-1} \frac{d}{dt} \|u(t)\|_{2p}, \end{aligned} \quad (21)$$

and for the rest, we have

$$\begin{aligned} \left| \int_{-\infty}^{\infty} u^{2p-1} u u_x dx \right| &= \left| \int_{-\infty}^{\infty} u^{2p} u_x dx \right| \\ &\leq \|u_x(t)\|_{\infty} \|u(t)\|_{2p}^{2p}, \\ \left| \int_{-\infty}^{\infty} u^{2p-1} \partial_x G * F(u) dx \right| &\leq \|u(t)\|_{2p}^{2p-1} \|\partial_x G * F(u)(t)\|_{2p}. \end{aligned} \quad (22)$$

From the above inequalities, we get

$$\frac{d}{dt} \|u(t)\|_{2p} \leq \|u_x(t)\|_{\infty} \|u(t)\|_{2p} + \|\partial_x G * F(u)\|_{2p}, \quad (23)$$

and therefore, by Sobolev embedding theorem and Gronwall's inequality, there exists a constant M such that

$$\|u(t)\|_{2p} \leq \left(\|u(0)\|_{2p} + \int_0^t \|\partial_x G * F(u)\|_{2p} d\tau \right) e^{Mt}. \quad (24)$$

Since $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ implies

$$\lim_{q \rightarrow \infty} \|f\|_q = \|f\|_{\infty}, \quad (25)$$

taking the limits in (24) (note that $\partial_x G \in L^1$ and $F(u) \in L^1 \cap L^\infty$) from (25) we get

$$\|u(t)\|_{\infty} \leq \left(\|u(0)\|_{\infty} + \int_0^t \|\partial_x G * F(u)\|_{\infty} d\tau \right) e^{Mt}. \quad (26)$$

We will now repeat the above arguments using the barrier function

$$\varphi_N(x) = \begin{cases} 1, & x \leq 1, \\ x^\theta, & x \in (1, N), \\ N^\theta, & x \geq N, \end{cases} \quad (27)$$

where $N \in \mathbb{Z}^+$. Observe that for all N we have

$$0 \leq \varphi'_N(x) \leq \theta \varphi_N(x) \quad \text{a.e. } x \in \mathbb{R}. \quad (28)$$

Using notation in (18), from (10) we obtain

$$(u\varphi_N)_t + (u\varphi_N)u_x + \varphi_N \partial_x G * F(u) = 0. \quad (29)$$

Hence, as in the weightless case (26), we get

$$\begin{aligned} \|u(t)\varphi_N\|_{\infty} &\leq e^{Mt} \|u(0)\varphi_N\|_{\infty} \\ &+ e^{Mt} \int_0^t \|\varphi_N \partial_x G * F(u)\|_{\infty} d\tau. \end{aligned} \quad (30)$$

A simple calculation shows that there exists $C_0 > 0$ depending only on θ such that, for any $N \in \mathbb{Z}^+$,

$$\frac{1}{2} \varphi_N(x) \int_{-\infty}^{\infty} e^{-|x-y|} \frac{1}{\varphi_N(y)} dy \leq C_0. \quad (31)$$

Thus, for any appropriate function f one finds that

$$\begin{aligned} &|\varphi_N \partial_x G * f^2(x)| \\ &= \left| \frac{1}{2} \varphi_N(x) \int_{-\infty}^{\infty} \operatorname{sgn}(x-y) e^{-|x-y|} f^2(y) dy \right| \\ &\leq \frac{\varphi_N(x)}{2} \int_{-\infty}^{\infty} e^{-|x-y|} \frac{1}{\varphi_N(y)} \varphi_N(y) f(y) f(y) dy \\ &\leq \left(\frac{\varphi_N(x)}{2} \int_{-\infty}^{\infty} \frac{e^{-|x-y|}}{\varphi_N(y)} dy \right) \|\varphi_N f\|_{\infty} \|f\|_{\infty} \\ &\leq C_0 \|\varphi_N f\|_{\infty} \|f\|_{\infty}. \end{aligned} \quad (32)$$

Combining with (30), we get

$$\|u(t)\varphi_N\|_{\infty} \leq C_1 \left(\|u_0\varphi_N\|_{\infty} + \int_0^t \|\varphi_N u\|_{\infty} d\tau \right), \quad (33)$$

where $C_1 = C_1(M; T, \cdot) > 0$. By Gronwall's inequality, there exists a constant $\bar{C}$ for any $t \in [0, T]$ such that

$$\|\varphi_N u\|_{\infty} \leq \bar{C} \|u_0\varphi_N\|_{\infty} \leq \bar{C} \|u_0 \cdot \max(1, x^\theta)\|_{\infty}. \quad (34)$$

Finally, taking the limit as N goes to infinity in (34) we find that for any $t \in [0, T]$

$$|u(x, t) x^\theta| \leq \bar{C} \|u_0 \cdot \max(1, x^\theta)\|_{\infty}. \quad (35)$$

From (15), we get $|u(x, t)| = O(x^{-\theta})$ as $x \uparrow \infty$.

Next, differentiating (10) in the x -variable produces the equation

$$u_{xt} + uu_{xx} + u_x^2 + \partial_x^2 G * \left(\frac{3}{2} u^2 \right) = 0. \quad (36)$$

Again, multiplying (36) by u_x^{2p-1} , ($p \in \mathbb{Z}^+$), integrating the result in the x -variable, and using integration by parts

$$\begin{aligned} \int_{-\infty}^{\infty} uu_{xx}(u_x)^{2p-1} dx &= \int_{-\infty}^{\infty} u \frac{(u_x)^{2p}}{2p} dx \\ &= -\frac{1}{2p} \int_{-\infty}^{\infty} u_x (u_x)^{2p} dx, \end{aligned} \quad (37)$$

one gets the inequality

$$\frac{d}{dt} \|u_x(t)\|_{2p} \leq 2 \|u_x(t)\|_{\infty} \|u_x(t)\|_{2p} + \|\partial_x^2 G * F(u)\|_{2p}, \quad (38)$$

and therefore as before

$$\|u_x(t)\|_{2p} \leq \left(\|u_x(0)\|_{2p} + \int_0^t \|\partial_x^2 G * F(u)\|_{2p} d\tau \right) e^{2Mt}. \quad (39)$$

Since $\partial_x^2 G = G - \delta$, we can use (25) and pass to the limit in (39) to obtain

$$\|u_x(t)\|_\infty \leq \left(\|u_x(0)\|_\infty + \int_0^t \|\partial_x^2 G * F(u)\|_\infty d\tau \right) e^{2Mt}; \quad (40)$$

from (36) we get

$$\partial_t(u_x \varphi_N) + uu_{xx} \varphi_N + (u_x \varphi_N) u_x + \varphi_N \partial_x^2 G * F(u) = 0. \quad (41)$$

We need to eliminate the second derivatives in the second term in (41). Thus, combining integration by parts and (28), we find

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} uu_{xx} \varphi_N (u_x \varphi_N)^{2p-1} dx \right| \\ &= \left| \int_{-\infty}^{\infty} u (u_x \varphi_N)^{2p-1} (\partial_x (u_x \varphi_N) - u_x \varphi_N') dx \right| \\ &= \left| \int_{-\infty}^{\infty} u \left(\partial_x \left(\frac{(u_x \varphi_N)^{2p}}{2p} \right) - u_x \varphi_N' (u_x \varphi_N)^{2p-1} \right) dx \right| \\ &\leq \kappa \cdot (\|u(t)\|_\infty + \|\partial_x u(t)\|_\infty) \|\partial_x u \varphi_N\|_{2p}^{2p}. \end{aligned} \quad (42)$$

Since $\partial_x^2 G = G - \delta$, the argument in (32) also shows that

$$|\varphi_N \partial_x^2 G * f^2(x)| \leq C_0 \|\varphi_N f\|_\infty \|f\|_\infty. \quad (43)$$

Similarly, we get

$$\begin{aligned} & \|u_x(t) \varphi_N\|_\infty \\ &\leq C_2 \left(\|u_x(0) \varphi_N\|_\infty + \int_0^t \|u(\tau) \varphi_N\|_\infty d\tau \right), \end{aligned} \quad (44)$$

where $C_2 = C_2(M; T)$.

Then, taking the limit as N goes to infinity, we find that for any $t \in [0, T]$

$$|u_x(t) x^\theta| \leq C_2 \left(\|u_x(0) x^\theta\|_\infty + \int_0^t \|u(\tau) x^\theta\|_\infty d\tau \right). \quad (45)$$

Since $|u(x, t)| = O(x^{-\theta})$ as $x \uparrow \infty$ and (15), we get

$$|\partial_x u(x, t)| = O(x^{-\theta}), \quad \text{as } x \uparrow \infty. \quad (46)$$

This completes the proof. $\square$

3. Measure of Momentum Support

It is known that, for the Degasperis-Procesi equation, the momentum density $y(x, t)$ with compactly supported initial data $y_0(x)$ will retain this property; that is, $y(x, t)$ is also compactly supported [21]. However, the same argument for $u(x, t)$ is false [21]. Note that a detailed description of solution $u(x, t)$ outside of the support of $y(x, t)$ is given in [26, 27].

Moreover, the exponential behavior of u in x outside this support is obvious. The comparison of the DP equation and the incompressible Euler equation above implies that the momentum $y(x, t)$ in (12) plays a similar role as the vorticity does in (13). This motivates us to estimate the size of $\text{supp } y(t, \cdot)$ for strong solutions. The approach is inspired by the work of Kim [28] and the recent work [29].

We first introduce the particle trajectory method. Let $u \in C([0, T], H^3(\mathbb{R})) \cap C^1([0, T], H^2(\mathbb{R}))$ be a strong solution of (4) guaranteed by the well posedness Theorem 1. Let $s \in [0, T]$, $q(t; \alpha, s)$ be the solution of the following initial value problem:

$$\begin{aligned} \frac{dq(t; \alpha, s)}{dt} &= u(s + t, q(t; \alpha, s)), \quad s, s + t \in [0, T], \quad \alpha \in \mathbb{R}, \\ q(0; \alpha, s) &= \alpha, \quad \alpha \in \mathbb{R}. \end{aligned} \quad (47)$$

Then, $q(t; \cdot, s) : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing diffeomorphism. It is shown [21, 23] that

$$y(q(t; x, 0), t) q_x^3(t; x, 0) = y(x, 0); \quad (48)$$

this implies that the support of y propagates along the flow. Set $D(t)$ to be the support of $y(\cdot, t)$. Let $\psi \in L^2(D(s))$, and let $\psi^t \in L^2(D(s + t))$ be given by the following:

$$\psi^t(q(t; \alpha, s)) = \psi(\alpha). \quad (49)$$

Moreover, we also want to mention the standard argument on the first Dirichlet eigenvalue problem. Let Ω be an open interval in $\mathbb{R}$, and, $\lambda_1(\Omega)$ be the first Dirichlet eigenvalue of the Laplacian on Ω . Then we have

$$\lambda_1(\Omega) = \inf \left\{ \|\phi'\|_{L^2(\Omega)}^2 \mid \phi \in H_0^1(\Omega) \text{ with } \|\phi\|_{L^2(\Omega)} = 1 \right\}. \quad (50)$$

It is just $(\pi/|\Omega|)^2$ and the normalized eigenfunctions are the suitable translations of

$$\pm \left(\frac{2}{|\Omega|} \right)^{1/2} \sin \left(\frac{\pi x}{|\Omega|} \right). \quad (51)$$

Theorem 3. Let $y \in C([0, T]; H^1(\mathbb{R})) \cap C^1([0, T]; L^2(\mathbb{R}))$ be a strong solution of (12). Let $D(t)$ be the support of $y(\cdot, t)$ for $t \in [0, T]$ with its initial $D(0)$ being connected.

(I) Suppose there exists a positive constant K such that $u_x(x, k) > -K$ for $(x, t) \in \mathbb{R} \times [0, T]$. Then

$$\begin{aligned} & |D(0)| e^{-(\exp(5KT/2) \|y_0\|_{L^2(\mathbb{R})})^t} \\ &\leq |D(t)| \leq |D(0)| e^{(\exp(5KT/2) \|y_0\|_{L^2(\mathbb{R})})^t}. \end{aligned} \quad (52)$$

(II) y_0 does not change sign or

$$\begin{aligned} y_0(x) &\leq 0, & x &\in (-\infty, x_0), \\ y_0(x) &\geq 0, & x &\in (x_0, \infty), \end{aligned} \quad (53)$$

and $y_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$; then, for all $t \geq 0$

$$\begin{aligned} |D(0)| e^{-\|y_0\|_{L^1(\mathbb{R})} t} &\leq |D(t)| \\ &\leq |D(0)| e^{\|y_0\|_{L^1(\mathbb{R})} t}. \end{aligned} \quad (54)$$

Proof. (I) The relation of momenta y and u gives

$$u(x, t) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-\xi|} y(\xi, t) d\xi, \quad (55)$$

$$u_x(x, t) = \frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn}(\xi - x) e^{-|x-\xi|} y(\xi, t) d\xi. \quad (56)$$

Then, we have by (12) and the lower bound of u_x

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} y^2(x, t) dx \\ = -5 \int_{\mathbb{R}} u_x(x, t) y^2(x, t) dx \leq 5K \int_{\mathbb{R}} y^2(x, t) dx. \end{aligned} \quad (57)$$

Thus

$$\frac{d}{dt} \|y(x, t)\|_{L^2}^2 \leq 5K \|y(x, t)\|_{L^2}^2. \quad (58)$$

Therefore, (56), (58), and Gronwall inequality imply that

$$|u_x(x, t)| \leq \frac{1}{2} \|y(x, t)\|_{L^2} \leq \frac{1}{2} e^{5KT/2} \|y_0\|_{L^2}. \quad (59)$$

On the other hand, due to Propositions A.2 and A.3, $\lambda_1(D(s))$ is Lipschitz and differentiable almost everywhere. Moreover, we have

$$-4M_1 \lambda_1(D(s)) \leq \frac{d}{ds} \lambda_1(D(s)) \leq 4M_1 \lambda_1(D(s)). \quad (60)$$

Then, it follows that

$$e^{-4M_1 s} \lambda_1(D(0)) \leq \lambda_1(D(s)) \leq e^{4M_1 s} \lambda_1(D(0)) \quad (61)$$

with $\lambda_1(D(s)) = \pi^2/|D(s)|^2$. So (52) follows from (61) and (59).

(II) If $y_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ does not change sign, we conclude that solutions of (10) exist globally in time. Equality (56) and the conservation of $\int_{\mathbb{R}} y(x, t) dx$ yield

$$|u_x(x, t)| \leq \frac{1}{2} \|y(x, t)\|_{L^1(\mathbb{R})} = \frac{1}{2} \|y_0(x)\|_{L^1(\mathbb{R})}. \quad (62)$$

By similar arguments of (I), constant M_1 in (61) can be replaced by $\|y_0(x)\|_{L^1(\mathbb{R})}/2$; then (54) follows. If (53) is satisfied, we know that the solution of (10) exists globally in time [21, 30]. From (53) and (48), it is easy to get

$$\begin{aligned} y(x, t) &\leq 0, & x &\in (-\infty, q(x_0, t)), \\ y(x, t) &\geq 0, & x &\in (q(x_0, t), \infty), \end{aligned} \quad (63)$$

where we denote $q(t; x, s)$ with $s = 0$ by $q(x, t)$. By direct computation, we have

$$\int_{\mathbb{R}} |y(x, t)| dx = \int_{q(x_0, t)}^{\infty} y(x, t) dx - \int_{-\infty}^{q(x_0, t)} y(x, t) dx. \quad (64)$$

Next, we prove that $\|y(x, t)\|_{L^1(\mathbb{R})}$ is decreasing with respect to time. To this end, one gets, by differentiating (64) with respect to t and integrating by parts,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} |y(x, t)| dx &= \int_{q(x_0, t)}^{\infty} y_t(x, t) dx \\ &\quad - \int_{-\infty}^{q(x_0, t)} y_t(x, t) dx \\ &\quad - 2(yu)(q(x_0, t), t) \\ &= - \int_{q(x_0, t)}^{\infty} (y_x u + 3yu_x) dx \\ &\quad + \int_{-\infty}^{q(x_0, t)} (y_x u + 3yu_x) dx \\ &\quad - 2(yu)(q(x_0, t), t) \\ &= -2 \int_{q(x_0, t)}^{\infty} yu_x dx + 2 \int_{-\infty}^{q(x_0, t)} yu_x dx \\ &= u^2(q(x_0, t), t) - u_x^2(q(x_0, t), t) \\ &= \int_{q(x_0, t)}^{\infty} e^{-\xi} y(\xi, t) dx \int_{-\infty}^{q(x_0, t)} e^{\xi} y(\xi, t) dx \\ &\leq 0. \end{aligned} \quad (65)$$

This implies that

$$|u_x(x, t)| \leq \frac{1}{2} \|y(x, t)\|_{L^1(\mathbb{R})} \leq \frac{1}{2} \|y_0(x)\|_{L^1(\mathbb{R})}. \quad (66)$$

Therefore, (54) follows by replacing M_1 with $\|y_0(x)\|_{L^1(\mathbb{R})}/2$ in (61). $\square$

Appendix

The following propositions with standard proofs are known in [29]; we list them here only for convenience of readers.

Proposition A.1. Let $s, s+t \in [0, T]$, $\alpha \in D(s)$, and $\psi \in H_0^1(D(s))$; u_x can be bounded by a constant M_1 ; then

$$(a) \quad e^{-M_1|t|} \leq q_{\alpha}(t; \alpha, s) \leq e^{M_1|t|}, \quad (A.1)$$

$$\begin{aligned} (b) \quad &|\psi'(\alpha)| e^{-M_1|t|} \leq \left| (\psi^t)'(q(t; \alpha, s)) \right| \\ &\leq |\psi'(\alpha)| e^{M_1|t|}, \end{aligned} \quad (A.2)$$

(c)

$$\begin{aligned} \|\psi\|_{L^2(D(s))} e^{-M_1|t|/2} &\leq \|\psi^t\|_{L^2(D(s+t))} \\ &\leq \|\psi\|_{L^2(D(s))} e^{M_1|t|/2}. \end{aligned} \quad (\text{A.3})$$

Proof. (a) Differentiating (47) with respect to α , we obtain

$$\frac{dq_t}{d\alpha} = u_q q_\alpha. \quad (\text{A.4})$$

Since $q(t; \cdot, s) : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing diffeomorphism, then $q_\alpha > 0$. Combining the bound of u_x , there holds

$$-M_1 q_\alpha \leq q_{\alpha t} \leq M_1 q_\alpha. \quad (\text{A.5})$$

This can be solved as (a).

(b) Differentiating (49) with respect to α to get

$$\psi_q^t q_\alpha = \psi'(\alpha), \quad (\text{A.6})$$

then (A.2) is a direct consequence of (A.1).

(c) Equation (49) and the definition of Sobolev norm give that

$$\|\psi^t\|_{L^2(D(s+t))}^2 = \int_{D(s+t)} \psi^t(x)^2 dx = \int_{D(s)} \psi^2(\alpha) q_\alpha d\alpha, \quad (\text{A.7})$$

where we have used the change of variable $x = q(t; \alpha, s)$. So (A.3) follows from (A.1). $\square$

Proposition A.2. Under the hypothesis of Theorem 3, for $s, s+t \in [0, T]$,

$$\begin{aligned} \limsup_{t \rightarrow 0^+} \frac{\lambda_1(D(s+t)) - \lambda_1(D(s))}{t} &\leq 4M_1 \lambda_1(D(s)), \\ \liminf_{t \rightarrow 0^-} \frac{\lambda_1(D(s+t)) - \lambda_1(D(s))}{t} &\geq -4M_1 \lambda_1(D(s)). \end{aligned} \quad (\text{A.8})$$

Proof. Let $t > 0$, $\phi_1 \in H_0^1(D(s))$ with $\|\phi_1\|_{L^2(D(s))} = 1$ be a first normalized eigenfunction on $D(s)$. Then, for $\varphi \in H_0^1(D(s+t))$ with $\|\varphi\|_{L^2(D(s+t))} = 1$, we have

$$\begin{aligned} \lambda_1(D(s+t)) - \lambda_1(D(s)) &= \inf \|\varphi'\|_{L^2(D(s+t))}^2 - \|\phi_1'\|_{L^2(D(s))}^2 \\ &\leq \|\phi_1^t\|_{L^2(D(s+t))}^{-2} \left\| (\phi_1^t)' \right\|_{L^2(D(s+t))}^2 \\ &\quad - \|\phi_1'\|_{L^2(D(s))}^2. \end{aligned} \quad (\text{A.9})$$

Furthermore

$$\begin{aligned} &\|\phi_1^t\|_{L^2(D(s+t))}^{-2} \left\| (\phi_1^t)' \right\|_{L^2(D(s+t))}^2 \\ &= \|\phi_1^t\|_{L^2(D(s+t))}^{-2} \int_{D(s)} \left[(\phi_1^t)' \right]^2 q_\alpha d\alpha \\ &\leq \|\phi_1^t\|_{L^2(D(s+t))}^{-2} e^{3M_1 t} \|\phi_1'\|_{L^2(D(s))}^2 \\ &\leq e^{4M_1 t} \|\phi_1'\|_{L^2(D(s))}^2. \end{aligned} \quad (\text{A.10})$$

Combing (A.9) and (A.10) together yields

$$\begin{aligned} \limsup_{t \rightarrow 0^+} \frac{\lambda_1(D(s+t)) - \lambda_1(D(s))}{t} &\leq \limsup_{t \rightarrow 0^+} \frac{e^{4M_1 t} \|\phi_1'\|_{L^2(D(s))}^2 - \|\phi_1'\|_{L^2(D(s))}^2}{t} \\ &= 4M_1 \lambda_1(D(s)). \end{aligned} \quad (\text{A.11})$$

The second one follows by similar arguments for $t < 0$. $\square$

Proposition A.3. Under the hypothesis of Theorem 3, for $s, s+t \in [0, T]$,

$$\begin{aligned} \limsup_{t \rightarrow 0^-} \frac{\lambda_1(D(s+t)) - \lambda_1(D(s))}{t} &\leq 4M_1 \lambda_1(D(s)), \\ \liminf_{t \rightarrow 0^+} \frac{\lambda_1(D(s+t)) - \lambda_1(D(s))}{t} &\geq -4M_1 \lambda_1(D(s)). \end{aligned} \quad (\text{A.12})$$

Proof. Let $\phi_1 \in H_0^1(D(s))$ with $\|\phi_1\|_{L^2(D(s))} = 1$ be a first normalized eigenfunction on $D(s)$, and let $\phi_2 \in L^2(D(s))$ be such that its t -transport is a normalized first eigenfunction on $D(s+t)$. For $t > 0$, using the left halves of (A.1) and (A.2) and then the right half of (A.3) we get

$$\begin{aligned} \left\| (\phi_2^t)' \right\|_{L^2(D(s+t))}^2 &= \int_{D(s+t)} \left[(\phi_2^t(x))' \right]^2 dx \\ &= \int_{D(s)} \left[(\phi_2^t)' \right]^2 q_\alpha d\alpha \\ &\geq e^{-3M_1 t} \int_{D(s)} [\phi_2'(\alpha)]^2 d\alpha \\ &= e^{-3M_1 t} \|\phi_2\|_{L^2(D(s))}^2 \left\| \left(\frac{\phi_2}{\|\phi_2\|_{L^2(D(s))}} \right)' \right\|_{L^2(D(s))}^2 \\ &\geq e^{-4M_1 t} \|\phi_2^t\|_{L^2(D(s+t))}^2 \lambda_1(D(s)) \\ &= e^{-4M_1 t} \lambda_1(D(s)). \end{aligned} \quad (\text{A.13})$$

Hence

$$\begin{aligned} \liminf_{t \rightarrow 0^+} \frac{\lambda_1(D(s+t)) - \lambda_1(D(s))}{t} &\geq \liminf_{t \rightarrow 0^+} \frac{e^{-4M_1 t} - 1}{t} \lambda_1(D(s)) \\ &= -4M_1 \lambda_1(D(s)). \end{aligned} \quad (\text{A.14})$$

The other part is similar. $\square$

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