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*Research Article*

# Existence of Prescribed $L^2$-Norm Solutions for a Class of Schrödinger-Poisson Equation

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By using the standard scaling arguments, we show that the infimum of the following minimization problem: $I_{\rho^2} = \inf\{(1/2)\int_{\mathbb{R}^3}|\nabla u|^2dx + (1/4)\iint_{\mathbb{R}^3}(|u(x)|^2|u(y)|^2/|x-y|)dx\,dy - (1/p)\int_{\mathbb{R}^3}|u|^p dx : u \in B_\rho\}$ can be achieved for $p \in (2,3)$ and $\rho > 0$ small, where $B_\rho := \{u \in H^1(\mathbb{R}^3) : \|u\|_2 = \rho\}$. Moreover, the properties of $I_{\rho^2}/\rho^2$ and the associated Lagrange multiplier $\lambda_\rho$ are also given if $p \in (2, 8/3]$.

## 1. Introduction

In this paper, we consider the nonlinear Schrödinger-Poisson type equation:

$$-\Delta u + \left(|x|^{-1} * |u|^2\right)u - |u|^{p-2}u = \lambda u, \quad \text{in } \mathbb{R}^3, \tag{1}$$

where $\lambda \in \mathbb{R}$ is a parameter, $p \in (2,6)$, and $*$ denotes the convolution. Problems like (1) have attracted considerable attentions recently since a pair $(u,\lambda)$, solution of (1), corresponds to a solitary wave of the form $\psi(x,t) = e^{-i\lambda t}u(x)$ of the evolution equation:

$$i\psi_t + \Delta\psi - \left(|x|^{-1} * |\psi|^2\right)\psi + |\psi|^{p-2}\psi = 0, \quad \text{in } \mathbb{R}^3 \times \mathbb{R}^+, \tag{2}$$

which was obtained by approximation of a special case of Hatree-Fock equation with the frequency $\lambda$ describing a quantum mechanical system of many particles. For more mathematical and physical background of (2), we refer to [1–4] and the references therein.

In the case that the frequency $\lambda$ is a fixed and assigned parameter, the critical points of the following functional defined in $H^1(\mathbb{R}^3;\mathbb{R})$:

$$\begin{aligned} E(u) = {} & \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx - \frac{\lambda}{2}\int_{\mathbb{R}^3}|u|^2dx \\ & + \frac{1}{4}\iint_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}dx\,dy - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p dx, \end{aligned} \tag{3}$$

are the solutions of (1), where $E(u)$ is obviously well defined and is a $C^1$ functional for each $p \in (2,6)$ (cf. [5]). Such case has been extensively studied by using variational methods in the past decades including the existence, nonexistence, and multiplicity of solutions; see, for example, [5–12] and the references therein.

On the other hand, the physicists are often interested in the solutions with prescribed $L^2$-norm and unknown frequency $\lambda$, such a solution is called a "normalized solution," which is associated with the existence of stable standing waves. Precisely, by a "normalized solution" $(u_\rho, \lambda_\rho)$ of (1), we mean that

$$\left(u_\rho, \lambda_\rho\right) \in H^1\left(\mathbb{R}^3;\mathbb{C}\right) \times \mathbb{R} \text{ solves (1) with } \left\|u_\rho\right\|_2 = \rho. \tag{4}$$

Clearly, this kind of solutions can be obtained as the constrained critical points of the $C^1$ functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}\iint_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx\, dy - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx, \tag{5}$$

on the constraint

$$B_\rho = \left\{ u \in H^1\left(\mathbb{R}^3; \mathbb{C}\right) : \|u\|_2 = \rho \right\}. \tag{6}$$

Thus, the frequency $\lambda_\rho \in \mathbb{R}$ cannot be fixed any longer and it will appear as a Lagrange multiplier associated with the critical point $u_\rho$ on $B_\rho$. Among all the critical points of $I$ constrained on $B_\rho$, we are interested in the ones with minimal energy since the corresponded standing waves are orbitally stable under the flow of (2) and can provide us some information on the dynamics of (2). Therefore, we are reduced to study the minimization problem

$$I_{\rho^2} = \min_{u \in B_\rho} I(u), \tag{7}$$

for $p \in (2, 10/3)$. Here we note that, for each $\rho > 0$, $I_{\rho^2} \in (-\infty, 0]$ if $p \in (2, 10/3)$, and $I_{\rho^2} = -\infty$ if $p \in (10/3, 6)$ (cf. [13, Remark 1.1] or (15) below). When $p \in (10/3, 6)$ (now $I_{\rho^2} = -\infty$), by using a mountain pass argument, it was proved in [14] that $I$ has a critical point constrained on $B_\rho$ at a strictly positive energy level for $\rho > 0$ small, and this critical point is orbitally unstable.

The main difficulty of considering (7) is the lack of compactness for the (bounded) minimizing sequence $\{u_n\} \subset B_\rho$. We recall that the necessary and sufficient condition due to Lions [15, 16] in order that any minimizing sequence for (7) is relatively compact is the strong subadditivity inequality:

$$I_{\rho^2} < I_{\mu^2} + I_{\rho^2-\mu^2}, \quad \forall 0 < \mu < \rho. \tag{8}$$

In the range $p \in \{8/3\} \cup (3, 10/3)$, by using the standard scaling arguments, Bellazzini and Siciliano in [17] proved that (8) holds for $\rho > 0$ large. In the range $p \in (2, 3)$, Bellazzini and Siciliano also showed in [18] that (8) holds for $\rho > 0$ small, where they developed a new abstract theorem which guarantees the following condition $(MD)$ for $s > 0$ small:

$(MD)$ The function $s \mapsto I_{s^2}/s^2$ is monotone decreasing.

We remark that their abstract theorem heavily relies on the behavior of $I_{\rho^2}$ near zero; that is, to use the abstract theorem, one has to verify some extra conditions, such as

$$\rho \longmapsto I_{\rho^2} \text{ is continuous}; \quad \lim_{\rho \to 0} \frac{I_{\rho^2}}{\rho^2} = 0; \tag{9}$$

these are unnecessary if one can show (8) by using the standard scaling arguments like [17]. However, as mentioned in [18], the authors were not sure whether (8) can be proved or not by using the standard scaling arguments if $p \in (2, 3)$. Therefore, the first aim of this paper is to show that (8) holds for $\rho > 0$ small when $p \in (2, 3)$ by using the standard scaling arguments. To achieve this aim, we introduce a new subset $B_\rho \cap \mathscr{P}$ of $B_\rho$ (see details in Section 3), then we consider the minimization problem (7) constrained on $B_\rho \cap \mathscr{P}$ instead of $B_\rho$, and we use the standard scaling arguments to prove that (8) holds for $\rho > 0$ small. Moreover, we can get a specific estimate on $\rho$ that allows us to obtain the sign and the behavior of the Lagrange multiplier $\lambda_\rho$ if $p \in (2, 8/3]$; these are not considered in [18].

The other aim of this paper is to study the properties of the Lagrange multiplier $\lambda_\rho$ and the ratio $I_{\rho^2}/\rho^2$ corresponding to the solution $(u_\rho, \lambda_\rho)$ of (1) with $\|u_\rho\|_2 = \rho$. It is known that $\lambda_\rho$ and $I_{\rho^2}/\rho^2$ are interpreted in physics as the frequency and the ratio between the infimum of the energy of the standing waves with fixed charge and the charge itself, respectively, and the relevance of the energy/charge ratio for the existence of standing waves in field theories has been discussed under a general framework in [19].

Our main results read as follows.

**Theorem 1.** *All the minimizing sequences for* (7) *are precompact in* $H^1(\mathbb{R}^3; \mathbb{C})$ *up to translations provided that one of the following conditions holds*

(1) $p \in (2, 8/3]$ *and* $0 < \rho < \overline{\rho}_1 := 3^{(3p-8)/8(3-p)} \pi^{(3p-10)/8(3-p)} S^{3/2}((p-2)/2p)^{1/4(3-p)}$, *where* $S$ *is defined by* (12);

(2) $p \in (8/3, 3)$ *and* $0 < \rho < \overline{\rho}_2$ *for some* $\overline{\rho}_2 > 0$.

*In particular,* (1) *has a solution* $(u_\rho, \lambda_\rho) \in H^1(\mathbb{R}^3; \mathbb{C}) \times \mathbb{R}$ *such that* $I(u_\rho) = I_{\rho^2}$ *and* $\|u_\rho\|_2 = \rho$. *Moreover, if the above assumption (1) holds and* $(u_\rho, \lambda_\rho)$ *is a solution of* (1) *with* $\|u_\rho\|_2 = \rho > 0$ *and* $I(u_\rho) = I_{\rho^2}$, *then* $\lambda_\rho < 0$, $\lambda_\rho \to 0$ *and* $I_{\rho^2}/\rho^2 \to 0$ *as* $\rho \to 0$, *respectively.*

**Theorem 2.** *Let* $p \in (2, 12/5]$ *and let* $\rho > 0$. *If* $(u_\rho, \lambda_\rho)$ *is a solution of* (1) *with* $\|u_\rho\|_2 = \rho$, *then we have*

(i) $\lambda_\rho < 0$, $I(u_\rho) < 0$, $\lambda_\rho \to 0$ *as* $\rho \to 0$ *and there exists a positive constant* $C_1$, *independent of* $\rho$, *such that* $\lambda_\rho \in (-C_1, 0)$;

(ii) *there exists a positive constant* $C_2$, *independent of* $\rho$, *such that* $I(u_\rho)/\rho^2 \in (-C_2, 0)$. *In particular, if* $I(u_\rho) = I_{\rho^2}$, *then* $I_{\rho^2}/\rho^2 \in (-C_2, 0)$.

*Remarks.* (a) We point out that parts of Theorem 1 are already contained in [18, Theorem 4.1]. In the proof of Theorem 1, with $\overline{\rho}_1$ in hand, we can obtain some additional information of the Lagrange multiplier $\lambda_\rho$ and the ratio $I_{\rho^2}/\rho^2$ when $p \in (2, 8/3]$, and these are not contained in [18, Theorem 4.1]. However, we do not know whether $\overline{\rho}_1$ is optimal or not.

(b) Theorem 2(i) shows that (1) has only the zero solution if $p \in (2, 12/5]$ and $\lambda \geq 0$. In the case of $p \in (2, 3)$, it was shown in [5, 20] (see also [13, Remark 1.4]) that there exists $\lambda_0 < 0$ such that (1) has only the zero solution for $\lambda \in (-\infty, \lambda_0)$. The nonexistence results of nonzero solutions of (1) were also discussed in [13] for $p \in [3, 10/3]$.

(c) As we have anticipated, the existence of minimizers for $I_{\rho^2}$ is related to the existence and stability of the standing

wave solutions to (2). For the existence of stable standing wave solutions to (2), we refer to [4, 14, 17, 18, 20, 21] and the references therein.

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proof of the main theorems, especially in the proof of Theorem 1, we first define a new subset of $B_\rho$ and then analyze the properties of minimizing sequences for $I_{\rho^2}$ constrained on the new subset, and finally, we prove that (8) holds when $p \in (2, 8/3]$ and $p \in (8/3, 3)$, respectively.

## 2. Preliminaries

Throughout this paper, all the functions, unless otherwise stated, are complex valued, but for simplicity we will write $L^q(\mathbb{R}^3)$, $H^1(\mathbb{R}^3)$ and $\mathscr{D}^{1,2}(\mathbb{R}^3)$ defined in the following:

(i) $L^q(\mathbb{R}^3)$ is the usual Lebesgue space endowed with the norm $\|u\|_q := (\int_{\mathbb{R}^3} |u|^q dx)^{1/q}$, where $q \in [1, \infty)$;

(ii) $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the norm

$$\|u\| := \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + |u|^2\right) dx\right)^{1/2}; \tag{10}$$

(iii) $\mathscr{D}^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|\nabla u\|_2 = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{1/2}; \tag{11}$$

(iv) $S$ is the best Sobolev imbedding constant of $\mathscr{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ defined as

$$S := \inf\left\{\|\nabla u\|_2 : u \in \mathscr{D}^{1,2}\left(\mathbb{R}^3\right), \|u\|_6 = 1\right\}. \tag{12}$$

Moreover, the letter $C$ will denote a suitable positive constant, whose value may change in the same line, and the symbol $o(1)$ denotes a quantity which goes to zero. We also use $O(1)$ to denote a bounded quantity.

Let $\phi_u(x) = |x - y|^{-1} * |u|^2$, and then, for each $u \in H^1(\mathbb{R}^3)$, $\phi_u$ is the unique solution of the Poisson equation $-\Delta\phi = 4\pi|u|^2$ and is usually interpreted as the Coulombian potential of the electrostatic field generated by the charge density $|u|^2$. Evidently, see, for example [5],

$$\begin{aligned}\int_{\mathbb{R}^3} \left|\nabla\phi_u\right|^2 dx &= 4\pi \int_{\mathbb{R}^3} \phi_u |u|^2 dx \\ &= 4\pi \iint_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx\, dy,\end{aligned} \tag{13}$$

$$\int_{\mathbb{R}^3} \phi_u |u|^2 dx \le \|\phi_u\|_6 \|u\|_{12/5}^2 \le \frac{4\pi}{S^2} \|u\|_{12/5}^4. \tag{14}$$

For each $\rho > 0$, let $u \in B_\rho$ and $u^t(x) = t^{3/2} u(tx)$ $(t > 0)$, and we have $\|u^t\|_2 = \|u\|_2 = \rho$, that is, $u^t \in B_\rho$. Let

$$\begin{aligned} f_u(t) = I\left(u^t\right) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{t}{4} \int_{\mathbb{R}^3} \phi_u |u|^2 dx \\ - \frac{t^{(3/2)(p-2)}}{p} \int_{\mathbb{R}^3} |u|^p dx; \end{aligned} \tag{15}$$

it is clear that $I_{\rho^2} \le 0$ for all $\rho > 0$ since $f_u(t) \to 0$ as $t \to 0$.

We now recall an abstract result on the constrained minimization problem

$$J_{\rho^2} = \inf_{u \in B_\rho} J(u), \quad \left(\text{we agree } J_0 = 0\right), \tag{16}$$

where $\rho > 0$, $B_\rho = \{u \in H^1(\mathbb{R}^3) : \|u\|_2 = \rho\}$, $J_{\rho^2} > -\infty$ is assumed, and

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 + T(u), \tag{17}$$

for some functional $T \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$.

**Lemma 3** (see [17, 18, Lemma 2.1]). *Let $T \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$. Let $\rho > 0$ and $\{u_n\} \subset B_\rho$ be a minimizing sequence for $J_{\rho^2}$ weakly convergent, up to translations, to a nonzero function $\overline{u}$. Assume that* (8) *holds and that*

$$T\left(u_n - \overline{u}\right) + T\left(\overline{u}\right) = T\left(u_n\right) + o(1); \tag{18}$$

$$\begin{aligned} T\left(\alpha_n\left(u_n - \overline{u}\right)\right) - T\left(u_n - \overline{u}\right) = o(1) \\ \text{where } \alpha_n = \frac{\rho^2 - \|\overline{u}\|_2^2}{\left\|u_n - \overline{u}\right\|_2}; \end{aligned} \tag{19}$$

$$\left\langle T'\left(u_n\right), u_n\right\rangle = O(1); \tag{20}$$

$$\left\langle T'\left(u_n\right) - T'\left(u_m\right), u_n - u_m\right\rangle = o(1) \quad \text{as } n, m \longrightarrow \infty. \tag{21}$$

*Then $\|u_n - \overline{u}\| \to 0$. In particular it follows that $\overline{u} \in B_\rho$ and $J(\overline{u}) = J_{\rho^2}$.*

As pointed out in [18], Lemma 3 is a variant of the concentration-compactness principle of Lions [15, 16]. Assumption (18) shows that $T$ possesses the Brizis-Lieb splitting property and (19) is the homogeneity of $T$. If, in addition, the condition (8) holds, then one can show that dichotomy does not occur; that is, $\overline{u} \in B_\rho$. Furthermore, if (20) and (21) are also fulfilled, then $\{u_n\}$ strongly converges to $\overline{u}$ in $H^1(\mathbb{R}^3)$. Finally we recall the following results obtained in [17, 18].

**Lemma 4** (see [18]). *If $p \in (2, 3)$, then $I_{\rho^2} < 0$ for all $\rho > 0$.*

**Lemma 5** (see [17, Lemma 3.1]). *If $p \in (2, 10/3)$, then, for every $\rho > 0$, the functional $I$ is bounded below and coercive on $B_\rho$.*

*Remark 6.* For $p \in (2,3)$, it follows from Lemmas 4 and 5 that each minimizing sequence for $I_{\rho^2}$ is bounded from below and above by two positive constants in $\mathscr{D}^{1,2}(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$, up to a subsequence, respectively.

## 3. Proof of the Main Theorems

Before proving our main theorems, we need some preliminary lemmas. First, we set

$$\mathscr{P} := \left\{u \in H^1\left(\mathbb{R}^3\right) : Q(u) = 0, I(u) = \min_{t>0} I\left(u^t\right)\right\}, \quad (22)$$

where $u^t(x) = t^{3/2}u(tx)$ with $t > 0$ and $Q(u)$ is a functional on $H^1(\mathbb{R}^3)$ defined as

$$Q(u) = \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{3(p-2)}{2p}\int_{\mathbb{R}^3} |u|^p dx. \quad (23)$$

It was shown in [13, Lemma 2.1] that if $u_\rho$ is a constrained critical point of $I$ on $B_\rho$ associated with the Lagrange multiplier $\lambda_\rho$,then $Q(u_\rho) = 0$, which is nothing but a linear combination of $\langle E'(u_\rho), u_\rho\rangle = 0$ (recall that $E(u)$ is given by (3)) and the following Pohozaev identity for (1) (cf. [5, 9])

$$\frac{1}{2}\|\nabla u_\rho\|_2^2 - \frac{3\lambda_\rho}{2}\|u_\rho\|_2^2 + \frac{5}{4}\int_{\mathbb{R}^3} \phi_{u_\rho}|u_\rho|^2 dx - \frac{3}{p}\|u_\rho\|_p^p = 0. \quad (24)$$

The following lemma shows that $B_\rho \cap \mathscr{P}$ is well defined.

**Lemma 7.** *Let $p \in (2,3)$ and let $\rho > 0$. For each $u \in B_\rho$ with $I(u) < 0$, there exists a unique $t_u > 0$ such that $I(u^{t_u}) = \min\{I(u^t) : t > 0\}$; moreover, $u^{t_u} \in B_\rho \cap \mathscr{P}$.*

*Proof.* We divide the proof into two cases.

*Case 1* ($p \in (2, 8/3)$). Let $u \in B_\rho$, for simplicity, and we will write $f_u'(t)$, $f_u''(t)$ and $f_u'''(t)$, the derivatives of $f_u(t)$ on $t$, instead of $df_u(t)/dt$, $d^2 f_u(t)/dt^2$ and $d^3 f_u(t)/dt^3$. From (15), we have

$$f_u'(t) = t\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{3(p-2)}{2p} t^{(3p-8)/2}\int_{\mathbb{R}^3} |u|^p dx. \quad (25)$$

Noting that $(3p-8)/2 \in (-1,0)$ since $p \in (2,8/3)$, then, by (25), $\lim_{t\to 0} f_u'(t) = -\infty$ and $\lim_{t\to\infty} f_u'(t) = +\infty$; thus there exists $t_u > 0$ such that $f_u'(t_u) = 0$. If there exists another $s_u > 0$ such that $f_u'(s_u) = 0$, without loss of generality, we may assume that $s_u > t_u$, and then we get

$$\begin{aligned} 0 &= f_u'(s_u) - f_u'(t_u) \\ &= (s_u - t_u)\int_{\mathbb{R}^3} |\nabla u|^2 dx \\ &\quad + \frac{3(p-2)}{2p}\left(t_u^{(3p-8)/2} - s_u^{(3p-8)/2}\right)\int_{\mathbb{R}^3} |u|^p dx > 0, \end{aligned} \quad (26)$$

a contradiction. Therefore, $t_u$ is unique and it is clear that $I(u^{t_u}) = \min\{I(u^t) : t > 0\}$. Moreover, $u^{t_u} \in B_\rho \cap \mathscr{P}$ because of $f_u'(t_u)t_u = 0$.

*Case 2* ($p \in [8/3, 3)$). By Lemma 4, we know that the set $A_\rho := \{u \in B_\rho : I(u) < 0\} \neq \emptyset$. Let $u \in A_\rho$, if $f_u'(t) > 0$ for all $t > 0$; that is, $f_u(t)$ is strictly increasing, then we obtain that $f_u(t) < f_u(1) = I(u) < 0$ for all $t \in (0,1)$. However, it is easy to see that $\lim_{t\to 0} f_u(t) = 0$; this is a contradiction. On the other hand, we know that $f_u(t) \to \infty$ as $t \to \infty$; hence there is a $t_u > 0$ such that $f_u'(t_u) = 0$, $u^{t_u} \in B_\rho \cap \mathscr{P}$ and

$$f_u(t_u) = \min\{f_u(t) : t > 0\} \le f_u(1) = I(u) < 0. \quad (27)$$

Next, we will show that $t_u$ is unique. Arguing by contradiction, suppose that there is another $s_u > 0$ such that $f_u(t_u) = f_u(s_u) = \min\{f_u(t) : t > 0\}$, without loss of generality, we may assume that $s_u > t_u$, and then we have

$$f_u'(t_u) = f_u'(s_u) = 0, \quad f_u''(t_u) \ge 0, \quad f_u''(s_u) \ge 0. \quad (28)$$

According to (28), there exists $\omega_u \in (t_u, s_u)$ such that $f_u''(\omega_u) = 0$. After a simple calculation, we get

$$f_u''(t) = \|\nabla u\|_2^2 - \frac{3(p-2)(3p-8)}{4p} t^{(3p-10)/2}\int_{\mathbb{R}^3} |u|^p dx, \quad (29)$$

$$f_u'''(t) = -\frac{3(p-2)(3p-8)(3p-10)}{8p} t^{(3p-12)/2} \times \int_{\mathbb{R}^3} |u|^p dx. \quad (30)$$

If $p = 8/3$,then, by (29), $f_u''(t) > 0$ for all $t > 0$, which contradicts $f_u''(\omega_u) = 0$. If $p \in (8/3, 3)$, then, by (30), $f_u'''(t) > 0$ for all $t > 0$. Noting that $\omega_u \in (t_u, s_u)$, we have

$$0 \le f_u''(t_u) < f_u''(\omega_u) = 0, \quad (31)$$

again a contradiction. Therefore, $t_u > 0$ is unique. □

**Lemma 8.** *Let $p \in (2,3)$ and $\rho > 0$. For each $\{u_n\} \subset B_\rho$ such that $I(u_n) \to I_{\rho^2} < 0$ as $n \to \infty$ and $I(u_n) < 0$ for all $n \in \mathbb{N}$, there exists a bounded sequence $\{t_n\} \subset \mathbb{R}^+$ such that $\{u_n^{t_n}\} \subset B_\rho \cap \mathscr{P}$ and $I(u_n^{t_n}) \to I_{\rho^2}$ as $n \to \infty$ with $I(u_n^{t_n}) < 0$ for all $n \in \mathbb{N}$; that is, $\{u_n^{t_n}\}$ is also a minimizing sequence for $I_{\rho^2}$ constrained on $B_\rho$.*

*Proof.* It follows from Lemma 7 that, for each $u_n$, there exists $t_n > 0$ such that $u_n^{t_n} \in B_\rho \cap \mathscr{P}$ and $I(u_n^{t_n}) = \min\{I(u_n^t) : t > 0\} \le I(u_n) < 0$; therefore, we have

$$I_{\rho^2} \le I\left(u_n^{t_n}\right) \le I\left(u_n\right) \longrightarrow I_{\rho^2}, \tag{32}$$

as $n \to \infty$, that is, $\{u_n^{t_n}\}$ is a minimizing sequence. Next, we will show that $\{t_n\}$ is bounded. Indeed, from Remark 6, $\{u_n\}$ and $\{u_n^{t_n}\}$ are bounded from below and above by two positive constants in $\mathscr{D}^{1,2}(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$, respectively. Noting that $\int_{\mathbb{R}^3} |\nabla u_n^{t_n}|^2 dx = t_n^2 \int_{\mathbb{R}^3} |\nabla u_n|^2 dx$; therefore, $\{t_n\}$ is bounded from below and above by two positive constants. □

*Remark 9.* Thanks to the Lemma 8, we know that $I_{\rho^2} = \inf\{I(u) : u \in B_\rho\} = \inf\{I(u) : u \in B_\rho \cap \mathscr{P}\}$, and, in the following, we will consider the minimization problem (7) restricted to $B_\rho \cap \mathscr{P}$ instead of $B_\rho$. By Lemmas 4 and 8, for each $\rho > 0$, if $\{u_n\} \subset B_\rho \cap \mathscr{P}$ satisfying $I(u_n) \to I_{\rho^2}$ as $n \to \infty$, then, up to a subsequence, we may assume that $I(u_n) < 0$. It follows from Lemma 5 that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$; by the results of [17, 18], we may assume that $u_n \rightharpoonup u \ne 0$ as $n \to \infty$ in $H^1(\mathbb{R}^3)$.

The following estimates of the elements of $B_\rho \cap \mathscr{P}$ are crucial to proving the strong subadditivity inequality (8).

**Lemma 10.** *Let $p \in (2,3)$ and $\rho > 0$. For each $u \in B_\rho \cap \mathscr{P}$, it holds*

$$\begin{gathered}\|\nabla u\|_2^2 \le \left(\frac{3(p-2)}{2p}\right)^{4/(10-3p)} S^{-6(p-2)/(10-3p)} \rho^{2(6-p)/(10-3p)}, \\ \|u\|_p^p \le \left(\frac{3(p-2)}{2p}\right)^{3(p-2)/(10-3p)} S^{-6(p-2)/(10-3p)} \rho^{2(6-p)/(10-3p)}.\end{gathered} \tag{33}$$

*Proof.* Since $u \in B_\rho \cap \mathscr{P}$,

$$\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{3(p-2)}{2p}\|u\|_p^p = 0. \tag{34}$$

Noting that $\int_{\mathbb{R}^3} \phi_u |u|^2 dx \ge 0$ (see (13)), by using the Hölder inequality, we get

$$\begin{aligned}\|\nabla u\|_2^2 &\le \frac{3(p-2)}{2p}\|u\|_p^p \le \frac{3(p-2)}{2p}\|u\|_2^{(6-p)/2}\|u\|_6^{3(p-2)/2} \\ &\le \frac{3(p-2)}{2p} S^{-3(p-2)/2}\|u\|_2^{(6-p)/2}\|\nabla u\|_2^{3(p-2)/2},\end{aligned} \tag{35}$$

which implies that

$$\|\nabla u\|_2^2 \le \left(\frac{3(p-2)}{2p}\right)^{4/(10-3p)} S^{-6(p-2)/(10-3p)} \rho^{2(6-p)/(10-3p)}. \tag{36}$$

On the other hand, we have

$$\begin{aligned}\|u\|_p^p &\le \|u\|_2^{(6-p)/2}\|u\|_6^{3(p-2)/2} \\ &\le S^{-3(p-2)/2}\|u\|_2^{(6-p)/2}\|\nabla u\|_2^{3(p-2)/2} \\ &\le \left(\frac{3(p-2)}{2p}\right)^{3(p-2)/(10-3p)} \\ &\quad \times S^{-6(p-2)/(10-3p)} \rho^{2(6-p)/(10-3p)};\end{aligned} \tag{37}$$

this concludes the proof of this lemma. □

*Remark 11.* Let $p \in (3, 10/3)$. It was shown in [13, Theorem 1.1] that $I_{\rho^2} < 0$ if and only if $\rho \in (\overline{\rho}, \infty)$, where the positive number $\overline{\rho}$ is defined as

$$\overline{\rho} = \inf\left\{\rho > 0 : I_{\rho^2} < 0\right\}. \tag{38}$$

Therefore, after a simple calculation, we can show that both of Lemmas 7 and 10 hold if $p \in (3, 10/3)$ and $\rho \in (\overline{\rho}, \infty)$.

Motivated by [17], we will use the standard scaling arguments to prove that the strong subadditivity inequality (8) holds for $p \in (2,3)$. First, we consider the case of $p \in (2, 8/3]$.

**Lemma 12.** *For $p \in (2, 8/3]$, let*

$$\overline{\rho}_1 = 3^{(3p-8)/8(3-p)} \pi^{(3p-10)/8(3-p)} S^{3/2} \left(\frac{p-2}{2p}\right)^{1/4(3-p)} > 0. \tag{39}$$

*Then*

$$I_{\rho^2} < I_{\mu^2} + I_{\rho^2-\mu^2} \quad \forall 0 < \mu < \rho < \overline{\rho}_1. \tag{40}$$

*Proof.* By Lemma 8 and Remark 9, for each $\{u_n\} \subset B_\rho \cap \mathscr{P}$ satisfying $I(u_n) \to I_{\rho^2} < 0$ as $n \to \infty$, we may assume that, for all $n$, $I(u_n) \le I_{\rho^2}/2$, which implies that

$$\|u_n\|_p^p \ge -\frac{p}{2} I_{\rho^2}. \tag{41}$$

Noting that $tu_n \in B_{t\rho}$ $(t > 0)$, we have

$$\begin{aligned}I(tu_n) &= \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^2 dx \\ &\quad - \frac{t^p}{p}\int_{\mathbb{R}^3} |u_n|^p dx \\ &= t^2\left(I(u_n) + \frac{t^2-1}{4}\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^2 dx \right. \\ &\quad \left. - \frac{t^{p-2}-1}{p}\int_{\mathbb{R}^3} |u_n|^p dx\right) \\ &= t^2\left(I(u_n) + g(t, u_n)\right),\end{aligned} \tag{42}$$

where

$$g(t,u) = \frac{t^2-1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{t^{p-2}-1}{p}\int_{\mathbb{R}^3} |u|^p dx. \tag{43}$$

We calculate the derivative of $g(t,u)$ on $t$:

$$\frac{dg(t,u)}{dt} = \frac{t}{2}\int_{\mathbb{R}^3}\phi_u|u|^2dx - \frac{p-2}{p}t^{p-3}\|u\|_p^p. \tag{44}$$

Letting $dg(t,u)/dt = 0$, we see from (14) that

$$t^{p-4} = \frac{p\int_{\mathbb{R}^3}\phi_u|u|^2dx}{2(p-2)\|u\|_p^p} \le \frac{2\pi p\|u\|_{12/5}^4}{S^2(p-2)\|u\|_p^p}. \tag{45}$$

Furthermore,

$$\frac{d^2g(t,u)}{dt^2} = \frac{1}{2}\int_{\mathbb{R}^3}\phi_u|u|^2dx - \frac{(p-2)(p-3)}{p}t^{p-4}\|u\|_p^p > 0. \tag{46}$$

Now we divide the value of $p$ into two cases to discuss $dg(t,u_n)/dt$.

*Case 1* ($p \in (2, 12/5)$). It follows from Lemma 10, (14), and the Hölder inequality that

$$\begin{aligned}
&\left.\frac{dg(t,u_n)}{dt}\right|_{t=1} \\
&= \frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2dx - \frac{p-2}{p}\int_{\mathbb{R}^3}|u_n|^pdx \\
&\le \frac{2\pi}{S^2}\|u_n\|_{12/5}^4 - \frac{p-2}{p}\|u_n\|_p^p \\
&\le \frac{2\pi}{S^2}\|u_n\|_p^{6p/(6-p)}\|u_n\|_6^{2(12-5p)/(6-p)} \\
&\quad - \frac{p-2}{p}\|u_n\|_p^p \\
&= \|u_n\|_p^p\left(2\pi S^{12(p-3)/(6-p)}\|\nabla u_n\|_2^{2(12-5p)/(6-p)}\right. \\
&\qquad \left.\times\|u_n\|_p^{p^2/(6-p)} - \frac{p-2}{p}\right) \\
&\le \|u_n\|_p^p\left(2\pi S^{12(p-3)/(10-3p)}\left(\frac{3(p-2)}{2p}\right)^{(8-3p)/(10-3p)}\right. \\
&\qquad \left.\times\rho^{8(3-p)/(10-3p)} - \frac{p-2}{p}\right).
\end{aligned} \tag{47}$$

*Case 2* ($p \in [12/5, 8/3]$). Again by Lemma 10, (14), and the Hölder inequality, we have

$$\begin{aligned}
&\left.\frac{dg(t,u_n)}{dt}\right|_{t=1} \\
&= \frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2dx - \frac{p-2}{p}\int_{\mathbb{R}^3}|u_n|^pdx \\
&\le \frac{2\pi}{S^2}\|u_n\|_{12/5}^4 - \frac{p-2}{p}\|u_n\|_p^p \\
&\le \frac{2\pi}{S^2}\|u_n\|_2^{2(5p-12)/3(p-2)}\|u_n\|_p^{2p/3(p-2)} - \frac{p-2}{p}\|u_n\|_p^p \\
&\le \|u_n\|_p^p \\
&\quad\times\left(\frac{2\pi}{S^2}\|u_n\|_2^{2(5p-12)/3(p-2)}\|u_n\|_p^{p(8-3p)/3(p-2)} - \frac{p-2}{p}\right) \\
&\le \|u_n\|_p^p\left(2\pi S^{12(p-3)/(10-3p)}\left(\frac{3(p-2)}{2p}\right)^{(8-3p)/(10-3p)}\right. \\
&\qquad \left.\times\rho^{8(3-p)/(10-3p)} - \frac{p-2}{p}\right).
\end{aligned} \tag{48}$$

Let

$$\overline{\rho}_1 = 3^{(3p-8)/8(3-p)}\pi^{(3p-10)/8(3-p)}S^{3/2}\left(\frac{p-2}{2p}\right)^{1/4(3-p)}. \tag{49}$$

Then by (47), (48), and (49), we know that, for each $0 < \rho < \overline{\rho}_1$, there holds

$$\begin{aligned}
\widetilde{\rho}_1(\rho) &:= \frac{2\pi p}{p-2}S^{12(p-3)/(10-3p)} \\
&\quad\times\left(\frac{3(p-2)}{2p}\right)^{(8-3p)/(10-3p)}\rho^{8(3-p)/(10-3p)} < 1, \\
&\left.\frac{dg(t,u_n)}{dt}\right|_{t=1} < 0,
\end{aligned} \tag{50}$$

for all $n \in \mathbb{N}$. On the other hand, for each $\varepsilon \in (0, 1-\widetilde{\rho}_1(\rho))$, it follows from (41), (44), (45), (46), and Lemma 10 that, for all $t \in (1, (\widetilde{\rho}_1(\rho)/(1-\varepsilon))^{1/(p-4)})$ and all $n \in \mathbb{N}$,

$$\begin{aligned}
\frac{dg(t,u_n)}{dt} &= \frac{p-2}{p}\|u_n\|_p^p t^{p-3}\left(\frac{p\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2dx}{2(p-2)\|u_n\|_p^p}t^{4-p} - 1\right) \\
&\le \frac{\varepsilon(p-2)}{2}I_{\rho^2} < 0.
\end{aligned} \tag{51}$$

This, together with the mean value theorem and (41), yields that for all $t \in (1, ((1-\varepsilon)^{-1}\widetilde{\rho}_1(\rho))^{1/(p-4)})$ and all $n \in \mathbb{N}$,

$$g(t,u_n) = g(1,u_n) + \left.\frac{dg(t,u_n)}{dt}\right|_{t=\theta_{t_n}}(t-1) < -C(t-1), \tag{52}$$

where $\theta_{t_n} \in (1,t)$ and $C > 0$ depend only on $\varepsilon$, $p$,and $\rho$. By (42), we have

$$\begin{aligned}
I_{(t\rho)^2} &\le I(tu_n) = t^2(I(u_n) + g(t,u_n)) \\
&\le t^2I(u_n) - Ct^2(t-1) \\
&= t^2I_{\rho^2} - Ct^2(t-1) + o(1),
\end{aligned} \tag{53}$$

then

$$I_{(t\rho)^2} < t^2 I_{\rho^2} \quad \forall t \in \left(1, \left(\frac{\widetilde{\rho}_1(\rho)}{1-\varepsilon}\right)^{1/(p-4)}\right), \ \rho \in (0, \overline{\rho}_1). \tag{54}$$

Clearly, $\widetilde{\rho}_1$ (cf. (50)) is strictly increasing on $\rho$, and then $(\widetilde{\rho}_1/(1-\varepsilon))^{1/(p-4)}$ is strictly decreasing on $\rho$ since $p \in (2,3)$.

Let

$$h(\rho) := \left(\frac{\widetilde{\rho}_1(\rho)}{1-\varepsilon}\right)^{1/(p-4)}. \tag{55}$$

For each $\rho \in (0, \overline{\rho}_1)$, let $\mu \in (0, \rho)$ without loss of generality, we may assume that $\mu > \sqrt{\rho^2 - \mu^2}$. Choosing $\varepsilon \in (0, \min\{\overline{\rho}_1 - \rho, 1 - \widetilde{\rho}_1(\rho), \mu - \sqrt{\rho^2-\mu^2}\})$, then by (50) we know that $h(\sqrt{\rho^2-\mu^2}) > h(\mu) > h(\rho) > 1$.

(a) If $\rho/\mu \in (1, h(\mu))$, then by (54)

$$\begin{aligned} I_{\rho^2} &= I_{(\rho^2/\mu^2)\mu^2} < \frac{\rho^2}{\mu^2} I_{\mu^2} = I_{\mu^2} + \frac{\rho^2-\mu^2}{\mu^2} I_{\mu^2} \\ &= I_{\mu^2} + \frac{\rho^2-\mu^2}{\mu^2} I_{(\mu^2/(\rho^2-\mu^2))(\rho^2-\mu^2)}. \end{aligned} \tag{56}$$

(b) If $\rho/\mu \notin (1, h(\mu))$, then there exists $k \in \mathbb{N}$ such that $(\rho/\mu)^{1/k} \in (1, h(\rho))$. Therefore

$$\left(\frac{\rho}{\mu}\right)^{1/k} \in \left(1, h\left(\left(\frac{\rho}{\mu}\right)^{(k-i)/k}\mu\right)\right), \quad \forall i = 1, 2, \ldots, k. \tag{57}$$

It follows from (54) that

$$\begin{aligned} I_{\rho^2} &= I_{(\rho/\mu)^{2/k}((\rho/\mu)^{2(k-1)/k}\mu^2)} \\ &< \left(\frac{\rho}{\mu}\right)^{2/k} I_{(\rho/\mu)^{2(k-1)/k}\mu^2} < \cdots < \frac{\rho^2}{\mu^2} I_{\mu^2}. \end{aligned} \tag{58}$$

Combining the above cases (a) and (b), we can show that

$$I_{\rho^2} < I_{\mu^2} + I_{\rho^2-\mu^2} \quad \forall 0 < \mu < \rho < \overline{\rho}_1. \tag{59}$$

Thus the conclusion of this lemma holds. □

*Remark 13.* For the case of $p = 8/3$, it has been proved in [4, 17] that the strong subadditivity inequality (8) holds for $\rho > 0$ small. By using the result of [17], we can give a specific estimate of lower bound of $\rho$ such that (8) holds; that is, (8) holds for all $\rho \in (0, (8\pi)^{-3/4} S^{3/2})$. However, if we plug $p = 8/3$ into (49), then we have $\overline{\rho}_1 = (8\pi)^{-3/4} S^{3/2}$, which coincides with the one given in [17].

Next, we will show (8) for $p \in (8/3, 3)$. We point out that the case of $p \in (8/3, 3)$ is quite different from the case of $p \in (2, 8/3]$ since the inequality (48) does not hold anymore. Inspired by [18], we will give some estimates for $I_{\rho^2}$ in Lemmas 14 and 15, and these are crucial for the proof of (8) if $p \in (8/3, 3)$.

**Lemma 14.** *Let $p \in (8/3, 3)$ and $\rho > 0$ be fixed. If there exists $u \in B_\rho \cap \mathscr{P}$ such that $I(u) \le I_{\rho^2}/2$ and*

$$\|u\|_p^p > 3\|\nabla u\|_2^2, \tag{60}$$

*then there exist positive constants $C_3$ and $C_4$ dependent on $p$ and $\rho$, such that*

$$I_{\mu^2} \le -C_3 \mu^{2(6-p)/(10-3p)} + C_4 \mu^{2(18-5p)/(10-3p)} \quad \forall \mu > 0. \tag{61}$$

*Proof.* From the assumptions of the lemma, we see that

$$\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{3(p-2)}{2p}\|u\|_p^p = 0, \tag{62}$$

$$\frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{1}{p}\|u\|_p^p = I(u) \le \frac{I_{\rho^2}}{2}. \tag{63}$$

By (60), (62), and (63), we deduce that

$$\frac{I_{\rho^2}}{2} \ge I(u) = -\frac{1}{2}\|\nabla u\|_2^2 + \frac{3p-8}{2p}\|u\|_p^p > \frac{4(p-3)}{p}\|\nabla u\|_2^2. \tag{64}$$

Combining (62) and (64), and using Lemma 10, we also obtain

$$\begin{aligned} &\frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx \\ &\quad = \frac{3(p-2)}{2p}\|u\|_p^p - \|\nabla u\|_2^2 \\ &\quad \le \left(\frac{3(p-2)}{2p}\right)^{4/(10-3p)} S^{-6(p-2)/(10-3p)} \\ &\qquad \times \rho^{2(6-p)/(10-3p)} + \frac{p}{8(3-p)} I_{\rho^2}. \end{aligned} \tag{65}$$

For each $t > 0$, let $u_t(x) = t^{4/(10-3p)} u(t^{2(p-2)/(10-3p)} x)$, we have $\|u_t\|_2 = t\|u\|_2 = t\rho$. It follows from (60), (64), and (65) that

$$\begin{aligned} I_{(t\rho)^2} \le I(u_t) &= \frac{1}{2} t^{2(6-p)/(10-3p)} \|\nabla u\|_2^2 \\ &\quad + \frac{1}{4} t^{2(18-5p)/(10-3p)} \int_{\mathbb{R}^3} \phi_u |u|^2 dx \\ &\quad - \frac{1}{p} t^{2(6-p)/(10-3p)} \|u\|_p^p \\ &\le \left(\frac{1}{2} - \frac{3}{p}\right) t^{2(6-p)/(10-3p)} \|\nabla u\|_2^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}t^{2(18-5p)/(10-3p)}\int_{\mathbb{R}^3}\phi_u|u|^2dx\\
&\le \frac{6-p}{16(3-p)}\frac{I_{\rho^2}}{\rho^{2(6-p)/(10-3p)}}(t\rho)^{2(6-p)/(10-3p)}\\
&+\left(\left(\frac{3(p-2)}{2p}\right)^{4/(10-3p)}S^{-6(p-2)/(10-3p)}\right.\\
&\times\rho^{8(p-3)/(10-3p)}\\
&\left.+\frac{p}{8(3-p)}\frac{I_{\rho^2}}{\rho^{2(18-5p)/(10-3p)}}\right)\\
&\times(t\rho)^{2(18-5p)/(10-3p)}.
\end{aligned}
\tag{66}
$$

Set $t\rho=\mu$, then $\mu\in(0,\infty)$ since $t\in(0,\infty)$ and $\rho$ is a fixed positive constant. From the above inequality, we see that

$$
I_{\mu^2}\le -C_3\mu^{2(6-p)/(10-3p)}+C_4\mu^{2(18-5p)/(10-3p)},
\tag{67}
$$

for some positive constants $C_3$ and $C_4$ depending on $p$ and $\rho$. □

**Lemma 15.** *Suppose that $p\in(8/3,3)$ and $\{u_k\}\subset B_{\rho_k}\cap\mathscr{P}$ satisfying $\|u_k\|_p^p>3\|\nabla u_k\|_2^2$ and $I(u_k)\le I_{\rho_k^2}/2$ for all $k\in\mathbb{N}$. Then there exists a positive constant $C$ dependent on $p$, such that*

$$
I_{\rho_k^2}\ge -C\rho_k^{2(5p-12)/(3p-8)}\quad\forall k\in\mathbb{N}.
\tag{68}
$$

*Proof.* Following the line of the proof of Lemma 14, we arrive that

$$
\begin{aligned}
\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_k}|u_k|^2dx&=\frac{3(p-2)}{2p}\|u_k\|_p^p-\|\nabla u_k\|_2^2\\
&>\frac{7p-18}{6p}\|u_k\|_p^p,
\end{aligned}
\tag{69}
$$

which, together with (14) and the Höder inequality, implies that

$$
\begin{aligned}
\|u_k\|_p^p&\le\frac{3p}{2(7p-18)}\int_{\mathbb{R}^3}\phi_{u_k}|u_k|^2dx\\
&\le\frac{6\pi p}{(7p-18)S^2}\|u_k\|_{12/5}^4\\
&\le\frac{6\pi p}{(7p-18)S^2}\|u_k\|_2^{2(5p-12)/3(p-2)}\|u_k\|_p^{2p/3(p-2)},
\end{aligned}
\tag{70}
$$

and then

$$
\begin{aligned}
\|u_k\|_p^p&\le\left(\frac{6\pi p}{(7p-18)S^2}\right)^{3(p-2)/(3p-8)}\|u_k\|_2^{2(5p-12)/(3p-8)}\\
&=\left(\frac{6\pi p}{(7p-18)S^2}\right)^{3(p-2)/(3p-8)}\rho_k^{2(5p-12)/(3p-8)}.
\end{aligned}
\tag{71}
$$

Combining (62), (69), and (71), we have

$$
\begin{aligned}
\frac{I_{\rho_k^2}}{2}>I(u_k)&=\frac{1}{2}\|\nabla u_k\|_2^2+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_k}|u_k|^2dx-\frac{1}{p}\|u_k\|_p^p\\
&=\frac{3p-10}{4p}\|u_k\|_p^p+\frac{1}{8}\int_{\mathbb{R}^3}\phi_{u_k}|u_k|^2dx\\
&\ge\frac{4(p-3)}{3p}\|u_k\|_p^p\\
&\ge\frac{4(p-3)}{3p}\left(\frac{6\pi p}{(7p-18)S^2}\right)^{3(p-2)/(3p-8)}\\
&\quad\times\rho_k^{2(5p-12)/(3p-8)},
\end{aligned}
\tag{72}
$$

and this completes the proof. □

**Lemma 16.** *If $p\in(8/3,3)$, then there exists a positive constant $\overline{\rho}_2$ such that*

$$
I_{\rho^2}<I_{\mu^2}+I_{\rho^2-\mu^2}\quad\forall 0<\mu<\rho<\overline{\rho}_2.
\tag{73}
$$

*Proof.* Suppose that $\rho>0$ and $\{u_n\}\subset B_\rho\cap\mathscr{P}$ satisfying $I(u_n)\to I_{\rho^2}$ as $n\to\infty$. It follows from Remark 9 that, up to a subsequence, $I(u_n)\le I_{\rho^2}/2<0$ for all $n\in\mathbb{N}$. By Lemma 5, it is easy to see that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Noting that $tu_n\in B_{t\rho}$, then, by (42), we have

$$
I(tu_n)=t^2(I(u_n)+g(t,u_n)),
\tag{74}
$$

where $g(t,u)$ is given by (43). Obviously,

$$
\begin{aligned}
\left.\frac{dg(t,u_n)}{dt}\right|_{t=1}&=\frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2dx-\frac{(p-2)}{p}\|u_n\|_p^p\\
&=I(u_n)-\frac{1}{2}\|\nabla u_n\|_2^2\\
&\quad+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2dx+\frac{(3-p)}{p}\|u_n\|_p^p\\
&=I(u_n)-\frac{3}{2}\|\nabla u_n\|_2^2+\frac{1}{2}\|u_n\|_p^p,
\end{aligned}
\tag{75}
$$

since $u_n\in B_\rho\cap\mathscr{P}$ and (34) holds. Moreover,

$$
\begin{aligned}
\frac{d^2g(t,u_n)}{dt^2}&=\frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2dx\\
&\quad-\frac{(p-2)(p-3)}{p}t^{p-4}\|u_n\|_p^p>0,
\end{aligned}
\tag{76}
$$

for all $t>0$ and $n\in\mathbb{N}$.

We claim that there exists $\overline{\rho}_2>0$ such that for each $\rho\in(0,\overline{\rho}_2)$ and for each $\{u_n\}\subset B_\rho\cap\mathscr{P}$ satisfying $I(u_n)\le I_{\rho^2}/2<0$ and $I(u_n)\to I_{\rho^2}$ as $n\to\infty$, we have

$$
\|u_n\|_p^p\le 3\|\nabla u_n\|_2^2.
\tag{77}
$$

Indeed, if not, we can find $\{\rho_k\}$ and $\{u_n^k\} \subset B_{\rho_k} \cap \mathscr{P}$ such that $\rho_k \to 0$ as $k \to \infty$ and for each $k \in \mathbb{N}$, $I(u_n^k) \to I_{\rho_k^2} < 0$ as $n \to \infty$, but $\|u_n^k\|_p^p > 3\|\nabla u_n^k\|_2^2$. For $k = 1$, there exists $n_1 > 0$ such that $I(u_{n_1}^1) < I_{\rho_1^2}/2 < 0$, and it can be deduced from Lemma 14 that

$$I_{\mu^2} \le -C_3\mu^{2(6-p)/(10-3p)} + C_4\mu^{2(18-5p)/(10-3p)} \quad \forall \mu > 0, \tag{78}$$

where $C_3$ and $C_4$ are positive constants dependent on $p$ and $\rho_1$. On the other hand, we know that for each $k \in \mathbb{N}$ there exists $n_k > 0$ such that $I(u_{n_k}^k) < I_{\rho_k^2}/2 < 0$. Then by Lemma 15, we obtain

$$I_{\rho_k^2} \ge -C\rho_k^{2(5p-12)/(3p-8)} \quad \forall k \in \mathbb{N}, \tag{79}$$

where $C$ is a positive constant depending only on $p$. Noting that (78) holds for all $\mu > 0$, by (78) and (79), we deduce that

$$\begin{aligned} -C\rho_k^{2(5p-12)/(3p-8)} \le I_{\rho_k^2} \le & -C_3\rho_k^{2(6-p)/(10-3p)} \\ & + C_4\rho_k^{2(18-5p)/(10-3p)}, \end{aligned} \tag{80}$$

which is a contradiction for $k$ large since $p \in (8/3, 3)$ implies

$$\frac{2(5p-12)}{3p-8} > \frac{2(6-p)}{10-3p}. \tag{81}$$

Thus we have shown the claim. Now for each $\rho \in (0, \overline{\rho}_2)$ and for all $\{u_n\} \subset B_\rho \cap \mathscr{P}$ with $I(u_n) \le I_{\rho^2}/2$ and $I(u_n) \to I_{\rho^2}$ as $n \to \infty$, using (77), we have

$$\left.\frac{dg(t,u_n)}{dt}\right|_{t=1} \le I(u_n) \le \frac{I_{\rho^2}}{2} < 0. \tag{82}$$

By (76), similarly as in the proofs of (45) and (51), we get that

$$\begin{aligned} t^{4-p} &= \frac{2(p-2)}{p}\frac{\|u_n\|_p^p}{\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2dx} \\ &= \frac{2(p-2)}{p}\frac{\|u_n\|_p^p}{(6(p-2)/p)\|u_n\|_p^p - 4\|\nabla u_n\|_2^2} \\ &\ge \frac{3(p-2)}{7p-18} > 1. \end{aligned} \tag{83}$$

Now, we can choose $\varepsilon > 0$ so small that there exists a positive constant $C$ dependent on $p$, $\rho$,and $\varepsilon$, such that

$$\frac{dg(t,u_n)}{dt} \le -C < 0 \quad \forall t \in \left(1, \left(\frac{3(1-\varepsilon)(p-2)}{7p-18}\right)^{1/(4-p)}\right). \tag{84}$$

Since, for each $n$, $g(1,u_n) = 0$, it follows that, for each $t \in (1, (3(1-\varepsilon)(p-2)/(7p-18))^{1/(4-p)})$,

$$\begin{aligned} I_{(t\rho)^2} &\le I(tu_n) = t^2(I(u_n) + g(t,u_n)) \\ &\le t^2\left(I(u_n) + \left.\frac{dg(t,u_n)}{dt}\right|_{t=\theta_{t_n}}(t-1)\right) \\ &\le t^2(I(u_n) - C(t-1)) \\ &= t^2 I_{\rho^2} - Ct^2(t-1) + o(1), \end{aligned} \tag{85}$$

where $\theta_{t_n} \in (1,t)$, namely, $I_{(t\rho)^2} < t^2 I_{\rho^2}$ for all $t \in (1, (3(1-\varepsilon)(p-2)/(7p-18))^{1/(4-p)})$. Thus we complete the proof of this lemma by using the arguments in the proof of Lemma 12. □

**Lemma 17.** *Let $\rho > 0$. Assume that $(u_\rho, \lambda_\rho)$ is a solution of (1) with $\|u_\rho\|_2 = \rho$.*

(a) *If $p \in (2, 12/5]$, then $\lambda_\rho < 0$.*

(b) *If $p \in (12/5, 8/3]$ and $\lambda_\rho \ge 0$, then*

$$\begin{aligned} \rho \ge \overline{\rho}_3 := & \left(\frac{6-p}{6\pi p}\right)^{(10-3p)/8(3-p)} \\ & \times S^{3/2}\left(\frac{3(p-2)}{2p}\right)^{(8-3p)/8(3-p)}. \end{aligned} \tag{86}$$

*Proof.* Since $(u_\rho, \lambda_\rho)$ is a solution of (1), it follows that

$$\|\nabla u_\rho\|_2^2 + \int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx - \|u_\rho\|_p^p = \lambda_\rho\|u_\rho\|_2^2, \tag{87}$$

$$\|\nabla u_\rho\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx - \frac{3(p-2)}{2p}\|u_\rho\|_p^p = 0. \tag{88}$$

Thus, from (87) and (88), after a simple calculation, we have

$$\lambda_\rho\|u_\rho\|_2^2 = \frac{p-6}{3(p-2)}\|\nabla u_\rho\|_2^2 + \frac{5p-12}{6(p-2)}\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx, \tag{89}$$

which yields that (a) holds. Moreover, if $p \in (12/5, 8/3]$ and $\lambda_\rho \ge 0$, then (89) implies that

$$\frac{5p-12}{6(p-2)}\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx \ge \frac{6-p}{3(p-2)}\|\nabla u_\rho\|_2^2. \tag{90}$$

Thus we get from (88) that

$$\begin{aligned} \frac{3(p-2)}{2p}\|u_\rho\|_p^p &= \|\nabla u_\rho\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx \\ &\le \frac{9(p-2)}{4(6-p)}\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx \\ &\le \frac{9(p-2)\pi}{(6-p)S^2}\|u_\rho\|_{12/5}^4. \end{aligned} \tag{91}$$

By using the Hölder inequality, it can be deduced from (91) and Lemma 10 that

$$\begin{aligned}
&\frac{6-p}{6\pi p}S^2\|u_\rho\|_p^p \\
&\le \|u_\rho\|_{12/5}^4 \le \|u_\rho\|_2^{2(5p-12)/3(p-2)}\|u_\rho\|_p^{2p/3(p-2)} \\
&\le \left(S^{-2(8-3p)/(10-3p)}\left(\frac{3(p-2)}{2p}\right)^{(8-3p)/(10-3p)}\right. \\
&\qquad \left.\times \rho^{8(3-p)/(10-3p)}\right)\|u_\rho\|_p^p,
\end{aligned} \tag{92}$$

and this means that

$$\rho \ge \left(\frac{6-p}{6\pi p}\right)^{(10-3p)/8(3-p)} S^{3/2}\left(\frac{3(p-2)}{2p}\right)^{(3p-8)/8(3-p)}. \tag{93}$$

Thus (b) holds. At this point, the lemma is proved. □

*Proof of Theorem 1.* It follows from Lemmas 12 and 16 that (8) holds. Let $T(u) = \int_{\mathbb{R}^3}\phi_u|u|^2dx/4 - \|u\|_p^p/p$. From the results of [17, 18], we know that (18), (19), (20), and (21) hold. Therefore, by Lemma 3, all the minimizing sequences for (7) are precompact and then (1) has a solution $(u_\rho, \lambda_\rho)$. Lemma 17 shows that, for $p \in (2, 8/3]$, $\lambda_\rho < 0$ since $\overline{\rho}_1 < \overline{\rho}_3$, where $\overline{\rho}_1$ and $\overline{\rho}_3$ are given by (49) and (86), respectively.

To complete the proof of Theorem 1, we need to show that $\lambda_\rho \to 0$ and $I_{\rho^2}/\rho^2 \to 0$ as $\rho \to 0$ provided that the assumption (1) of Theorem 1 holds. Indeed, since $(u_\rho, \lambda_\rho)$ is the solution of (1), it follows from (87), (88), and Lemma 10 that

$$\begin{aligned}
0 &\le \frac{3}{4}\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx - \lambda_\rho\rho^2 \\
&= \frac{6-p}{2p}\|u_\rho\|_p^p \le C\rho^{2(6-p)/(10-3p)},
\end{aligned} \tag{94}$$

which implies that

$$0 \le -\lambda_\rho \le C\rho^{4(p-2)/(10-3p)}, \tag{95}$$

and that is, $\lambda_\rho \to 0$ as $\rho \to 0$. On the other hand, we have

$$I(u_\rho) = \frac{1}{2}\|\nabla u_\rho\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_\rho|u_\rho|^2dx - \frac{1}{p}\|u_\rho\|_p^p, \tag{96}$$

this, together with (87) and (88), gives

$$\|u_\rho\|_p^p = \frac{3p}{2(p-3)}I(u_\rho) + \frac{p}{4(3-p)}\lambda_\rho\rho^2. \tag{97}$$

Therefore, $I(u_\rho) < 0$ since $\lambda_\rho < 0$ and $p \in (2, 8/3)$. Noting that $I(u_\rho) = I_{\rho^2}$, by Lemma 10 and (97), we obtain

$$\begin{aligned}
0 &< \frac{3p}{2(p-3)}\frac{I_{\rho^2}}{\rho^2} \\
&\le \left(\frac{3(p-2)}{2p}\right)^{3(p-2)/(10-3p)} S^{-6(p-2)/(10-3p)} \\
&\quad \times \rho^{4(p-2)/(10-3p)} - \frac{p}{4(3-p)}\lambda_\rho \longrightarrow 0 \quad \text{as } \rho \longrightarrow 0.
\end{aligned} \tag{98}$$

□

*Proof of Theorem 2.* Suppose that $p \in (2, 12/5]$ and $(u_\rho, \lambda_\rho)$ is a solution of (1) with $\|u_\rho\|_2 = \rho$. Then Lemma 17 and the above proof of Theorem 1 show that $\lambda_\rho < 0$, $I(u_\rho) < 0$ and $\lambda_\rho \to 0$ as $\rho \to 0$. It was proved in [5, 20] (see also [13, Remark 1.4]) that there exists $\lambda_0 < 0$ such that (1) has only the zero solution when $p \in (2, 3)$ and $\lambda \in (-\infty, \lambda_0)$. Therefore, $\lambda_\rho$ must be bounded; that is, (i) holds. For (ii), it is clear that (87), (88), and (96) hold; after a simple calculation, we have

$$\begin{aligned}
\|\nabla u_\rho\|_2^2 &= \frac{5p-12}{2(p-3)}I(u_\rho) + \frac{3p-8}{4(3-p)}\lambda_\rho\rho^2, \\
\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx &= \frac{6-p}{p-3}I(u_\rho) + \frac{10-3p}{2(3-p)}\lambda_\rho\rho^2, \\
\|u_\rho\|_p^p &= \frac{3p}{2(p-3)}I(u_\rho) + \frac{p}{4(3-p)}\lambda_\rho\rho^2.
\end{aligned} \tag{99}$$

On the other hand, since $\phi_{u_\rho}$ is the solution of the Poisson equation $-\Delta\phi = 4\pi|u_\rho|^2$, multiplying this equation by $|u_\rho|$ and integrating, we obtain

$$\begin{aligned}
4\pi\|u_\rho\|_3^3 &= \int_{\mathbb{R}^3}\nabla\phi_\rho\nabla|u_\rho|\,dx \\
&\le \frac{1}{2}\|\nabla\phi_\rho\|_2^2 + \frac{1}{2}\|\nabla u_\rho\|_2^2 \\
&= 2\pi\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx + \frac{1}{2}\|\nabla u_\rho\|_2^2.
\end{aligned} \tag{100}$$

It follows from (99) and (100) that

$$\begin{aligned}
0 &< \frac{3p}{2(p-3)}I(u_\rho) + \frac{p}{4(3-p)}\lambda_\rho\rho^2 \\
&= \|u_\rho\|_p^p \le \|u_\rho\|_2^{2(3-p)}\|u_\rho\|_3^{3(p-2)} \\
&\le \|u_\rho\|_2^{2(3-p)}\left(\frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx + \|\nabla u_\rho\|_2^2\right)^{p-2} \\
&= \left(\frac{2}{p}\right)^{p-2}\|u_\rho\|_2^{2(3-p)} \\
&\quad \times\left(\frac{p(2p-3)}{2(p-3)}I(u_\rho) + \frac{p}{4(3-p)}\lambda_\rho\rho^2\right)^{p-2}
\end{aligned}$$

$$\leq \left(\frac{2}{p}\right)^{p-2} \|u_\rho\|_2^{2(3-p)} \times \left(\frac{3p}{2(p-3)} I(u_\rho) + \frac{p}{4(3-p)} \lambda_\rho \rho^2\right)^{p-2}, \tag{101}$$

which implies that

$$\begin{aligned} 0 &< \frac{3p}{2(p-3)} I(u_\rho) + \frac{p}{4(3-p)} \lambda_\rho \rho^2 \\ &\leq \left(\frac{2}{p}\right)^{(p-2)/(3-p)} \|u_\rho\|_2^2 = \left(\frac{2}{p}\right)^{(p-2)/(3-p)} \rho^2. \end{aligned} \tag{102}$$

Therefore we get

$$0 < \frac{3p}{2(p-3)} \frac{I(u_\rho)}{\rho^2} + \frac{p}{4(3-p)} \lambda_\rho \leq \left(\frac{2}{p}\right)^{(p-2)/(3-p)}, \tag{103}$$

so that there exists $C > 0$ such that $I(u_\rho)/\rho^2 \in (-C, 0)$ since, by (i), $\lambda_\rho < 0$ is bounded. □

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