Research Article

Notes on the Global Well-Posedness for the Maxwell-Navier-Stokes System

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Masmoudi (2010) obtained global well-posedness for 2D Maxwell-Navier-Stokes system. In this paper, we reprove global existence of regular solutions to the 2D system by using energy estimates and Brezis-Gallouet inequality. Also we obtain a blow-up criterion for solutions to 3D Maxwell-Navier-Stokes system.

1. Introduction

In this paper, we consider Maxwell-Navier-Stokes equations in $\mathbb{R}^d$ ($d = 2, 3$) as follows:

$$\begin{aligned} \frac{\partial v}{\partial t} + (v \cdot \nabla) v - \Delta v + \nabla p &= j \times B \quad \text{in } \mathbb{R}^d \times (0, T), \\ \frac{\partial E}{\partial t} - \nabla \times B &= -j \quad \text{in } \mathbb{R}^d \times (0, T), \\ \frac{\partial B}{\partial t} + \nabla \times E &= 0 \quad \text{in } \mathbb{R}^d \times (0, T), \\ \nabla \cdot v &= \nabla \cdot B = 0 \quad \text{in } \mathbb{R}^d \times (0, T), \\ j &= E + v \times B, \end{aligned} \quad (1)$$

subject to the initial data

$$\begin{aligned} v(x, 0) &= v_0(x), \quad E(x, 0) = E_0(x), \\ B(x, 0) &= B_0(x). \end{aligned} \quad (2)$$

Here v , E , and $B : \mathbb{R}^d \times (0, T) \rightarrow \mathbb{R}^3$ are vector fields defined on $\mathbb{R}^d$ ($d = 2$ or 3). Vector fields v , E , and B denote fluid velocity, electric fields and magnetic fields, respectively. p denotes the scalar pressure and j is the electric current given by Ohm's law. $j \times B$ represents the Lorentz force. Here we put the viscosity and the electric resistivity to be 1 for the simplification. Note that in 2D case, vector fields v , E , and B can be understood as $v(x, t) = (v_1(x_1, x_2, t), v_2(x_1, x_2, t), 0)$, and so forth.

For the compatibility of the initial data, we assume that

$$\nabla \cdot v_0 = \nabla \cdot B_0 = 0. \quad (3)$$

Since the divergence-free condition of the magnetic field is conserved, $\nabla \cdot B = 0$ in (1) is not necessary in general if we assume the divergence-free condition for the initial data of the magnetic field in $\mathbb{R}^d$. In many physical situations, current displacement term $\partial_t E$ is neglected because the physical coefficient for this term is very small ($\sim 1/c^2$, where c denotes the speed of light). But mathematically, the presence of the term $\partial_t E$ in the second equation (Ampere-Maxwell equation) preserves the hyperbolic nature of the Maxwell equation in the Maxwell-Navier-Stokes equations (see [1, 2] and references therein). Also we remark that full Maxwell-Navier-Stokes equations have been used for the accurate computation of electromagnetic hypersonics in aerothermodynamics (see [3, 4] and references therein). For further physical motivations, see [5].

Neglecting the current displacement term, Maxwell-Navier-Stokes system is reduced to the usual MHD system. There have been many extensive mathematical studies for the existence, blow-up criterion, and regularity criterion of MHD and related models (see [6–12] and references therein). Recently, Maxwell-Navier-Stokes system has been receiving much mathematical attention after pioneering work of Masmoudi [2]. In [2], global existence of regular solutions to (1) in $\mathbb{R}^2$ is proved by using the Besov-type $\tilde{L}$ space technique

developed by Chemin and Lerner [13]. In [1, 14], the local existence of mild solution and the global existence of (1) with small data have been studied. Duan [15] studied large time behaviour of solutions to (1). In [16], Ibrahim and Yoneda obtained local-in-time existence for nondecaying initial data in torus. Also Germain and Masmoudi [17] studied global existence of solutions to Euler-Maxwell equations with small data and Jang and Masmoudi [18] mathematically derived Ohm's law from the kinetic equation.

The aim of this paper is to study the global well-posedness for (1) using the standard energy estimates. We obtain the local-in-time existence of H^2 solution by using the standard mollifier technique (see Proposition 4) and re-prove the global existence of H^2 solution for 2D Maxwell-Navier-Stokes system (see Theorem 1) by using standard energy estimates and Brezis-Gallouet inequality, which was used to prove global existence of regular solution for the partial viscous Boussinesq equations by Chae [19]. Also we provide blow-up criterion of regular solutions to 3D Maxwell-Navier-Stokes equations (see Theorem 2).

We state our main results in the following.

Theorem 1. Assume that $(v_0, E_0, B_0) \in H^2(\mathbb{R}^2)$ and $\nabla \cdot v_0 = \nabla \cdot B_0 = 0$. Then, for any $T > 0$, there exists a solution to 2D Maxwell-Navier-Stokes system (1) such that $(v, E, B) \in C((0, T]; H^2)$ and $(\nabla v, j) \in L^2(0, T; H^2)$.

Theorem 2. Suppose that $(v_0, E_0, B_0) \in H^2(\mathbb{R}^3)$ and $\nabla \cdot v_0 = \nabla \cdot B_0 = 0$. If T^* , the maximal existence time of the local existence of regular solution to 3D Maxwell-Navier-Stokes system (1), is finite, then

$$\int_0^{T^*} \|v(t)\|_{L^\infty}^2 + \|B(t)\|_{L^\infty}^{8/3} dt = \infty. \quad (4)$$

Remark 3. (1) As logarithmic inequality has been used in [2], Brezis-Gallouet inequality gives logarithmic-type estimates. But it provides double exponential bound compared with exponential bound in [2].

(2) The presence of the current displacement term $\partial_t E$ makes Maxwell-Navier-Stokes system do not enjoy the scaling invariance property of the usual Navier-Stokes system, $v_\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t)$. In Theorem 2, $\int_0^T \|v(t)\|_{L^\infty}^2 dt$ is concurrent with the usual scaling invariant norm of solutions to 3D Navier-Stokes equations.

The rest of this paper is organized as follows. In Section 2, we provide the local-in-time existence of regular solution to 2D and 3D Maxwell-Navier-Stokes systems and global existence of 2D Maxwell-Navier-Stokes system with large data. In Section 3, we provide the blow-up criterion for H^2 solution to 3D Maxwell-Navier-Stokes system.

2. Local Existence and Global Well-Posedness

At first, we note that one can have the energy identity in two or three dimensions:

$$\frac{1}{2} \frac{d}{dt} (\|v\|_{L^2}^2 + \|B\|_{L^2}^2 + \|E\|_{L^2}^2) + \|j\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 = 0. \quad (5)$$

The previously energy inequality can be justified for local in time regular solution in the following proposition. In the following, C denotes a harmless constant which may change from one line to the other. We prove local-in-time existence of H^2 solution using the standard energy estimates.

Proposition 4. Let $(u_0, E_0, B_0) \in H^2(\mathbb{R}^d)$ ($d = 2$ or 3) with $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$. Then there exists $T = T(\|u_0\|_{H^2}, \|E_0\|_{H^2}, \|B_0\|_{H^2})$ such that there exists a unique solution $(u, E, B) \in L^\infty(0, T; H^2(\mathbb{R}^d)) \cap Lip(0, T; L^2)$.

Proof. We use the mollifier method as described in [20]. Although the details are similar to [20], we provide some a priori estimates for the reader's sake. We consider the standard mollifier operator

$$\mathcal{J}_\epsilon f = \rho_\epsilon * f, \quad \rho_\epsilon(\cdot) = \frac{1}{\epsilon^d} \rho\left(\frac{\cdot}{\epsilon}\right), \quad (6)$$

where $\rho \in C_0^\infty(\mathbb{R}^d)$, and $\rho \geq 0$, $\int_{\mathbb{R}^d} \rho dx = 1$.

We introduce the following regularized system of (1):

$$\begin{aligned} \partial_t v^\epsilon + \mathcal{J}_\epsilon (\mathcal{J}_\epsilon v^\epsilon \cdot \nabla) \mathcal{J}_\epsilon v^\epsilon - \Delta \mathcal{J}_\epsilon^2 v^\epsilon + \nabla p^\epsilon \\ = \mathcal{J}_\epsilon (\mathcal{J}_\epsilon^2 j^\epsilon \times \mathcal{J}_\epsilon B^\epsilon) \quad \text{in } \mathbb{R}^d \times (0, T), \\ \partial_t E^\epsilon - \nabla \times \mathcal{J}_\epsilon^2 B^\epsilon = -\mathcal{J}_\epsilon^2 j^\epsilon \quad \text{in } \mathbb{R}^d \times (0, T), \\ \partial_t B^\epsilon + \nabla \times \mathcal{J}_\epsilon^2 E^\epsilon = 0 \quad \text{in } \mathbb{R}^d \times (0, T), \\ \nabla \cdot v^\epsilon = \nabla \cdot B^\epsilon = 0 \quad \text{in } \mathbb{R}^d \times (0, T), \\ j^\epsilon = E^\epsilon + \mathcal{J}_\epsilon v^\epsilon \times \mathcal{J}_\epsilon B^\epsilon, \end{aligned} \quad (7)$$

with initial data $(v_0^\epsilon, E_0^\epsilon, B_0^\epsilon) = (\mathcal{J}_\epsilon v_0, \mathcal{J}_\epsilon E_0, \mathcal{J}_\epsilon B_0)$.

Taking the L^2 inner product of (7)₁, (7)₂, and (7)₃ with v^ϵ , E^ϵ , B^ϵ , respectively, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|v^\epsilon\|_{L^2}^2 + \|E^\epsilon\|_{L^2}^2 + \|B^\epsilon\|_{L^2}^2) \\ + \|\nabla \mathcal{J}_\epsilon v^\epsilon\|_{L^2}^2 + \|\mathcal{J}_\epsilon j^\epsilon\|_{L^2}^2 \\ = -\frac{1}{2} \int_{\mathbb{R}^d} (\mathcal{J}_\epsilon v^\epsilon) \cdot \nabla (\mathcal{J}_\epsilon v^\epsilon)^2 dx \\ + \int_{\mathbb{R}^d} (\nabla \times \mathcal{J}_\epsilon B^\epsilon) \cdot \mathcal{J}_\epsilon E^\epsilon dx \\ - \int_{\mathbb{R}^d} (\nabla \times \mathcal{J}_\epsilon E^\epsilon) \cdot \mathcal{J}_\epsilon B^\epsilon dx \\ + \int_{\mathbb{R}^d} (\mathcal{J}_\epsilon^2 j^\epsilon \times \mathcal{J}_\epsilon B^\epsilon) \cdot \mathcal{J}_\epsilon v^\epsilon dx \\ + \int_{\mathbb{R}^d} \mathcal{J}_\epsilon^2 j^\epsilon \cdot (\mathcal{J}_\epsilon v^\epsilon \times \mathcal{J}_\epsilon B^\epsilon) dx = 0. \end{aligned} \quad (8)$$

We compute the derivative D^α , α is a multi-index such that $|\alpha| \leq 2$, of (7), multiply them by $D^\alpha v^\epsilon$, $D^\alpha E^\epsilon$, and $D^\alpha B^\epsilon$, respectively, and integrate them over $\mathbb{R}^d$ to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|v^\epsilon\|_{H^2}^2 + \|E^\epsilon\|_{H^2}^2 + \|B^\epsilon\|_{H^2}^2) \\ & + \|\nabla \mathcal{F}_\epsilon v^\epsilon\|_{H^2}^2 + \|\mathcal{F}_\epsilon j^\epsilon\|_{H^2}^2 \\ & \leq C \|\mathcal{F}_\epsilon v^\epsilon \otimes \mathcal{F}_\epsilon v^\epsilon\|_{H^2} \|\nabla \mathcal{F}_\epsilon v^\epsilon\|_{H^2} \\ & + C \|\mathcal{F}_\epsilon^2 j^\epsilon \times \mathcal{F}_\epsilon B^\epsilon\|_{H^2} \|\mathcal{F}_\epsilon v^\epsilon\|_{H^2} \\ & + C \|\mathcal{F}_\epsilon^2 j^\epsilon\|_{H^2} \|\mathcal{F}_\epsilon v^\epsilon \times \mathcal{F}_\epsilon B^\epsilon\|_{H^2} \\ & \leq C (\|\mathcal{F}_\epsilon v^\epsilon\|_{H^2}^4 + \|\mathcal{F}_\epsilon B^\epsilon\|_{H^2}^4) \\ & + \frac{1}{2} (\|\mathcal{F}_\epsilon \nabla v^\epsilon\|_{H^2}^2 + \|\mathcal{F}_\epsilon j^\epsilon\|_{H^2}^2). \end{aligned} \quad (9)$$

In the previously mentioned, $\mathcal{F}_\epsilon v^\epsilon \otimes \mathcal{F}_\epsilon v^\epsilon$ denotes a tensor $(\mathcal{F}_\epsilon v_i^\epsilon \mathcal{F}_\epsilon v_j^\epsilon)_{1 \leq j \leq d}$.

Using Picard's theorem, these estimates imply local existence of solution. $\square$

The main ingredient of the proof of Theorem 1 is the following Brezis-Gallouet inequality (logarithmic Sobolev inequality):

$$\begin{aligned} \|f\|_{L^\infty} & \leq C \left(1 + \|f\|_{L^2} + \|\nabla f\|_{L^2} (\log^+ \|\Delta f\|_{L^2})^{1/2} \right), \\ f & \in H^2(\mathbb{R}^2). \end{aligned} \quad (10)$$

Here $\log^+ a$ denotes $\log(e + a)$.

Proof of Theorem 1. We provide a priori estimates on the regular solutions. Let T be a finite maximal time of existence in Proposition 4. By obtaining H^2 bound on $(0, T]$ of solution, we can continue solution beyond T by using Proposition 4.

Taking curl operator on $(1)_1$ and $\partial_i = \partial/\partial x_i$ ($i = 1, 2$) operator on $(1)_2$ and $(1)_3$, we have

$$\begin{aligned} \frac{\partial \omega}{\partial t} + (v \cdot \nabla) \omega - \Delta \omega & = \nabla \times (j \times B), \quad \text{in } \mathbb{R}^2 \times (0, T), \\ \frac{\partial (\partial_i E)}{\partial t} - \nabla \times \partial_i B & = -\partial_i j, \quad \text{in } \mathbb{R}^2 \times (0, T), \\ \frac{\partial (\partial_i B)}{\partial t} + \nabla \times \partial_i E & = 0, \quad \text{in } \mathbb{R}^2 \times (0, T). \end{aligned} \quad (11)$$

(i) H^1 Estimates. Taking scalar product (11) with ω , $\partial_i E$, and $\partial_i B$, respectively, and summing over $i = 1, 2$, we have

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 = \int_{\mathbb{R}^2} \nabla \times (j \times B) \cdot \omega \, dx, \quad (12)$$

$$\frac{1}{2} \frac{d}{dt} \|\nabla E\|_{L^2}^2 = \sum_i \int_{\mathbb{R}^2} \nabla \times \partial_i B \cdot \partial_i E \, dx - \int_{\mathbb{R}^2} \nabla j \cdot \nabla E \, dx,$$

$$\frac{1}{2} \frac{d}{dt} \|\nabla B\|_{L^2}^2 = - \sum_i \int_{\mathbb{R}^2} \nabla \times \partial_i E \cdot \partial_i B \, dx. \quad (13)$$

Using the identity

$$\int_{\mathbb{R}^2} \nabla \times \partial_i B \cdot \partial_i E \, dx = \int_{\mathbb{R}^2} \nabla \times \partial_i E \cdot \partial_i B \, dx \quad (14)$$

and $E = j - v \times B$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla E\|_{L^2}^2 + \|\nabla B\|_{L^2}^2) + \|\nabla j\|_{L^2}^2 \\ & = \int_{\mathbb{R}^2} \nabla j \cdot \nabla (v \times B) \, dx. \end{aligned} \quad (15)$$

In the following, ϵ denotes a sufficiently small positive number. Since it holds that $\nabla \times (j \times B) = (B \cdot \nabla)j$, we estimate the right-hand side of (12) using Young's inequality and interpolation inequality:

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} \nabla \times (j \times B) \cdot \omega \, dx \right| \\ & \leq \|B\|_{L^4} \|\nabla j\|_{L^2} \|\omega\|_{L^4} \\ & \leq C \|B\|_{L^2}^{1/2} \|\nabla B\|_{L^2}^{1/2} \|\omega\|_{L^2}^{1/2} \|\nabla \omega\|_{L^2}^{1/2} \|\nabla j\|_{L^2} \\ & \leq C \|\omega\|_{L^2}^2 \|\nabla B\|_{L^2}^2 + \epsilon \|\nabla \omega\|_{L^2}^2 + \epsilon \|\nabla j\|_{L^2}^2, \end{aligned} \quad (16)$$

where ϵ is a small positive number. Also we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} \nabla j \cdot \nabla (v \times B) \, dx \right| \\ & \leq \int_{\mathbb{R}^2} |\nabla j| |v| |\nabla B| \, dx \\ & + \int_{\mathbb{R}^2} |\nabla j| |B| |\nabla v| \, dx = I + II. \end{aligned} \quad (17)$$

We estimate

$$\begin{aligned} I & \leq C \|v\|_{L^\infty}^2 \|\nabla B\|_{L^2}^2 + \epsilon \|\nabla j\|_{L^2}^2, \\ II & \leq \|B\|_{L^4} \|\nabla v\|_{L^4} \|\nabla j\|_{L^2} \leq C \|B\|_{L^2}^2 \|\nabla v\|_{L^2}^2 \|\nabla B\|_{L^2}^2 \\ & + \epsilon \|\Delta v\|_{L^2}^2 + \epsilon \|\nabla j\|_{L^2}^2. \end{aligned} \quad (18)$$

Collecting previous estimates, we have

$$\begin{aligned} & \frac{d}{dt} (\|\omega\|_{L^2}^2 + \|\nabla E\|_{L^2}^2 + \|\nabla B\|_{L^2}^2) \\ & + \|\nabla \omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 \leq C \|\omega\|_{L^2}^2 \|\nabla B\|_{L^2}^2 \\ & + C \|v\|_{L^\infty}^2 \|\nabla B\|_{L^2}^2 + C \|\nabla B\|_{L^2}^2 \|\nabla v\|_{L^2}^2 \|\nabla B\|_{L^2}^2. \end{aligned} \quad (19)$$

(ii) H^2 Estimates. Taking Δ operator on $(1)_1$, $(1)_2$, and $(1)_3$ and L^2 scalar product with Δv , ΔE , and ΔB , respectively, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta v\|_{L^2}^2 + \|\nabla \Delta v\|_{L^2}^2 \\ & \leq C \int_{\mathbb{R}^2} |\nabla v| |D^2 v| dx \\ & \quad + \int_{\mathbb{R}^2} |\Delta(j \times B)| |\Delta v| dx := I_1 + I_2, \quad (20) \\ & \frac{1}{2} \frac{d}{dt} (\|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2) + \|\Delta j\|_{L^2}^2 \\ & \leq \int_{\mathbb{R}^2} |\Delta j| |\Delta(v \times B)| dx := I_3. \end{aligned}$$

We estimate I_1 , I_2 , and I_3 using interpolation inequality, Young's inequality, and Hölder's inequality:

$$\begin{aligned} I_1 & \leq C \|\nabla v\|_{L^4} \|\Delta v\|_{L^4} \|\Delta v\|_{L^2} \\ & \leq C \|\nabla v\|_{L^2}^{1/2} \|\Delta v\|_{L^2}^{3/2} \|\nabla \Delta v\|_{L^2}^{1/2} \\ & \leq C \|\nabla v\|_{L^2}^{2/3} \|\Delta v\|_{L^2}^2 + \epsilon \|\nabla \Delta v\|_{L^2}^2, \end{aligned} \quad (21)$$

$$\begin{aligned} I_2 & \leq C \int_{\mathbb{R}^2} |\nabla j| |\nabla B| |\Delta v| dx \\ & \quad + C \int_{\mathbb{R}^2} |\Delta j| |B| |\Delta v| dx \\ & \quad + C \int_{\mathbb{R}^2} |j| |\Delta B| |\Delta v| dx := I_{21} + I_{22} + I_{23}. \end{aligned} \quad (22)$$

Each term can be estimated by the standard interpolation inequality and Young's inequality as follows:

$$\begin{aligned} I_{21} & \leq C \|j\|_{L^4} \|\nabla B\|_{L^4} \|\Delta v\|_{L^2} \\ & \leq C \|j\|_{L^2}^{1/4} \|\Delta j\|_{L^2}^{3/4} \|B\|_{L^2}^{1/4} \|\Delta B\|_{L^2}^{3/4} \|\nabla v\|_{L^2}^{1/2} \|\nabla \Delta v\|_{L^2}^{1/2} \\ & \leq C \|j\|_{L^2}^{2/3} \|B\|_{L^2}^{2/3} \|\nabla v\|_{L^2}^{4/3} \|\Delta B\|_{L^2}^2 + \epsilon \|\Delta j\|_{L^2}^2 \\ & \leq C (\|j\|_{L^2}^2 + \|\nabla v\|_{L^2}^2) \|\Delta B\|_{L^2}^2 + \epsilon \|\Delta j\|_{L^2}^2, \\ I_{22} & \leq C \|\Delta j\|_{L^2} \|B\|_{L^4} \|\Delta v\|_{L^4} \\ & \leq C \|\Delta j\|_{L^2} \|B\|_{L^2}^{3/4} \|\Delta B\|_{L^2}^{1/4} \|\Delta v\|_{L^2}^{1/2} \|\nabla \Delta v\|_{L^2}^{1/2} \\ & \leq \epsilon \|\Delta j\|_{L^2}^2 + C \|\Delta B\|_{L^2}^{1/2} \|\Delta v\|_{L^2}^{1/2} \|\nabla \Delta v\|_{L^2} \\ & \leq \epsilon \|\Delta j\|_{L^2}^2 + \epsilon \|\nabla \Delta v\|_{L^2}^2 + C \|j\|_{L^2} \|\nabla v\|_{L^2} \|\Delta B\|_{L^2}^2, \\ I_{23} & \leq C \|j\|_{L^\infty} \|\Delta B\|_{L^2} \|\Delta v\|_{L^2} \\ & \leq C \|j\|_{L^2} \|\Delta j\|_{L^2}^{1/2} \|\Delta B\|_{L^2} \|\nabla v\|_{L^2}^{1/2} \|\nabla \Delta v\|_{L^2}^{1/2} \\ & \leq \epsilon \|\Delta j\|_{L^2}^2 + C \|j\|_{L^2}^{2/3} \|\Delta B\|_{L^2}^{4/3} \|\nabla v\|_{L^2}^{2/3} \|\nabla \Delta v\|_{L^2}^{2/3} \\ & \leq \epsilon \|\Delta j\|_{L^2}^2 + \epsilon \|\nabla \Delta v\|_{L^2}^2 + C \|j\|_{L^2} \|\nabla v\|_{L^2} \|\Delta B\|_{L^2}^2. \end{aligned} \quad (23)$$

I_3 can be written as

$$\begin{aligned} I_3 & \leq C \int_{\mathbb{R}^2} |\Delta j| |\nabla v| |\nabla B| dx \\ & \quad + C \int_{\mathbb{R}^2} |\Delta j| |\Delta v| |B| dx \\ & \quad + C \int_{\mathbb{R}^2} |\Delta j| |v| |\Delta B| dx := I_{31} + I_{32} + I_{33}, \quad (24) \\ I_{31} & \leq C \|\Delta j\|_{L^2} \|\nabla v\|_{L^4} \|\nabla B\|_{L^4} \\ & \leq C \|\Delta j\|_{L^2} \|\nabla v\|_{L^2}^{3/4} \|\nabla \Delta v\|_{L^2}^{1/4} \|B\|_{L^2}^{1/4} \|\Delta B\|_{L^2}^{3/4} \\ & \leq \epsilon \|\Delta j\|_{L^2}^2 + \epsilon \|\nabla \Delta v\|_{L^2}^2 + C \|\nabla v\|_{L^2}^2 \|\Delta B\|_{L^2}^2. \end{aligned}$$

The same as the estimate of I_{22} , we obtain

$$\begin{aligned} I_{32} & \leq \epsilon \|\Delta j\|_{L^2}^2 + \epsilon \|\nabla \Delta v\|_{L^2}^2 \\ & \quad + C \|j\|_{L^2} \|\nabla v\|_{L^2} \|\Delta B\|_{L^2}^2. \end{aligned} \quad (25)$$

Also we have

$$I_{33} \leq C \|\Delta j\|_{L^2} \|v\|_{L^\infty} \|\Delta B\|_{L^2} \leq \epsilon \|\Delta j\|_{L^2}^2 + C \|v\|_{L^\infty}^2 \|\Delta B\|_{L^2}^2. \quad (26)$$

Therefore, we have

$$\begin{aligned} & \frac{d}{dt} (\|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2) \\ & \quad + \|\nabla \Delta v\|_{L^2}^2 + \|\Delta j\|_{L^2}^2 \\ & \leq C (1 + \|\nabla v\|_{L^2}^2 + \|j\|_{L^2}^2 + \|v\|_{L^\infty}^2) \\ & \quad \times (\|\Delta v\|_{L^2}^2 + \|\Delta B\|_{L^2}^2). \end{aligned} \quad (27)$$

(iii) *Use of Brezis-Gallouet Inequality.* Using Brezis-Gallouet inequality, we obtain

$$\begin{aligned} & \frac{d}{dt} (\|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2) \\ & \quad + \|\nabla \Delta v\|_{L^2}^2 + \|\Delta j\|_{L^2}^2 \\ & \leq C (1 + \|\nabla v\|_{L^2}^2 + \|j\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2) \\ & \quad \times (\|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2) \\ & \quad \times \log^+ (\|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2). \end{aligned} \quad (28)$$

Let $y(t) = \|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2$, and let $z(t) = 1 + \|\Delta v\|_{L^2}^2 + \|j\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2$. Hence one has

$$\frac{d}{dt} y(t) \leq C z(t) y(t) \log^+ y(t). \quad (29)$$

Since

$$\int_0^T z(t) dt \leq C(1+T), \quad (30)$$

the bound of $y(t)$ is immediate as follows:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2) \\ & \leq (\|\Delta v_0\|_{L^2}^2 + \|\Delta E_0\|_{L^2}^2 + \|\Delta B_0\|_{L^2}^2) \\ & \quad \times \exp(\exp(C(T+1))). \end{aligned} \quad (31)$$

This completes the proof of Theorem 1. $\square$

3. Blow-Up Criterion for 3D Maxwell-Navier-Stokes System

In this section, we provide a blow-up criterion for H^2 solution in Proposition 4 to 3D Maxwell-Navier-Stokes system.

Proof of Theorem 2. Assume that

$$\int_0^{T^*} \|v(t)\|_{L^\infty}^2 + \|B(t)\|_{L^\infty}^{8/3} dt < \infty, \quad (32)$$

where T^* is the finite maximal existence time of a classical solution.

Similar to the computation in Section 2, one has H^1 estimates of E and B as follows:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla E\|_{L^2}^2 + \|\nabla B\|_{L^2}^2) + \|\nabla j\|_{L^2}^2 \\ & = \int_{\mathbb{R}^3} \nabla j \cdot \nabla (v \times B) dx \\ & \leq C \|\nabla(v \times B)\|_{L^2}^2 + \epsilon \|\nabla j\|_{L^2}^2 \\ & \leq C \|B\|_{L^\infty}^2 \|\nabla v\|_{L^2}^2 + C \|v\|_{L^\infty}^2 \|\nabla B\|_{L^2}^2 + \epsilon \|\nabla j\|_{L^2}^2. \end{aligned} \quad (33)$$

H^1 estimates of v are as follows:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 \\ & \leq \int_{\mathbb{R}^3} |v| |\nabla v| |\Delta v| dx \\ & \quad + \int_{\mathbb{R}^3} |j \times B| |\Delta v| dx \\ & \leq C \|v\|_{L^\infty}^2 \|\nabla v\|_{L^2}^2 \\ & \quad + C \|j \times B\|_{L^2}^2 + \epsilon \|\Delta v\|_{L^2}^2. \end{aligned} \quad (34)$$

The estimate of $\|j \times B\|_{L^2}^2$ is provided in the following:

$$\begin{aligned} \|j \times B\|_{L^2}^2 & \leq C \|E \times B\|_{L^2}^2 + C \|(v \times B) \times B\|_{L^2}^2 \\ & \leq C \|E\|_{L^2}^2 \|B\|_{L^\infty}^2 + C \|v\|_{L^6}^2 \|B\|_{L^6}^4 \\ & \leq C \|E\|_{L^2}^2 \|B\|_{L^\infty}^2 + C \|\nabla v\|_{L^2}^2 \|B\|_{L^2}^{4/3} \|B\|_{L^\infty}^{8/3}. \end{aligned} \quad (35)$$

Thus we have

$$\begin{aligned} & \frac{d}{dt} (\|\nabla v\|_{L^2}^2 + \|\nabla E\|_{L^2}^2 + \|\nabla B\|_{L^2}^2) \\ & \quad + \|\Delta v\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 \\ & \leq C (1 + \|v\|_{L^\infty}^2 + \|B\|_{L^\infty}^{8/3}) \\ & \quad \times (\|\nabla v\|_{L^2}^2 + \|\nabla E\|_{L^2}^2 + \|\nabla B\|_{L^2}^2) + C \|B\|_{L^\infty}^2. \end{aligned} \quad (36)$$

Gronwall's inequality gives us that

$$\|(\nabla v, \nabla E, \nabla B)\|_{L^\infty(0, T^*; L^2)}^2 + \|(\Delta v, \nabla j)\|_{L^2(0, T^*; L^2)}^2 \leq C < \infty. \quad (37)$$

Next, we consider H^2 estimates.

Integrating by parts and using Young's inequality, it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2) \\ & \quad + \|\Delta j\|_{L^2}^2 \leq C \|\Delta(v \times B)\|_{L^2}^2 + \epsilon \|\Delta j\|_{L^2}^2 \\ & \leq C \|\Delta v\|_{L^2}^2 \|B\|_{L^\infty}^2 + C \|v\|_{L^\infty}^2 \|\Delta B\|_{L^2}^2 \\ & \quad + C \|\nabla v\|_{L^4}^2 \|\nabla B\|_{L^4}^2 + \epsilon \|\Delta j\|_{L^2}^2 \\ & \leq C \|\Delta v\|_{L^2}^2 \|B\|_{L^\infty}^2 + C \|v\|_{L^\infty}^2 \|\Delta B\|_{L^2}^2 \\ & \quad + C \|v\|_{L^\infty} \|B\|_{L^\infty} \|\Delta v\|_{L^2} \|\Delta B\|_{L^2} + \epsilon \|\Delta j\|_{L^2}^2. \end{aligned} \quad (38)$$

Similarly, it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta v\|_{L^2}^2 \leq C \|\nabla(v \cdot \nabla v)\|_{L^2}^2 \\ & \quad + C \|\nabla(j \times B)\|_{L^2}^2 + \epsilon \|\nabla \Delta v\|_{L^2}^2 \\ & \leq C \|\nabla v\|_{L^4}^4 + C \|v\|_{L^\infty}^2 \|\Delta v\|_{L^2}^2 \\ & \quad + C \|\nabla E\|_{L^6}^2 \|B\|_{L^3}^2 + C \|E\|_{L^6}^2 \|\nabla B\|_{L^3}^2 \\ & \quad + C \|\nabla(v \times B)\|_{L^2}^2 \|B\|_{L^\infty}^2 + C \|v \times B\|_{L^3}^2 \|\nabla B\|_{L^6}^2. \end{aligned} \quad (39)$$

Using the interpolation inequality, one has

$$\|\nabla v\|_{L^4}^4 \leq C \|v\|_{L^\infty}^2 \|\Delta v\|_{L^2}^2. \quad (40)$$

Interpolation inequality and Young's inequality produce that

$$\begin{aligned} & \|E\|_{L^6}^2 \|\nabla B\|_{L^3}^2 \\ & \leq C \|\nabla E\|_{L^2}^2 \|\nabla B\|_{L^2} \|\Delta B\|_{L^2} \\ & \leq C \|\nabla E\|_{L^2}^2 (\|\nabla B\|_{L^2}^2 + \|\Delta B\|_{L^2}^2). \end{aligned} \quad (41)$$

Similarly, we estimate that

$$\begin{aligned} & \|\nabla(v \times B)\|_{L^2}^2 \|B\|_{L^\infty}^2 \\ & \leq C (\|\nabla v\|_{L^3}^2 \|\nabla B\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \|\nabla B\|_{L^3}^2), \\ & \|v \times B\|_{L^3}^2 \|\nabla B\|_{L^6}^2 \leq C \|v\|_{L^\infty}^2 \|B\|_{L^3}^2 \|\Delta B\|_{L^2}^2. \end{aligned} \quad (42)$$

We already know that

$$\|(\nabla v, \nabla E, \nabla B)\|_{L^\infty(0, T^*; L^2)}^2 < C. \quad (43)$$

Gathering all the estimates, we achieve

$$\begin{aligned} & \frac{d}{dt} \left(\|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2 \right) \\ & + \|\nabla \Delta v\|_{L^2}^2 + \|\Delta j\|_{L^2}^2 \\ & \leq C \left(1 + \|v\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \right) \\ & \times \left(1 + \|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2 \right). \end{aligned} \quad (44)$$

Using Gronwall's inequality, we conclude that

$$\|(\Delta v, \Delta E, \Delta B)\|_{L^\infty(0, T^*; L^2)}^2 + \|(\nabla \Delta v, \Delta j)\|_{L^2(0, T^*; L^2)}^2 \leq C < \infty. \quad (45)$$

This completes the proof of Theorem 2. $\square$

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References

- [1] P. Germain, S. Ibrahim, and N. Masmoudi, "On the well-posedness of the Navier-Stokes-Maxwell system," <http://arxiv.org/abs/1207.6187>.
- [2] N. Masmoudi, "Global well posedness for the Maxwell-Navier-Stokes system in 2D," *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, vol. 93, no. 6, pp. 559–571, 2010.
- [3] D. D'Ambrosio and D. Giordano, "Electromagnetic fluid dynamics for aerospace applications. Part I: classification and critical review of physical models," in *Proceedings of the 35th AIAA Plasmadynamics and Lasers Conference*, Portland, Ore, USA, June 2004, AIAA Paper 2004-2165.
- [4] D. D'Ambrosio, M. Pandol, and D. Giordano, "Electromagnetic fluid dynamics for aerospace applications. Part II: numerical simulations using different physical models," in *Proceedings of the 35th AIAA Plasmadynamics and Lasers Conference*, Portland, Ore, USA, June 2004, AIAA Paper 2004-2362.
- [5] P. A. Davidson, *An Introduction to Magnetohydrodynamics*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, UK, 2001.
- [6] C. Cao and J. Wu, "Two regularity criteria for the 3D MHD equations," *Journal of Differential Equations*, vol. 248, no. 9, pp. 2263–2274, 2010.
- [7] C. Cao and J. Wu, "Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion," *Advances in Mathematics*, vol. 226, no. 2, pp. 1803–1822, 2011.
- [8] G. Duvaut and J.-L. Lions, "Inéquations en thermoélasticité et magnétohydrodynamique," *Archive for Rational Mechanics and Analysis*, vol. 46, pp. 241–279, 1972.
- [9] Z. Lei and Y. Zhou, "BKM's criterion and global weak solutions for magnetohydrodynamics with zero viscosity," *Discrete and Continuous Dynamical Systems. Series A*, vol. 25, no. 2, pp. 575–583, 2009.
- [10] M. Sermange and R. Temam, "Some mathematical questions related to the MHD equations," *Communications on Pure and Applied Mathematics*, vol. 36, no. 5, pp. 635–664, 1983.
- [11] J. Wu, "Regularity criteria for the generalized MHD equations," *Communications in Partial Differential Equations*, vol. 33, no. 1–3, pp. 285–306, 2008.
- [12] Y. Zhou, "Regularity criteria for the generalized viscous MHD equations," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 24, no. 3, pp. 491–505, 2007.
- [13] J.-Y. Chemin and N. Lerner, "Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes," *Journal of Differential Equations*, vol. 121, no. 2, pp. 314–328, 1995.
- [14] S. Ibrahim and S. Keraani, "Global small solutions for the Navier-Stokes-Maxwell system," *SIAM Journal on Mathematical Analysis*, vol. 43, no. 5, pp. 2275–2295, 2011.
- [15] R. Duan, "Green's function and large time behavior of the Navier-Stokes-Maxwell system," *Analysis and Applications*, vol. 10, no. 2, pp. 133–197, 2012.
- [16] S. Ibrahim and T. Yoneda, "Local solvability and loss of smoothness of the Navier-Stokes-Maxwell equations with large initial data," *Journal of Mathematical Analysis and Applications*, vol. 396, no. 2, pp. 555–561, 2012.
- [17] P. Germain and N. Masmoudi, "Global existence for the Euler-Maxwell system," <http://arxiv.org/abs/1107.1595>.
- [18] J. Jang and N. Masmoudi, "Derivation of Ohm's law from the kinetic equations," *SIAM Journal on Mathematical Analysis*, vol. 44, no. 5, pp. 3649–3669, 2012.
- [19] D. Chae, "Global regularity for the 2D Boussinesq equations with partial viscosity terms," *Advances in Mathematics*, vol. 203, no. 2, pp. 497–513, 2006.
- [20] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow*, vol. 27 of *Cambridge Texts in Applied Mathematics*, Cambridge University Press, Cambridge, UK, 2002.