Research Article

Nodal Solutions of the $p$-Laplacian with Sign-Changing Weight

Ruyun Ma, 1 Xilan Liu, 2 and Jia Xu 1

1 Department of Mathematics, Northwest Normal University, Lanzhou 730070, China
2 Department of Mathematics, Qinghai University for Nationalities, Xining 810007, China

Correspondence should be addressed to Ruyun Ma; ruyun_ma@126.com

Received 27 July 2013; Accepted 16 October 2013

Copyright © 2013 Ruyun Ma et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We are concerned with determining values of $\gamma$, for which there exist nodal solutions of the boundary value problem

\[
|u''|^{p-2}u'' + \gamma m(t)f(u) = 0, \quad t \in (0, 1),
\]

\[
u(0) = u(1) = 0,
\]

where $m \in C[0, 1]$ is a sign-changing function, $f : \mathbb{R} \to \mathbb{R}$ with $f(s)s > 0$.

The proof of our main results is based upon global bifurcation techniques.

1. Introduction

In [1], Ma and Thompson considered determining values of $r$, for which there exist nodal solutions of the boundary value problem

\[
u'' + rm(t)f(u) = 0, \quad t \in (0, 1),
\]

\[u(0) = u(1) = 0,
\]

under the following assumptions:

(H1) $f \in C(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$;

(H2) $m : [0, 1] \to [0, +\infty)$ is continuous and does not vanish identically on any subinterval of $[0, 1]$;

(H3) there exist $f_0, f_\infty \in (0, +\infty)$ such that

\[
f_0 = \lim_{|s| \to 0} \frac{f(s)}{s}, \quad f_\infty = \lim_{|s| \to +\infty} \frac{f(s)}{s}.
\]

Using the bifurcation theory of Rabinowitz [2, 3], they proved the following.

**Theorem 1.** Let (H1), (H2), and (H3) hold. Assume that, for some $k \in \mathbb{N}$,

\[
\frac{\lambda_k}{f_\infty} < r < \frac{\lambda_k}{f_0} \text{ or } \frac{\lambda_{k-1}}{f_0} < r < \frac{\lambda_k}{f_\infty}.
\]

Then (1) has two solutions $u^+_k$ and $u^-_k$ such that $u^+_k$ has exactly $k - 1$ zeros in $(0, 1)$ and is positive near 0 and $u^-_k$ has exactly $k - 1$ zeros in $(0, 1)$ and is negative near 0.

The results of Theorem 1 have been extended to the case that the weight function changes its sign by Ma and Han [4]. Bifurcation methods have been applied to study the existence of nodal solutions of nonlinear two-point, multipoint, and periodic boundary value problems; see [5–9] and the references therein. The results they obtained extend some well-known theorems of the existence of positive solutions for the related problems [10].

However, no results on the existence of nodal solutions, even positive solutions, have been established for one-dimensional $p$-Laplacian equation with sign-changing weight $m(t)$. It is the purpose of this paper to establish a similar result to Theorem 1 for one-dimensional $p$-Laplacian equation with sign-changing weight. Problem with sign-changing weight arises from the selection-migration model in population genetics. In this model, $m(t)$ changes sign corresponding to the fact that an allele $A_1$ holds an advantage over a rival allele $A_2$ at the same points and is at a disadvantage at others; the parameter $r$ corresponds to the reciprocal of diffusion; for details see [11].

If $m(t) \equiv 1$, Del Pino et al. [12] established the global bifurcation theory for one-dimensional $p$-Laplacian eigenvalue problem. Peral [13] got the global bifurcation theory for $p$-Laplacian eigenvalue problem on the unite ball. In [14], Del Pino and Manásevich obtained the global bifurcation from...
the principal eigenvalue for $p$-Laplacian eigenvalue problem on the general domain. If $m(t) \geq 0$ and is singular at $t = 0$ or $t = 1$, Lee and Sim [15] also established the bifurcation theory for one-dimensional $p$-Laplacian eigenvalue problem. However, if $m(t)$ changes sign, there are a few papers dealing with the $p$-Laplacian eigenvalue problem via bifurcation techniques. In [16], Drábek and Huang established the global bifurcation from the principal eigenvalue for $p$-Laplacian eigenvalue problem in $\mathbb{R}^N$.

The purpose of this paper is to study the bifurcation behavior of one-dimensional $p$-Laplacian eigenvalue problem as follows:

$$\varphi_p(u')' + ym(t)f(u) = 0, \quad t \in (0, 1), \quad u(0) = u(1) = 0,$$

under the condition $(H_1)$ and

$$(H_2) \ m(t) \in C[0, 1] \text{ changes sign and } \max \{x \in [0, 1] \mid m(t) = 0\} = 0;$$

$$(H_3) \ \exists f_0 \in (0, \infty) \ \text{such that} \ f_0 = \lim_{|s| \to 0} \varphi_p(s),$$

where $\varphi_p(s) = |s|^{p-2}s$ with $1 < p < +\infty$;

$$(H_4) \ \exists f_{\infty} \in (0, +\infty) \ \text{such that} \ f_{\infty} = \lim_{|s| \to +\infty} \varphi_p(s).$$

Moreover, based on our global bifurcation theorem, we will prove the existence of nodal solutions for the corresponding nonlinear problem with a parameter (see Theorem 11).

The main tool is the global bifurcation techniques in [17]. The rest of this paper is arranged as follows. In Section 2, we establish the global bifurcation theory for one-dimensional $p$-Laplacian eigenvalue problem with sign-changing weight. In Section 3, we state and prove the main results of this paper.

2. Some Preliminaries

Let $E$ be the Banach space $C^1_0[0, 1]$ with the norm

$$\|u\| = \max \left\{\|u\|_{\infty}, \|u'\|_{\infty}\right\}. \quad (8)$$

Let $Y = L^1(0, 1)$ with its usual normal $\|\cdot\|_{L^1}$. We start by considering the following auxiliary problem:

$$\varphi_p(u')' = h, \quad t \in (0, 1), \quad u(0) = u(1) = 0,$$  \quad (9)

for a given $h \in L^1(0, 1)$. By a solution of problem (9), we understand a function $u \in E$ with $\varphi_p(u')$ absolutely continuous which satisfies (9). Problem (9) is equivalently written to

$$u(t) = G_p(h)(t) := \int_0^t \varphi_p^{-1}\left(a(h) + \int_0^s h(r) \, dr\right) \, ds, \quad (10)$$

where $a : Y \to \mathbb{R}$ is a continuous function satisfying

$$\int_0^1 \varphi_p^{-1}\left(a(h) + \int_0^s h(r) \, dr\right) \, ds = 0. \quad (11)$$

It is known that $G_p : Y \to E$ is continuous and maps equi-integrable sets of $Y$ into relatively compacts of $E$. One may refer to Lee and Sim [15] for details.

Since the bifurcation points of

$$\varphi_p(u'(t))' + \lambda m(t)f(u(t)) = 0 \quad \text{a.e. in } (0, 1), \quad u(0) = u(1) = 0$$

is related to the eigenvalues of the problem

$$\varphi_p(u'(t))' + \lambda m(t)f_p(u(t)) = 0 \quad \text{a.e. in } (0, 1), \quad u(0) = u(1) = 0. \quad (13)$$

We define the operator $T^p_\lambda : E \to E$ by

$$T^p_\lambda(u)(t) = \int_0^t \varphi_p^{-1}\left(a(\lambda m \varphi_p(u(t))) - \int_0^s \lambda m(z) \varphi_p(u(z)) \, dz\right) \, ds \quad (14)$$

$$=: G_p\left(-\lambda m \varphi_p(u)\right)(t).$$

Then $T^p_\lambda : E \to E$ is completely continuous and problem (13) is equivalent to

$$u = T^p_\lambda(u). \quad (15)$$

The following spectrum result plays a fundamental role in our study.

Lemma 2 (see [18, 19]). Let $(H_2)$ hold. Then

(i) the set of all eigenvalues of the problem (13) is two infinite sequences of simple eigenvalues as follows:

$$0 < \mu_1^- (p) < \mu_2^-(p) < \cdots \mu_k^- (p) < \cdots, \quad \lim_{k \to +\infty} \mu_k^- (p) = +\infty,$$

$$0 > \mu_1^+ (p) > \mu_2^+(p) > \cdots > \mu_k^+ (p) > \cdots, \quad \lim_{k \to -\infty} \mu_k^+ (p) = -\infty; \quad (16)$$

(ii) for $k \in \mathbb{N}$ and $v \in \{+,-\}$, Ker $(I - T^p_\mu(p))$ is a space of $E$ with dimensional 1;
(iii) the eigenfunction corresponding to \( \mu_k(p) \) has exactly \( k - 1 \) simple zeros in \((0, 1)\).

**Remark 3.** Using the Gronwall inequality, we can easily show that all zeros of eigenfunction corresponding to eigenvalue \( \mu_k(p) \) are simple.

It is very known that \( T_2^2 \) is completely continuous in \( C^1[0, 1] \). Thus, the Leray-Schauder degree \( d_{LS}(I - T_2^2, B_r(0), 0) \) is well-defined for arbitrary \( r \)-ball \( B_r(0) \) and \( \lambda \neq \mu_k, k \in \mathbb{Z} \) and \( v \in \{+, -\} \).

**Lemma 4.** For \( r > 0 \), we have
\[
d_{LS} \left(I - T_2^2, B_r(0), 0 \right) = (-1)^{m(\lambda)},
\]
where \( m(\lambda) \) is the sum of algebraic multiplicity of the eigenvalues of (13) satisfying \( \mu^{-1} \lambda > 1 \).

If \( \lambda \in (0, \mu_1(2)) \), then there are no such \( \mu \) at all; then
\[
d_{LS} \left(I - T_2^2, B_r(0), 0 \right) = (-1)^{m(\lambda)} = (-1)^{0} = 1.
\]
If \( \lambda \in (\mu_1(2), \mu_{k+2}(2)) \) for some \( k \in \mathbb{N} \), then
\[
(\mu_j(2)) \lambda > 1, \quad j \in \{1, \cdots, k\}.
\]
This together with Lemma 2 (ii) implies the following:
\[
d_{LS} \left(I - T_2^2, B_r(0), 0 \right) = (-1)^{k}.
\]

**Proof.** We divide the proof into two cases.

**Case 1.** \( \lambda \geq 0 \). Since \( T_2^2 \) is compact and linear, by [20, Theorem 8.10] and Lemma 2 (ii) with \( p = 2 \),
\[
d_{LS} \left(I - T_2^2, B_r(0), 0 \right) = (-1)^{m(\lambda)},
\]
where \( m(\lambda) \) is the sum of algebraic multiplicity of the eigenvalues of (13) satisfying \( \mu^{-1} \lambda > 1 \).

If \( \lambda \in (0, \mu_1(2)) \), then there are no such \( \mu \) at all; then
\[
d_{LS} \left(I - T_2^2, B_r(0), 0 \right) = (-1)^{m(\lambda)} = (-1)^{0} = 1.
\]
If \( \lambda \in (\mu_1(2), \mu_{k+2}(2)) \) for some \( k \in \mathbb{N} \), then
\[
(\mu_j(2)) \lambda > 1, \quad j \in \{1, \cdots, k\}.
\]
Thus, we may use the result obtained in Case 1 to deduce the desired result.

**Proposition 5.** The eigenvalue function \( \mu_k^+ \) : \((1, +\infty) \) is continuous.

Proof. We only show that \( \mu_k^+ \) : \((1, +\infty) \) is continuous since the case of \( \mu_k^- \) is similar. In the following proof, we will shorten \( \mu_k^+ \) to \( \mu_k \). From the variational characterization of \( \mu_k(p) \), it follows that
\[
\mu_k(p) = \sup \left\{ \mu > 0 \mid \mu \int_0^1 m(t) |u|^p \, dt \right\},
\]
where \( m(t) = \int_{\mathbb{R}} |\tilde{m}(t)| \, dt \) and Lemma 2 (ii) with \( \mu_k \).

Let \( \{p_j\}_{j=1}^\infty \) be a sequence in \((1, +\infty) \) convergent to \( p > 1 \). We will show that
\[
\lim \sup_{j \to \infty} \mu_k(p_j) = \mu_k(p).
\]
To do this, let \( u \in C^\infty_c(0, 1) \). Then, from (24),
\[
\mu_k(p_j) \int_0^1 m(t) |u|^p \, dt \leq \int_0^1 |u|^p \, dt.
\]
On applying the Dominated Convergence Theorem, we find that
\[
\lim \sup_{j \to \infty} \mu_k(p_j) \int_0^1 m(t) |u|^p \, dt \leq \int_0^1 |u|^p \, dt.
\]
Relation (27), the fact that \( u \) is arbitrary and (24) yield
\[
\lim \sup_{j \to \infty} \mu_k(p_j) \leq \mu_k(p).
\]
Thus, to prove (25), it suffices to show that
\[
\lim \inf_{j \to \infty} \mu_k(p_j) \geq \mu_k(p).
\]
Let \( \{p_k\}_{k=1}^\infty \) be a subsequence of \( \{p_j\}_{j=1}^\infty \) such that
\[
\lim_{k \to \infty} \mu_k(p_k) = \lim \inf_{j \to \infty} \mu_k(p_j).
\]
Let us fix \( \epsilon_0 > 0 \) so that \( p - \epsilon_0 > 1 \) and, for each \( 0 < \epsilon < \epsilon_0 \), \( W_0^{1,p+\epsilon}(0, 1) \) is compactly embedded into \( L^{p+\epsilon}(0, 1) \). For \( k \in \mathbb{N} \), let us choose \( u_k \in W_0^{1,p+\epsilon}(0, 1) \) such that
\[
\int_0^1 |u_k|^p \, dt = 1,
\]
\[
\int_0^1 |u_k|^p \, dt = \mu_k(p_k) \int_0^1 m(t) |u_k|^p \, dt.
\]
For \( 0 < \epsilon < \epsilon_0 \), there exists \( k_0 \in \mathbb{N} \) such that \( p - \epsilon < p_k < p + \epsilon \) for any \( k \geq k_0 \). Thus, for \( k \geq k_0 \), (30) and Hölder’s inequality imply that
\[
\int_0^1 |u_k| |u_k|^\epsilon \, dt \leq 1.
\]
This shows that \( \{u_k\}_{k=k_0}^\infty \) is a bounded sequence in \( W_0^{1,p+\epsilon}(0, 1) \). Passing to a subsequence if necessary, we can
assume that \( u_k \rightharpoonup u \) in \( W^{1,p-\varepsilon}_0(0,1) \) and hence that \( u_k \rightarrow u \) in \( L^{p+\varepsilon}(0,1) \). Furthermore, \( u \in L^p(0,1) \) and \( u_k \rightarrow u \) in \( L^{p_k}(0,1) \) for \( k \geq k_0 \). It follows that

\[
\left[ \int_0^1 |u_k|^{p_k} \, dt - \int_0^1 |u|^{p_k} \, dt \right] \\
\leq \int_0^1 p_k |u + \theta u_k|^{p_k-1} |u_k - u| \, dt \\
\leq (p + \varepsilon) \left( \int_0^1 |u + \theta u_k|^{p_k} \, dt \right)^{(p_k-1)/p_k} \\
\times \left( \int_0^1 |u_k - u|^{p_k} \, dt \right)^{1/p_k} \\
\leq (p + \varepsilon) \left( \|u\|_{p_k} + \|u_k\|_{p_k} \right)^{p_k-1} \left( \int_0^1 |u_k - u|^{p_k} \, dt \right)^{1/p_k} \\
\rightarrow 0
\]

as \( k \rightarrow +\infty \). It is clear that

\[
\int_0^1 |u|^{p_k} \, dt - \int_0^1 |u|^{p} \, dt \rightarrow 0 \quad \text{as} \quad k \rightarrow +\infty.
\]

Thus,

\[
\int_0^1 |u_k|^{p_k} \, dt \rightarrow \int_0^1 |u|^{p} \, dt.
\]

Similarly, we can also obtain that

\[
\int_0^1 m^+(t) |u_k|^{p_k} \, dt \rightarrow \int_0^1 m^+(t) |u|^{p} \, dt,
\]

\[
\int_0^1 m^-(t) |u_k|^{p_k} \, dt \rightarrow \int_0^1 m^-(t) |u|^{p} \, dt,
\]

where \( m^+(t) = \max\{m(t), 0\} \) and \( m^-(t) = -\min\{m(t), 0\} \). Therefore,

\[
\int_0^1 m(t) |u_k|^{p_k} \, dt \\
= \int_0^1 m^+(t) |u_k|^{p_k} \, dt - \int_0^1 m^-(t) |u_k|^{p_k} \, dt \\
\rightarrow \int_0^1 m^+(t) |u|^{p} \, dt - \int_0^1 m^-(t) |u|^{p} \, dt \\
= \int_0^1 m(t) |u|^{p} \, dt.
\]

We note that (30) and (31) imply that

\[
\mu_1(p_k) \int_0^1 m(t) |u_k|^{p_k} \, dt = 1
\]

for all \( k \in \mathbb{N} \). Thus, letting \( k \rightarrow +\infty \) in (38) and using (37), we find that

\[
\liminf_{j \rightarrow +\infty} \mu_1(p_k) \int_0^1 m(t) |u|^{p} \, dt = 1.
\]

On the other hand, since \( u_k \rightharpoonup u \) in \( W^{1,p-\varepsilon}_0(0,1) \), from (32) we obtain that

\[
\|u''\|^{p-\varepsilon}_{p-\varepsilon} \leq \liminf_{k \rightarrow +\infty} \|u_k''\|^{p-\varepsilon}_{p-\varepsilon} \leq \frac{1}{e} p.
\]

Now, letting \( \varepsilon \rightarrow 0^+ \) and applying Fatou’s Lemma, we find that

\[
\|u''\|_{p}^{p-1} \leq 1.
\]

Hence, \( u \in W^{1,p}(0,1) \); here \( W^{1,p}(0,1) \) denotes the radially symmetric subspace of \( W^{1,p}(0,1) \). We claim that actually \( u \in W^{1,p}_0(0,1) \). Indeed, we know that \( u \in W^{1,p-\varepsilon}_0(0,1) \) for each \( \varepsilon < e \). For \( \phi \in C_c(\mathbb{R}) \), it is easy to see that

\[
\int_0^1 |u\phi'\|^p \, dt \leq \|u''\|_{p-\varepsilon} \|\phi\|_{p-\varepsilon}, \quad i = 1, \ldots, N.
\]

Then, letting \( \varepsilon \rightarrow 0^+ \), we obtain that

\[
\int_0^1 |u\phi'\|^p \, dt \leq \|u''\|_{p} \|\phi\|_p, \quad i = 1, \ldots, N,
\]

where \( p' = p/(p - 1) \). Since \( \phi \) is arbitrary, from Proposition IX-18 of [21], we find that \( u \in W^{1,p}_0(0,1) \), as desired.

Finally, combining (39) and (41), we obtain that

\[
\liminf_{j \rightarrow +\infty} \mu_1(p_k) \int_0^1 m(t) |u|^{p} \, dt \geq \int_0^1 |u|^{p} \, dt.
\]

This and the variational characterization of \( \mu_1(p) \) imply (29) and hence (25). This concludes the proof of the lemma.

Using Remark 3, Lemma 2, and Proposition 5, we will show that all eigenvalue functions \( \mu^+_k : (1, +\infty) \rightarrow \mathbb{R} \), \( 2 \leq k \in \mathbb{N} \) are continuous.

**Lemma 6.** For fixed \( 2 \leq k \in \mathbb{N} \) and \( \nu \in \{+, -\} \), \( \mu^\nu_k(p) \) as a function of \( p \in (1, +\infty) \) is continuous.

**Proof.** Let \( u^\nu_k \) be an eigenfunction corresponding to \( \mu^\nu_k(p) \). By Lemma 2 and Remark 3, we know that \( u \) has exactly \( k - 1 \) simple zeros in \( I \); that is, there exist \( c_{k,1}, \ldots, c_{k,k-1} \in I \) such that \( u(c_{k,i}) = \cdots = u(c_{k,k-1}) = 0 \). For convenience, we set \( c_{k,0} = 0, c_{k,k} = 1, \) and \( I_i = (c_{k,i-1}, c_{k,i}) \) for \( i = 1, \ldots, k \). Let \( \mu^\nu_i(p, m_i, f_i) \) denote the first positive or negative eigenvalue of the restriction of the problem (13) on \( I_i \) for \( i = 1, \ldots, k \). Lemma 3 of [18] follows that \( \mu^\nu_i(p) = \mu^\nu_i(p, m_i, f_i) \) for \( i = 1, \ldots, k \). Using a similar proof to Proposition 5, we can show that \( \mu^\nu_i(p, m_i, f_i) \) is continuous with respect to \( p \) for \( i = 1, \ldots, k \). Therefore, \( \mu^\nu_k(p) \) is also continuous with respect to \( p \).
Lemma 7. (i) Let \( \{\mu_k^p(p)\}_{k \in \mathbb{N}} \) be the sequence of positive eigenvalues of (13). Let \( \lambda \) be a constant with \( \lambda \neq \mu_k^p(p) \) for all \( k \in \mathbb{N} \). Then, for arbitrary \( r > 0 \),

\[
\deg \left( T_{\lambda}^r, B_2(0), 0 \right) = (-1)^\beta, 
\]

where \( \beta \) is the number of eigenvalues \( \mu_k^p(p) \) of problem (13) less than \( \lambda \).

(ii) Let \( \{\mu_k(p)\}_{k \in \mathbb{N}} \) be the sequence of negative eigenvalues of (13). Consider \( \lambda \neq \mu_k(p) \), \( k \in \mathbb{N} \); then

\[
\deg \left( T_{\lambda}^r, B_2(0), 0 \right) = (-1)^\beta, \quad \forall r > 0, 
\]

where \( \beta \) is the number of eigenvalues \( \mu_k^p(p) \) of problem (25) larger than \( \lambda \).

Proof. We will only prove the case \( \lambda > \mu_1^p(p) \) since the proof for the other cases is similar. Also we only give the proof for the case \( p > 2 \). Proof for the case \( 1 < p < 2 \) is similar. Assume that \( \mu_1^p(p) < \lambda < \mu_{k+1}^p(p) \) for some \( k \in \mathbb{N} \). Since the eigenvalues depend continuously on \( p \), there exists a continuous function \( \chi : [2, p] \to \mathbb{R} \) and \( q \in [2, p] \) such that \( \mu_k^q(q) < \chi(q) < \mu_{k+1}^q(q) \) and \( \lambda = \chi(p) \). Define

\[
\Phi(q, u) = u - G_q(-\chi(q) m(t) \varphi_q(u)).
\]

It is easy to show that \( \Phi(q, u) \) is a compact perturbation of the identity such that, for all \( u \neq 0 \), by definition of \( \chi(q) \), \( \Phi(q, u) \neq 0 \), for all \( q \in [2, p] \). Hence, the invariance of the degree under homotopy and the classical result for \( p = 2 \) imply

\[
\deg \left( T_{\lambda}^r, B_2(0), 0 \right) = \deg \left( T_{\lambda}^r, B_2(0), 0 \right) = (-1)^\beta. 
\]

\( \square \)

For the existence of bifurcation branches for (12), we will make use of the following global bifurcation theorem results.

Lemma 8 (see [17]). Let \( X \) be a Banach space. Let \( F : \mathbb{R} \times X \to X \) be completely continuous such that \( F(\lambda, 0) = 0 \) for all \( \lambda \in \mathbb{R} \). Suppose that there exist constants \( \rho, \eta \in \mathbb{R} \), with \( \rho < \eta \), such that \( \rho, 0 \) and \( \eta, 0 \) are not bifurcation points for the equation

\[
u - F(\lambda, u) = 0.
\]

Furthermore, assume that

\[
\deg \left( I - F(\rho, \cdot), B_r(0), 0 \right) \neq \deg \left( I - F(\eta, \cdot), B_r(0), 0 \right),
\]

where \( B_r(0) = \{ u \in X : \| u \| < r \} \) is an isolating neighborhood of the trivial solution for both constants \( \rho \) and \( \eta \). Let

\[
\mathcal{S} = \{ (\lambda, u) : (\lambda, u) \text{ is a solution of (49) with } u \neq 0 \}
\]

\[\cup \{ (\rho, \eta) \times \{ 0 \} \}, \]

and let \( \mathcal{C} \) be the component of \( \mathcal{S} \) containing \( \{ \rho, \eta \} \times \{ 0 \} \). Then, either

(i) \( \mathcal{C} \) is unbounded in \( \mathbb{R} \times X \) or

(ii) \( \mathcal{C} \cap [(\mathbb{R} \setminus [\rho, \eta]) \times \{ 0 \}] \neq \emptyset \).

Define the Nemytskii operators \( \mathcal{H} : \mathbb{R} \times E \to Y \) by

\[
\mathcal{H} (\lambda, u)(t) := -\lambda m(t) f (u(t)).
\]

Then, it is clear that \( \mathcal{H} \) is continuous operator which sends bounded sets of \( \mathbb{R} \times E \) into an equi-integrable sets of \( Y \) and problem (12) can be equivalently written as

\[
u = G_p \circ \mathcal{H} (\lambda, u) := F(\lambda, u).
\]

\( F \) is completely continuous in \( \mathbb{R} \times E \to E \) and \( F(\lambda, 0) = 0 \), for all \( \lambda \in \mathbb{R} \).

Notice that (12) with \( \lambda = 0 \) has only the trivial solution. Applying this fact and Lemma 8 and the same method to prove [15, Theorem 2.1] with obvious changes, we may obtain the following.

Lemma 9. Assume that \( (H_2), (H_3), \) and \( (H_4) \) hold. Then, for fixed \( p > 1 \) and for fixed \( \sigma \in (+, -) \), each \( (\mu_k^p(p)/f_0, 0) \) is a bifurcation point of (12) and the associated bifurcation branch \((\mathcal{C}_k^p)^\sigma\) satisfies the following:

1. \((\mathcal{C}_k^p)^\sigma\) is unbounded in \( \mathbb{R} \);

2. \((\mathcal{C}_k^p)^\sigma\) \( \cup \left( [\mu_k^0(0), 0] \right) \), where \( \Phi_k^p \) is the set of function \( u \in C_{\sigma}[0, 1] \) which has exact \( k \) simple zeros in \( (0, 1) \) and \( \sigma u \) is positive near \( 0 \).

Finally, we give a key lemma that will be used in Section 3. Let

\[
I^+ := \{ t \in [0, 1] \mid m(t) > 0 \}, \\
I^- := \{ t \in [0, 1] \mid m(t) < 0 \}. 
\]

Lemma 10. Let \( (H_2) \) hold. Let \( I = [a, b] \) be such that \( I \subset I_+ \) and

\[
\text{meas } I > 0.
\]

Let \( g_n : [0, 1] \to (0, +\infty) \) be such that

\[
\lim_{n \to +\infty} g_n (t) = +\infty, \quad \text{uniformly on } I.
\]

Let \( y_n \in E \) be a solution of the equation

\[
q_p \left( y_n' \right) + m(t) g_n(t) q_p(y_n) = 0, \quad t \in (0, 1). 
\]

Then, the number of zeros of \( y_n |_{\partial I} \) goes to infinity as \( n \to +\infty \).

Proof. After taking a subsequence if necessary, we may assume that

\[
m(t) g_n(t) \geq j, \quad t \in I,
\]

as \( j \to +\infty \). It is easy to check that the distance between any two consecutive zeros of any nontrivial solution of the equation

\[
q_p \left( y' \right) + j q_p(u) = 0, \quad t \in I,
\]

goes to zero as \( j \to +\infty \). Using this with [21, Lemma 2.5], it follows the desired results. \( \square \)
3. Main Results and Its Proof

Let $\mu_k^\pm$ be the kth positive or negative eigenvalue of (13). By applying Lemma 9, we will establish the main results as follows.

Theorem 11. Let $(H_1)$, $(H_2)$, $(H_3)$, and $(H_4)$ hold. Assume that, for some $k \in \mathbb{N}$, either

$$\gamma \in \left( \frac{\mu^+_k(p)}{f_{co}}, \frac{\mu^-_k(p)}{f_0} \right) \cup \left( \frac{\mu^-_k(p)}{f_0}, \frac{\mu^+_k(p)}{f_{co}} \right)$$

or

$$\gamma \in \left( \frac{\mu^+_k(p)}{f_{co}}, \frac{\mu^-_k(p)}{f_0} \right) \cup \left( \frac{\mu^-_k(p)}{f_0}, \frac{\mu^+_k(p)}{f_{co}} \right).$$

Then, (4) has two solutions $u_k^+$ and $u_k^-$ such that $u_k^+$ has exactly $k - 1$ zeros in $(0, 1)$ and is positive near 0 and $u_k^-$ has exactly $k - 1$ zeros in $(0, 1)$ and is negative near 0.

Proof. We only prove the case of $\gamma > 0$. The case of $\gamma < 0$ is similar. Consider the problem

$$\varphi_p(u') + \lambda m(t) f(u) = 0, \quad t \in (0, 1),$$

$$u(0) = 0, \quad u(1) = 0. \quad (62)$$

Considering the results of Lemma 9, we have that, for each integer $k \geq 1$, $\sigma \in \{1, -1\}$, there exists a continuum $(C_k^\sigma)^\sigma \subseteq \Phi_k^\sigma$ of solutions of (62) joining $(\mu_k^\sigma(p)/\gamma f_0, 0)$ to infinity in $(0, \infty) \times \Phi_k^\sigma$. Moreover, $(C_k^\sigma)^\sigma \setminus \{\mu_k^\sigma(p)/\gamma f_0, 0\} \subset (0, \infty) \times \Phi_k^\sigma$.

It is clear that any solution of (62) of the form $(1, u)$ yields a solution $u$ of (4). We will show that $(C_k^\sigma)^\sigma$ crosses the hyperplane $\{(1, u) : (\mu_k^\sigma(p)/\gamma f_0, 0) \in (0, \infty) \times \Phi_k^\sigma\}$. To this end, it will be enough to show that $(C_k^\sigma)^\sigma$ joins $(\mu_k^\sigma(p)/\gamma f_0, 0)$ to $(\mu_k^\sigma(p)/\gamma f_{co}, +\infty)$. Let $\eta_n, \gamma_n \in (C_k^\sigma)^\sigma$ satisfy

$$\mu_n + \|\gamma_n\| \to +\infty. \quad (63)$$

We note that $\eta_n > 0$ for all $n \in \mathbb{N}$ since $(0, 0)$ is the only solution of (62) for $\lambda = 0$ and $(C_k^\sigma)^\sigma \cap \{(0, 0) \times E\} = \emptyset$.

Case 1. $\mu_k^\sigma(p)/\gamma_{co} < \gamma < \mu_k^\sigma(p)/\gamma_0$. In this case, we only need to show that

$$\left( \frac{\mu^+_k(p)}{\gamma f_{co}}, \frac{\mu^+_k(p)}{\gamma f_0} \right) \setminus \left\{ \mu \in \mathbb{R} : (\mu, u) \in (C_k^\sigma)^\sigma \right\}. \quad (64)$$

We divide the proof into two steps.

Step 1. We show that, if there exists a constant number $M > 0$ such that

$$\eta_n \in (0, M] \quad (65)$$

for $n \in \mathbb{N}$ large enough, then $(C_k^\sigma)^\sigma$ joins $(\mu_k^\sigma(p)/\gamma f_0, 0)$ to $(\mu_k^\sigma(p)/\gamma f_{co}, +\infty)$. In this case, it follows that

$$\|\gamma_n\| \to +\infty. \quad (66)$$

Let $\xi \in C(\mathbb{R})$ be such that

$$f(u) = f_{co} \varphi_p(u) + \xi(u). \quad (67)$$

Then,

$$\lim_{|u| \to +\infty} \varphi_p(u) = 0. \quad (68)$$

Let

$$\tilde{\xi}(u) = \max_{0 \leq s \leq 1} |\xi(s)|. \quad (69)$$

Then, $\tilde{\xi}$ is nondecreasing and

$$\lim_{|u| \to +\infty} \frac{\tilde{\xi}(u)}{|u|^{p-1}} = 0. \quad (70)$$

We divide the equation

$$\varphi_p(y_n') - \mu_n \gamma m(t) f(y_n) = \mu_n \gamma m(t) \xi(y_n) \quad (71)$$

by $\|\gamma_n\|$ and set $\tilde{y}_n = y_n/\|\gamma_n\|$. Since $\tilde{y}_n$ is bounded in $E$, after taking a subsequence if necessary, we have $\tilde{y}_n \to \tilde{y}$ in $E$ and $\tilde{y}_n \to \tilde{y}$ in $Y$ with $\|\tilde{y}\| = 1$. Moreover, from (70) and the fact that $\tilde{\xi}$ is nondecreasing, we have

$$\lim_{n \to +\infty} \frac{\tilde{\xi}(y_n(t))}{\|y_n\|^{p-1}} = 0. \quad (72)$$

By the continuity and compactness of $G_p$, it follows that

$$\tilde{y} = G_p \left( \tilde{\mu} \gamma m(t) f_{co} \varphi_p(\tilde{y}) \right), \quad (73)$$

where $\tilde{\mu} = \lim_{n \to +\infty} \mu_n$, again choosing a subsequence and relabeling if necessary.

We claim that

$$\tilde{y} \in (C_k^\sigma)^\sigma. \quad (75)$$

Suppose on the contrary that $\tilde{y} \not\in (C_k^\sigma)^\sigma$. Since $\tilde{y} \not\equiv 0$ is a solution of (74) and all zeros of $\tilde{y}$ in $[0, 1]$ are simple, it follows that $\tilde{y} \in (C_k^\sigma)^\sigma \setminus (C_k^\sigma)^\sigma$ for some $h \in \mathbb{N}$ and $\sigma \in \{1, -1\}$.

By the openness of $E \setminus (C_k^\sigma)^\sigma$, we have that there exists a neighborhood $U(\tilde{y}, \rho_0)$ such that

$$U(\tilde{y}, \rho_0) \subset E \setminus (C_k^\sigma)^\sigma, \quad (76)$$

which contradicts the facts that $\tilde{y} \to \tilde{y}$ in $E$ and $\tilde{y}_n \in (C_k^\sigma)^\sigma$. Therefore, $\tilde{y} \in C_k^\sigma$. Moreover, by Lemma 2, $\tilde{\mu} \gamma f_{co} = \mu_k^{\sigma}(p)$, so that

$$\tilde{\mu} = \frac{\lambda_k}{\gamma f_{co}}. \quad (77)$$

Therefore, $(C_k^\sigma)^\sigma$ joins $(\mu_k^\sigma(p)/\gamma f_0, 0)$ to $(\mu_k^\sigma(p)/\gamma f_{co}, +\infty)$. Abstract and Applied Analysis
Abstract and Applied Analysis

Step 2. We show that there exists a constant $M$ such that $\mu_n \in (0, M]$ for $n \in \mathbb{N}$ large enough.

On the contrary, we suppose that

$$\lim_{n \to +\infty} \mu_n = +\infty. \quad (78)$$

Since $(\eta_n, y_n) \in (C^1_{\kappa})^r$, it follows that

$$\varphi(y'_n) + y_n \eta_n m(t) f(y_n) \varphi(y_n) = 0. \quad (79)$$

Let

$$0 = \tau (0, n) < \tau (1, n) < \cdots < \tau (k, n) = 1 \quad (80)$$

be the zeros of $y_n$ in $[0, 1]$. Then, after taking a subsequence if necessary,

$$\lim_{n \to +\infty} \tau (l, n) := \tau (l, \infty), \quad l \in [0, 1, \cdots, k-1]. \quad (81)$$

Notice that Lemma 10 and the fact that $y_n$ has exactly $k-1$ simple zeros in $[0, 1]$ yield

$$\left\{ \cup_{l=0}^{k-1} \left( \tau (l, \infty), \tau (l + 1, \infty) \right) \right\} \cap I^n = \emptyset, \quad (82)$$

which implies that

$$\text{meas} \left\{ \left\{ \cup_{l=0}^{k-1} \left( \tau (l, \infty), \tau (l + 1, \infty) \right) \right\} \cap I^n \right\} = 1. \quad (83)$$

However, this contradicts $(H_2); 0 < \text{meas} I^n < 1$.

Case 2. $\mu_k(p)/f_0 < y < \mu_k(p)/f_{\infty}$. In this case, we have that

$$\frac{\mu_k(p)}{y_0} < 1 < \frac{\mu_k(p)}{y_{\infty}}. \quad (84)$$

Assume that $(\eta_n, y_n) \in (C^1_{\kappa})^r$ is such that

$$\lim_{n \to +\infty} (\mu_n + \|y_n\|) = +\infty. \quad (85)$$

If $\eta_n \to +\infty$, then we are done!

If there exists $M > 0$, such that, for $n \in \mathbb{N}$ sufficiently large,

$$\eta_n \in (0, M]. \quad (86)$$

Applying the same method used in Step 1 of Case 1, after taking a subsequence and relabeling if necessary, it follows that

$$(\eta_n, y_n) \longrightarrow \left( \frac{\mu_k(p)}{y_{\infty}}, +\infty \right) \quad \text{as } n \longrightarrow +\infty. \quad (87)$$

Thus, $(C^1_{\kappa})^r$ joins $(\mu_k(p)/y_0, 0)$ to $(\mu_k(p)/y_{\infty}, +\infty)$. \hfill \Box

Acknowledgments

This paper was supported by the NSFC (nos. 11061030, 11361047, and 11201378), SRDFP (no. 20126203100004), and Gansu Provincial National Science Foundation of China (no. 1208RJZA258).

References


[12] M. del Pino, M. Elgueta, and R. Manásevich, "A homotopic deformation along p of a Leray-Schauder degree result and existence for \((\mu)^{1/p} u' + f(t, u) = 0, u(0) = u(\chi(T)) = 0, p > 1\)," Journal of Differential Equations, vol. 80, no. 1, pp. 1–13, 1989.


