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Research Article

Improved (G'/G)-Expansion Method for the Space and Time Fractional Foam Drainage and KdV Equations

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The fractional complex transformation is used to transform nonlinear partial differential equations to nonlinear ordinary differential equations. The improved (G'/G)-expansion method is suggested to solve the space and time fractional foam drainage and KdV equations. The obtained results show that the presented method is effective and appropriate for solving nonlinear fractional differential equations.

1. Introduction

The soliton solutions of nonlinear evolution equations have made a major impact in the flesh. These solitons appear in various areas of physical and biological sciences. They show up in nonlinear optics, plasma physics, fluid dynamics, biochemistry, and mathematical chemistry. Fractional partial differential equations (FPDEs) have received considerable interest in recent years and have been extensively investigated. These equations were applied for many real problems which are modeled in various areas, for example, in mathematical physics [1], fluid and continuum mechanics [2], viscoplastic and viscoelastic flow [3], biology, chemistry, acoustics, and psychology [4, 5]. Some FPDEs do not have exact solutions, so approximation and numerical techniques must be used. There are several approximation and numerical methods. The most commonly used ones are the homotopy perturbation method [6, 7], the Adomian decomposition method [8, 9], the variational iteration method [10–12], the homotopy analysis method [13, 14], the generalized differential transform method [15], the finite difference method [16], and the finite element method [17]. In recent years, some authors have got exact solutions of FPDEs by using analytical methods. S. Zhang and H.-Q. Zhang [18] proposed to solve the nonlinear time fractional biological population model and (4 + 1)-dimensional space-time fractional Fokas equation by using the fractional subequation method. Guo et al. [19] presented the improved subequation method to solve the space-time fractional Whitham-Broer-Kaup and the generalized Hirota-Satsuma coupled KdV equations. Tang et al. [20] used the generalized fractional subequation method to obtain exact solutions of the space-time fractional Gardner equation with variable coefficients. Lu [21] investigated the exact solutions of the nonlinear fractional Klein-Gordon equation, the generalized time fractional Hirota-Satsuma coupled KdV system, and the nonlinear fractional Sharma-Tasso-Olver equation. Bin [22] solved the time-space fractional generalized Hirota-Satsuma coupled KdV equations and the time fractional fifth-order Sawada-Kotera equation by using the (G'/G)-expansion method. Omran and Gepreel [23] used the improved (G'/G)-expansion method to calculate the exact solutions to the time-space fractional foam drainage and KdV equations. In this paper, we will apply the improved (G'/G)-expansion method to obtain the exact solutions for the time-space fractional foam drainage and KdV equations with the modified Riemann-Liouville derivative defined by Jumarie [24–27]:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{2} u \frac{\partial^{2\beta} u}{\partial x^{2\beta}} + 2u^2 \frac{\partial^{\beta} u}{\partial x^{\beta}} + \left(\frac{\partial^{\beta} u}{\partial x^{\beta}}\right)^2, \quad t > 0,\ \alpha > 0,\ \beta \le 1, \tag{1}$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \alpha u \frac{\partial^\beta u}{\partial x^\beta} + \frac{\partial^{3\beta} u}{\partial x^{3\beta}} = 0, \quad t > 0,\ \alpha > 0,\ \beta \le 1, \tag{2}$$

where α is arbitrary constant. This paper is organized as follows. In Section 2, we introduce some basic definitions of Jumarie's modified Riemann-Liouville derivative. In Section 3, the main steps of the improved (G'/G)-expansion method are given. In Section 4, we construct the exact solutions of (1) and (2) by the proposed method. Some conclusions are given in Section 5.

2. Preliminaries

There are several definitions for fractional differential equations. These definitions include Riemann-Liouville, Weyl, Grünwald-Letnikov, Riesz, and Jumarie fractional derivatives. The Riemann-Liouville fractional derivative of a constant is not zero. So the fractional derivative is only defined for differentiable function. In order to deal with nondifferentiable functions, Jumarie [24–27] presented a modification of the Riemann-Liouville definition which appears to provide a framework for a fractional calculus. This modification was successfully applied in the probability calculus, fractional Laplace problem, exact solutions of the nonlinear fractional differential equations, and many other types of linear and nonlinear fractional differential equations [28–30].

Definition 1. The Riemann-Liouville fractional integral is defined [31] as

$$\begin{aligned} {}_0I_x^\alpha f(x) &= I^\alpha f(x) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^x f(\xi)(x-\xi)^{\alpha-1} d\xi, \quad \alpha > 0, \\ I^0 f(x) &= f(x). \end{aligned} \tag{3}$$

Definition 2. Jumarie [24–27] defined the fractional derivative in the limit form by

$$f^{(\alpha)}(x) = \lim_{h \to 0} \frac{\Delta^\alpha [f(x) - f(0)]}{h^\alpha}, \tag{4}$$

where $f(x)$ should be a continuous (but not necessarily differentiable) function and $h > 0$ denotes a constant discretization span. So, the modified form of the Riemann-Liouville derivative is defined as

$${}_0D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-\xi)^{(n-\alpha)} [f(\xi) - f(0)], \tag{5}$$

where $x \in [0,1]$, $n-1 \le \alpha < n$ and $n \ge 1$.

Lemma 3. *The integral with respect to $(dx)^\alpha$ is defined by Jumarie [24, 25] as follows:*

$$\begin{aligned} &\int_0^x f(\xi)(d\xi)^\alpha = \alpha \int_0^x (x-\xi) f(\xi)\, d\xi, \quad 0 < \alpha \le 1, \\ &\frac{d^\alpha}{dx^\alpha} \int_0^{u(x)} f(\xi)(d\xi)^\alpha = \Gamma(\alpha+1) f[u(\xi)] [u'(\xi)]^\alpha, \\ &\qquad 0 < \alpha \le 1. \end{aligned} \tag{6}$$

Theorem 4. *Assume that the continuous function $f(x)$ has a fractional derivative of order α; then*

$$\begin{aligned} &\frac{d^\alpha}{dx^\alpha} I^\alpha f(x) = f(x), \\ &I^\alpha \frac{d^\alpha}{dx^\alpha} f(x) = f(x) - f(0), \quad 0 < \alpha \le 1, \end{aligned} \tag{7}$$

hold.

3. Description of the Improved (G'/G)-Expansion Method

In this section, we give the description of the improved (G'/G)-expansion method for solving the nonlinear FPDEs as

$$\begin{aligned} &F\left(u, D_t^\alpha u, D_x^\beta u, D_y^\gamma u, D_z^\delta u, D_t^\alpha D_t^\alpha u, \right. \\ &\left. D_t^\alpha D_x^\beta u, D_x^\beta D_x^\beta u, \ldots\right) = 0, \\ &0 < \alpha, \beta, \gamma, \delta \le 1, \end{aligned} \tag{8}$$

where u is an unknown function and F is a polynomial of u and its partial fractional derivatives, in which the highest order derivatives and nonlinear terms are involved. We offer an improved (G'/G)-expansion method [32]. The essential steps of this method are described as follows.

Step 1. Li and He [33] and He and Li [34] presented a fractional complex transform to transform fractional differential equations into ordinary differential equations. So, all analytical methods devoted to advanced calculus can be easily dedicated to fractional calculus. The traveling wave variable is given as

$$\begin{aligned} &u(x,y,z,t) = u(\xi), \\ &\xi = \frac{Kx^\beta}{\Gamma(\beta+1)} + \frac{Ny^\gamma}{\Gamma(\gamma+1)} + \frac{Lt^\alpha}{\Gamma(\alpha+1)}, \end{aligned} \tag{9}$$

where K, N, and L are nonzero arbitrary constants. So, (9) is reduced to (10):

$$p\left(u, u', u'', u''', \ldots\right) = 0, \tag{10}$$

where $u = u(\xi)$.

Step 2. Suppose that (10) has the solution (11):

$$u(\xi) = \sum_{i=0}^n a_i F^i(\xi), \tag{11}$$

where a_i $(i = 0, 1, \ldots, n)$ are real constants to be determined, the balancing number n is a positive integer which can be determined by balancing the highest derivative terms with the highest power nonlinear terms in (10). More precisely, we define the degree of $u(\xi)$ as $D[u(\xi)] = m$, which gives rise to the degrees of other expressions, as follows:

$$\begin{aligned} &D\left[\frac{d^q u}{d\xi^q}\right] = m + q, \\ &D\left[u^p \left(\frac{d^q u}{d\xi^q}\right)^s\right] = mp + s(q+m). \end{aligned} \tag{12}$$

Therefore, we can obtain the value of m in (11).

Step 3. $F(\xi)$ is

$$F(\xi) = \frac{G'(\xi)}{G(\xi)}, \tag{13}$$

where $G(\xi)$ expresses the solution of the following auxiliary ordinary differential equation

$$G(\xi)G''(\xi) = AG^2(\xi) + BG(\xi)G'(\xi) + C[G(\xi)]^2, \tag{14}$$

where the prime denotes derivative with respect to ξ. A, B, and C are real parameters.

Step 4. Substituting (13) into (10), using (14), collecting all terms with the same order of $(G'(\xi)/G(\xi))$ together, and then equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for a_i $(i = 0, 1, \ldots, n)$, A, B, C, K, N, and L.

Step 5. Using the general solutions of (14), with the aid of Mathematica, we have the following four solutions of (13).

Case 1. If $B \neq 0$ and $\Delta = B^2 + 4A - 4AC \geq 0$, then

$$F(\xi) = \frac{B}{2(1-C)} + \frac{B\sqrt{\Delta}}{2(1-C)} \times \frac{c_1 \exp\left(\left(\sqrt{\Delta}/2\right)\xi\right) + c_2 \exp\left(\left(-\sqrt{\Delta}/2\right)\xi\right)}{c_1 \exp\left(\left(\sqrt{\Delta}/2\right)\xi\right) - c_2 \exp\left(\left(-\sqrt{\Delta}/2\right)\xi\right)}. \tag{15}$$

Case 2. If $B \neq 0$ and $\Delta = B^2 + 4A - 4AC < 0$, then

$$F(\xi) = \frac{B}{2(1-C)} + \frac{B\sqrt{-\Delta}}{2(1-C)} \times \frac{ic_1 \cos\left(\left(\sqrt{-\Delta}/2\right)\xi\right) - c_2 \sin\left(\left(\sqrt{-\Delta}/2\right)\xi\right)}{ic_1 \sin\left(\left(\sqrt{-\Delta}/2\right)\xi\right) + c_2 \cos\left(\left(\sqrt{-\Delta}/2\right)\xi\right)}. \tag{16}$$

Case 3. If $B = 0$ and $\Delta = A(C-1) \geq 0$, then

$$F(\xi) = \frac{\sqrt{\Delta}}{(1-C)} \frac{c_1 \cos\left(\sqrt{\Delta}\xi\right) + c_2 \sin\left(\sqrt{\Delta}\xi\right)}{c_1 \sin\left(\sqrt{\Delta}\xi\right) - c_2 \cos\left(\sqrt{\Delta}\xi\right)}. \tag{17}$$

Case 4. If $B = 0$ and $\Delta = A(C-1) < 0$, then

$$F(\xi) = \frac{\sqrt{-\Delta}}{(1-C)} \frac{ic_1 \cosh\left(\sqrt{-\Delta}\xi\right) - c_2 \sinh\left(\sqrt{-\Delta}\xi\right)}{ic_1 \sinh\left(\sqrt{-\Delta}\xi\right) - c_2 \cosh\left(\sqrt{-\Delta}\xi\right)}, \tag{18}$$

where

$$\xi = \frac{Kx^\beta}{\Gamma(\beta+1)} + \frac{Ny^\gamma}{\Gamma(\gamma+1)} + \frac{Lt^\alpha}{\Gamma(\alpha+1)} \tag{19}$$

and A, B, C, c_1, and c_2 are real parameters.

4. Applications

We use the improved (G'/G)-expansion method on the time-space fractional nonlinear foam drainage equation and the time-space fractional nonlinear KdV equation in this section.

4.1. The Time and Space-Fractional Nonlinear Foam Drainage Equation. We apply the improved (G'/G)-expansion method to construct the exact solutions for the time-space fractional nonlinear foam drainage equation in this subsection. Foams are of great importance in many technological processes and applications. Their properties are subject of intensive studies from practical and scientific points of view [27, 35–37]. Liquid foam is an example of soft matter with a very well-defined structure, described by Joseph Plateau in the 19th century. Foams are common in foods and personal care products such as lotions and creams. They have important applications in food and chemical industries, mineral processing, fire fighting, and structural material sciences [27, 35–37]. This equation is numerically and analytically taken into account by different authors [38–40]. The space-time fractional nonlinear foam drainage equation is solved analytically only by Omran and Gepreel [23]. We can see the fractional complex transform as

$$u(x,t) = u(\xi), \qquad \xi = \frac{Kx^\beta}{\Gamma(\beta+1)} + \frac{Lt^\alpha}{\Gamma(\alpha+1)}, \tag{20}$$

where K and L are constants. So, (20) reduces to (21):

$$-Lu' + \frac{1}{2}K^2uu'' + 2Ku^2u' + K^2\left(u'\right)^2 = 0. \tag{21}$$

Balancing the highest order nonlinear term and the highest order linear term, we get $n = 1$. Thus, we obtain

$$u(\xi) = a_0 + a_1F(\xi), \qquad F(\xi) = \frac{G'(\xi)}{G(\xi)}, \tag{22}$$

where a_0 and a_1 will be determined constants. Substituting (22) into (21), using (14), collecting all the terms of powers of (G'/G), and setting each coefficient to zero, we have the following system of algebraic equations:

$$\left(\frac{G'}{G}\right)^0 : 2Aa_0^2a_1K + A^2a_1^2K^2 + \frac{1}{2}Aa_0a_1BK^2 - Aa_1L = 0,$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^1 :& 2a_0^2a_1BK - Aa_0a_1K^2 + \frac{5}{2}Aa_1^2BK^2 \\ &+ \frac{1}{2}a_0a_1B^2K^2 + Aa_0a_1CK^2 - a_1BL = 0,\end{aligned}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^2 :& -2a_0^2a_1K + 2Aa_1^3K + 4a_0a_1^2BK \\ &+ 2a_0^2a_1CK - 3Aa_1^2K^2 + \frac{3}{2}a_0a_1BK^2 \\ &+ \frac{3}{2}a_1^2B^2K^2 + 3Aa_1^2CK^2 + \frac{3}{2}a_0a_1BCK^2 \\ &+ \frac{3}{2}a_1^2B^2K^2 + 3Aa_1^2CK^2 + \frac{3}{2}a_0a_1BCK^2 \\ &+ a_1L - a_1CL = 0,\end{aligned}$$

$$\left(\frac{G'}{G}\right)^3 : -4a_0a_1^2K + 2a_1^3BK + 4a_0a_1^2CK + a_0a_1K^2 - \frac{7}{2}a_1^2BK^2 + 2a_0a_1CK^2 + \frac{7}{2}a_1^2BCK^2 + a_0a_1C^2K^2 = 0,$$

$$\left(\frac{G'}{G}\right)^4 : -2a_1^3K + 2a_1^3CK + 2a_1^2K^2 - 4a_1^2CK^2 + 2a_1^2C^2K^2 = 0. \tag{23}$$

Solving the set of the above algebraic equations, we get the following result:

$$a_0 = \frac{8ABK(C-1)}{B^2+6A-6AC}, \qquad a_1 = K(C-1),$$
$$L = \frac{1}{2}\left(4a_0^2K + a_0BK^2 + 2AK^3 - 2ACK^3\right), \tag{24}$$
$$KB(C-1) \neq 0.$$

Substituting this value in (22) and by Cases 1–4, we obtain the following exponential, hyperbolic and triangular function solutions of (1).

(1) If we choose $B \neq 0$ and $\Delta = B^2 + 4A - 4AC \geq 0$, then the exponential function solutions can be found as

$$u(x,t) = \frac{16ABK(C-1) - KB[\Delta + 2A(C-1)]}{2\Delta + 2A(C-1)} - \frac{KB\sqrt{\Delta}}{2} \times \frac{c_1\exp\left(\left(\sqrt{\Delta}/2\right)\xi\right) + c_2\exp\left(\left(-\sqrt{\Delta}/2\right)\xi\right)}{c_1\exp\left(\left(\sqrt{\Delta}/2\right)\xi\right) - c_2\exp\left(\left(-\sqrt{\Delta}/2\right)\xi\right)}, \tag{25}$$

where

$$\xi = \frac{Kx^\beta}{\Gamma(\beta+1)} + \frac{1}{2}\left(4a_0^2K + a_0BK^2 + 2AK^3 - 2ACK^3\right) \times \frac{t^\alpha}{\Gamma(\alpha+1)}. \tag{26}$$

(2) If we choose $B \neq 0$ and $\Delta = B^2 + 4A - 4AC < 0$, then the triangular function solution will be

$$u(x,t) = \frac{16ABK(C-1) - KB[\Delta + 2A(C-1)]}{2\Delta + 2A(C-1)} - \frac{KB\sqrt{-\Delta}}{2} \times \frac{ic_1\cos\left(\left(\sqrt{-\Delta}/2\right)\xi\right) - c_2\sin\left(\left(\sqrt{-\Delta}/2\right)\xi\right)}{ic_1\sin\left(\left(\sqrt{-\Delta}/2\right)\xi\right) + c_2\cos\left(\left(\sqrt{-\Delta}/2\right)\xi\right)}, \tag{27}$$

where

$$\xi = \frac{Kx^\beta}{\Gamma(\beta+1)} + \frac{1}{2}\left(4a_0^2K + a_0BK^2 + 2AK^3 - 2ACK^3\right) \times \frac{t^\alpha}{\Gamma(\alpha+1)}. \tag{28}$$

(3) If we choose $B = 0$ and $\Delta_1 = A(C-1) \geq 0$, then we get another triangular function solution

$$u(x,t) = -K\sqrt{\Delta_1}\frac{c_1\cos\left(\sqrt{\Delta_1}\xi\right) + c_2\sin\left(\sqrt{\Delta_1}\xi\right)}{c_1\sin\left(\sqrt{\Delta_1}\xi\right) - c_2\cos\left(\sqrt{\Delta_1}\xi\right)}, \tag{29}$$

where $\xi = Kx^\beta/\Gamma(\beta+1) - \Delta_1(K^3t^\alpha/\Gamma(\alpha+1))$.

(4) If we choose $B = 0$ and $\Delta_1 = A(C-1) < 0$, then we obtain the hyperbolic function solution

$$u(x,t) = -K\sqrt{\Delta_1}\frac{ic_1\cosh\left(\sqrt{-\Delta_1}\xi\right) - c_2\sinh\left(\sqrt{-\Delta_1}\xi\right)}{ic_1\sinh\left(\sqrt{-\Delta_1}\xi\right) - c_2\cosh\left(\sqrt{-\Delta_1}\xi\right)}, \tag{30}$$

where $\xi = Kx^\beta/\Gamma(\beta+1) - \Delta_1(K^3t^\alpha/\Gamma(\alpha+1))$.

If we take $c_1 = -c_2$ and $c_1 = c_2$ in (25), respectively, then we get

$$u(x,t) = \frac{16ABK(C-1) - KB[\Delta + 2A(C-1)]}{2\Delta + 2A(C-1)} - \frac{KB\sqrt{\Delta}}{2}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right),$$
$$u(x,t) = \frac{16ABK(C-1) - KB[\Delta + 2A(C-1)]}{2\Delta + 2A(C-1)} - \frac{KB\sqrt{\Delta}}{2}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right). \tag{31}$$

4.2. The Nonlinear Space-Time Fractional KdV Equation. The KdV equation is the most popular soliton equation, and it has been largely investigated. In addition, the space and time fractional KdV equations with initial conditions were widely worked by [27, 38, 39]. Integrating (2) with respect to u and ignoring the integral constants leads to

$$\frac{1}{2}Lu^2 + \frac{1}{6}aKu^3 + \frac{1}{2}K^3\left(u'\right)^2 = 0. \tag{32}$$

Considering the homogeneous balance between the highest order derivatives and the nonlinear term in (32), we get $n = 2$. So, we can suppose that (32) has the following ansatz:

$$u(\xi) = a_0 + a_1F(\xi) + a_2F^2(\xi), \tag{33}$$

where a_0, a_1, a_2, L, and K are arbitrary constants to be determined later. Substituting (33) and (14), along with (13),

into (32) and using Mathematica yields a system of Equations of (G'/G):

$$\begin{aligned}
&\left(\frac{G'}{G}\right)^0: \frac{1}{3}aa_0^3K + A^2a_1^2K^3 + a_0^2L = 0,\\
&\left(\frac{G'}{G}\right)^1: \frac{1}{2}aa_0^2a_1K + 2A^2a_1a_2K^3\\
&\qquad + Aa_1^2BK^3 + a_0a_1L = 0,\\
&\left(\frac{G'}{G}\right)^2: \frac{1}{2}aa_0a_1^2K + \frac{1}{2}aa_0^2a_2K - Aa_1^2K^3\\
&\qquad + 2A^2a_2^2K^3 + 4Aa_1a_2BK^3 + \frac{1}{2a_1^2B^2K^3}\\
&\qquad + Aa_1^2CK^3 + \frac{1}{2}a_1^2L + a_0a_2L = 0,\\
&\left(\frac{G'}{G}\right)^3: \frac{1}{6}aa_1^3K + aa_0a_1a_2K - 4Aa_1a_2K^3\\
&\qquad - a_1^2BK^3 + 4Aa_2^2BK^3 + 2a_1a_2B^2K^3\\
&\qquad + 4Aa_1a_2CK^3 + a_1^2BCK^3 + a_1a_2L = 0,\\
&\left(\frac{G'}{G}\right)^4: \frac{1}{2}a_1^2a_2K + \frac{1}{2}aa_0a_2^2K + \frac{1}{2}a_1^2K^3\\
&\qquad - 4Aa_2^2K^3 - 4a_1a_2BK^3 + 2a_2^2B^2K^3\\
&\qquad - a_1^2CK^3 + 4a_1a_2BCK^3 + \frac{1}{2}a_1^2C^2K^3\\
&\qquad + \frac{1}{2}a_2^2L = 0,\\
&\left(\frac{G'}{G}\right)^5: \frac{1}{2}aa_1a_2^2K + 2a_1a_2K^3 - 4a_2^2BK^3\\
&\qquad - 4a_1a_2CK^3 + 4a_2^2BCK^3 + 2a_1a_2C^2K^3 = 0,\\
&\left(\frac{G'}{G}\right)^6: \frac{1}{6}a_1a_2^2K + 2a_2^2K^3 - 4a_2^2CK^3\\
&\qquad + 2a_2^2C^2K^3 = 0.
\end{aligned} \tag{34}$$

Solving the set of the above algebraic equations by use of Mathematica, we get the following results:

$$\begin{aligned}
&a \neq 0, \quad a_0 = -\frac{12AK^2}{a}(C-1),\\
&a_1 = \frac{12BK^2}{a}(C-1),\\
&a_2 = -\frac{12K^2}{a}(C-1),\\
&L = -K^3\left(B^2 - 4AC + 4A\right).
\end{aligned} \tag{35}$$

Substituting (35) into (33) and according to (15)–(18), we obtain the following exponential function solutions, hyperbolic function solutions, and triangular function solutions of (2), respectively.

(1) If we choose $B \neq 0$ and $\Delta = B^2 + 4A - 4AC \geq 0$, then the exponential function solution can be found as

$$\begin{aligned}
u(x,t) = {} & \frac{3K^2(\Delta - 2C)}{a(C-1)} - \frac{6CK^2\sqrt{\Delta}}{a(C-1)}\\
& \times \frac{c_1\exp\left(\sqrt{\Delta}\xi/2\right) + c_2\exp\left(\left(-\sqrt{\Delta}/2\right)\xi\right)}{c_1\exp\left(\left(\sqrt{\Delta}/2\right)\xi\right) - c_2\exp\left(\left(-\sqrt{\Delta}/2\right)\xi\right)}\\
& - \frac{6CK^2\Delta}{a(C-1)}\\
& \times \left[\frac{c_1\exp\left(\left(\sqrt{\Delta}/2\right)\xi\right) + c_2\exp\left(-\left(\sqrt{\Delta}/2\right)\xi\right)}{c_1\exp\left(\left(\sqrt{\Delta}/2\right)\xi\right) - c_2\exp\left(-\left(\sqrt{\Delta}/2\right)\xi\right)}\right]^2,
\end{aligned} \tag{36}$$

where $\xi = Kx^\beta/\Gamma(\beta+1) - K^3\Delta(t^\alpha/\Gamma(\alpha+1))$.

(2) If we choose $B \neq 0$ and $\Delta = B^2 + 4A - 4AC < 0$, then the triangular function solution will be

$$\begin{aligned}
u(x,t) = {} & \frac{3K^2(\Delta - 2C)}{a(C-1)} + \frac{6CK^2\sqrt{\Delta}}{a(C-1)}\\
& \times \frac{ic_1\cos\left(\sqrt{-\Delta}\xi/2\right) - c_2\sin\left(\left(\sqrt{-\Delta}/2\right)\xi\right)}{ic_1\sin\left(\left(\sqrt{-\Delta}/2\right)\xi\right) + c_2\cos\left(\left(\sqrt{-\Delta}/2\right)\xi\right)}\\
& + \frac{6CK^2\Delta}{a(C-1)}\\
& \times \left[\frac{ic_1\cos\left(\left(\sqrt{-\Delta}/2\right)\xi\right) - c_2\sin\left(\left(\sqrt{-\Delta}/2\right)\xi\right)}{ic_1\sin\left(\left(\sqrt{-\Delta}/2\right)\xi\right) + c_2\cos\left(\left(\sqrt{-\Delta}/2\right)\xi\right)}\right]^2,
\end{aligned} \tag{37}$$

where $\xi = Kx^\beta/\Gamma(\beta+1) - K^3\Delta(t^\alpha/\Gamma(\alpha+1))$.

(3) If we choose $B = 0$ and $\Delta_1 = A(C-1) \geq 0$, then the triangular function solution is given as

$$\begin{aligned}
u(x,t) = {} & -\frac{12K^2\Delta_1}{a} - \frac{12K^2\Delta_1}{a(C-1)}\\
& \times \left[\frac{c_1\cos\left(\sqrt{\Delta_1}\xi\right) + c_2\sin\left(\sqrt{\Delta_1}\xi\right)}{c_1\sin\left(\sqrt{\Delta_1}\xi\right) - c_2\cos\left(\sqrt{\Delta_1}\xi\right)}\right]^2,
\end{aligned} \tag{38}$$

where $\xi = Kx^\beta/\Gamma(\beta+1) + 4K^3\Delta_1(t^\alpha/\Gamma(\alpha+1))$.

(4) If we choose $B = 0$ and $\Delta_1 = A(C-1) < 0$, then the hyperbolic function solution is given as

$$u(x,t) = -\frac{12K^2\Delta_1}{a} + \frac{12K^2\Delta_1}{a(C-1)} \times \left[\frac{ic_1\cosh\left(\sqrt{-\Delta_1}\xi\right) - c_2\sinh\left(\sqrt{-\Delta_1}\xi\right)}{ic_1\sinh\left(\sqrt{-\Delta_1}\xi\right) - c_2\cosh\left(\sqrt{-\Delta_1}\xi\right)}\right]^2, \tag{39}$$

where $\xi = Kx^\beta/\Gamma(\beta+1) - 4K^3\Delta_1(t^\alpha/\Gamma(\alpha+1))$. Equation (36) can be rewritten at $c_1 = -c_2$; so we get the other hyperbolic function solution of (2):

$$\begin{aligned} u(x,t) = {} & \frac{3K^2(\Delta-2C)}{a(C-1)} - \frac{6CK^2\sqrt{\Delta}}{a(C-1)} \\ & \times \tanh\left[\frac{\sqrt{\Delta}}{2}\left(\frac{Kx^\beta}{\Gamma(\beta+1)} - K^3\Delta\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\right] \\ & + \frac{6CK^2\sqrt{\Delta}}{a(C-1)} \\ & \times \tanh^2\left[\frac{\sqrt{\Delta}}{2}\left(\frac{Kx^\beta}{\Gamma(\beta+1)} - K^3\Delta\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\right]. \end{aligned} \tag{40}$$

Equation (36) becomes

$$\begin{aligned} u(x,t) = {} & \frac{3K^2(\Delta-2C)}{a(C-1)} - \frac{6CK^2\sqrt{\Delta}}{a(C-1)} \\ & \times \coth\left[\frac{\sqrt{\Delta}}{2}\left(\frac{Kx^\beta}{\Gamma(\beta+1)} - K^3\Delta\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\right] \\ & + \frac{6CK^2\sqrt{\Delta}}{a(C-1)} \\ & \times \coth^2\left[\frac{\sqrt{\Delta}}{2}\left(\frac{Kx^\beta}{\Gamma(\beta+1)} - K^3\Delta\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\right] \end{aligned} \tag{41}$$

at $c_1 = c_2$.

Remark 5. Kudryashov et al. [41–44] have showed that every solution, which was obtained when soliton solutions have been found by some analytic methods, is not a new solution. They also showed that these methods are very similar. Furthermore, they mentioned that authors who used these methods should check the obtained results very carefully. The reason for using improved (G'/G)-expansion method in this work is to use nonlinear equation (14) instead of linear equation

$$G'' - \lambda G' - \mu G = 0, \tag{42}$$

which was used in standard (G'/G) method and to obtain lots of different solutions.

5. Conclusion

In this paper, we introduced an improved (G'/G)-expansion method and carried it out to obtain new travelling wave solutions of the space-time fractional foam drainage equation and the space-time fractional KdV equation. This method gives new exact solutions for nonlinear FPDEs. These solutions include the hyperbolic function solution, the exponential function solution, the triangular function solution, and the trigonometric function solution. These solutions are useful to understand the mechanisms of the complicated nonlinear physical phenomena.

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