Research Article

Coefficient Estimates and Other Properties for a Class of Spirallike Functions Associated with a Differential Operator

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For $0 \leq \eta < 1$, $0 \leq \lambda < 1$, $-\pi/2 < \gamma < \pi/2$, $0 \leq \beta \leq \alpha$, and $m \in \mathbb{N} \cup \{0\}$, a new class $\mathcal{S}_m^{\alpha,\beta}(\eta,\gamma,\lambda)$ of analytic functions defined by means of the differential operator $D_m^{\alpha,\beta}$ is introduced. Our main object is to provide sharp upper bounds for Fekete-Szegő problem in $\mathcal{S}_m^{\alpha,\beta}(\eta,\gamma,\lambda)$. We also find sufficient conditions for a function to be in this class. Some interesting consequences of our results are pointed out.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hfill (1)

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be in the class of $\gamma$-spirallike functions of order $\lambda$ in $\mathbb{U}$, denoted by $\mathcal{S}^*(\gamma,\lambda)$, if

$$\Re \left( e^{i\gamma} \frac{zf'(z)}{f(z)} \right) > \lambda \cos \gamma, \quad z \in \mathbb{U},$$  \hfill (2)

for $0 \leq \lambda < 1$ and some real $\gamma$ with $|\gamma| < \pi/2$.

The class $\mathcal{S}^*(\gamma,\lambda)$ was studied by Libera [1] and Keogh and Merkes [2].

Note that $\mathcal{S}^*_0(\gamma,\lambda)$ is the class of spirallike functions introduced by Špaček [3], $\mathcal{S}^*_0(0,\lambda) = \mathcal{S}^*(\lambda)$ is the class of starlike functions of order $\lambda$, and $\mathcal{S}^*_0(0,0) = \mathcal{S}^*$ is the familiar class of starlike functions.

For the constants $\lambda, \gamma$ with $0 \leq \lambda < 1$ and $|\gamma| < \pi/2$, denote

$$p_{\lambda,\gamma}(z) = \frac{1 + e^{-i\gamma} (e^{-i\gamma} - 2\lambda \cos \gamma)z}{1 - z}, \quad z \in \mathbb{U}. \hfill (3)$$

The function $p_{\lambda,\gamma}(z)$ maps the open unit disk onto the half-plane $H_{\lambda,\gamma} = \{z \in \mathbb{C} : \Re(e^{i\gamma}z) > \lambda \cos \gamma\}$. If

$$p_{\lambda,\gamma}(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \hfill (4)$$

then it is easy to check that

$$p_n = 2e^{-i\gamma} (1 - \lambda) \cos \gamma, \quad \forall n \geq 1. \hfill (5)$$

For $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$  \hfill (6)

the Hadamard product (or convolution), denoted by $f \ast g$, is defined by

$$(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \hfill (7)$$
Denote by $B$ the family of all analytic functions $w(z)$ that satisfy the conditions $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathcal{U}$.

A function $f \in \mathcal{A}$ is said to be subordinate to a function $g \in \mathcal{A}$, written $f \prec g$, if there exists a function $w \in B$ such that $f(z) = g(w(z)), z \in \mathcal{U}$.

A classical theorem of Fekete and Szegő (see [4]) states that if $f \in \mathcal{S}$ is given by (1), then
\[
\left| a_3 - \mu a_2^2 \right| \leq 1 + 2 \exp \left( \frac{-2\mu}{1-\mu} \right), \quad 0 \leq \mu < 1,
\]
and
\[
\left| a_3 \right| \leq \cos^{-1} \left( \frac{1-\delta}{1-\lambda} \right), \quad 0 \leq \lambda < 1,
\]
where $\delta < 1$ and $\lambda$ are real numbers with $0 \leq \delta < 1$.

It should be remarked that the operator $D^m_{\alpha,\beta} f$ generalizes other differential operators considered earlier. For $f \in \mathcal{A}$, we have
\[
D^m_{\alpha,\beta} f(z) = D^m f(z), \quad \text{the operator introduced by Sălăgean [14];}
\]
and
\[
D^m_{\alpha,0} f(z) = D^m f(z), \quad \text{the operator studied by Al-Ouboudi [15].}
\]

In view of (9), $D^m_{\alpha,\beta} f(z)$ can be written in terms of convolution as
\[
D^m_{\alpha,\beta} f(z) = (g_{\alpha,\beta} \ast f)(z), \quad z \in \mathcal{U},
\]
where
\[
g_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \Phi_n(\alpha, \beta, m) z^n, \quad z \in \mathcal{U}.
\]

Define the function $g^{(-1)}_{\alpha,\beta}$ such that
\[
g^{(-1)}_{\alpha,\beta}(z) = \frac{z}{1-z}, \quad z \in \mathcal{U}.
\]

It is easy to observe that
\[
f(z) = g^{(-1)}_{\alpha,\beta}(z) \ast D^m_{\alpha,\beta} f(z).
\]

Making use of the differential operator $D^m_{\alpha,\beta} f$, we define the following class of functions.

**Definition 1.** For $0 \leq \eta < 1, 0 \leq \lambda < 1$, and $|\gamma| < \pi/2$, denote by $\mathcal{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$ the class of functions $f \in \mathcal{A}$ which satisfy the condition
\[
\Re \left( e^{\gamma} \frac{z(D^m_{\alpha,\beta} f(z))^\nu}{(1-\eta)(1-\lambda)^m} \right) > \lambda \cos \gamma, \quad z \in \mathcal{U}.
\]

The class $\mathcal{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$ contains as particular cases the following classes of functions:
\[
\mathcal{S}^0_{\alpha,\beta}(0, \gamma, \lambda) = \mathcal{S}^*_{\eta, \gamma},
\]
\[
\mathcal{S}^0_{\alpha,\beta}(0, 0, \lambda) = \mathcal{S}^* \quad \text{for } \lambda < 1.
\]

Also, the class $\mathcal{S}^0_{\alpha,\beta}(\eta, \gamma, \lambda)$ consists of functions $f \in \mathcal{A}$ satisfying the inequality
\[
\Re \left( e^{\gamma} \frac{z f(z)}{(1-\lambda)^m} \right) > \lambda \cos \gamma, \quad z \in \mathcal{U}.
\]

An analogous of the class $\mathcal{S}^0_{\alpha,\beta}(\eta, \gamma, \lambda)$ has been recently studied by Murugusundaramoorthy [16].

The main object of this paper is to obtain sharp upper bounds for the Fekete-Szegő problem for the class $\mathcal{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$. We also find sufficient conditions for a function to be in this class.

## 2. Membership Characterizations

In this section, we obtain several sufficient conditions for a function $f \in \mathcal{A}$ to be in the class $\mathcal{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$.

**Theorem 2.** Let $f \in \mathcal{A}$, and let $\delta$ be a real number with $0 \leq \delta < 1$. If
\[
\left| \frac{z(D^m_{\alpha,\beta} f(z))^\nu}{(1-\eta)(1-\lambda)^m} \right| - 1 \leq 1 - \delta, \quad z \in \mathcal{U},
\]
then $f \in \mathcal{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$ provided that
\[
|\gamma| \leq \cos^{-1} \left( \frac{1-\delta}{1-\lambda} \right).
\]
Proof. From (18), it follows that
\[
\frac{z(D^m_{\alpha,\beta} f(z))'}{(1-\eta) D^m_{\alpha,\beta} f(z) + \eta z(D^m_{\alpha,\beta} f(z))'} = 1 + (1-\delta) w(z),
\]
where \(w(z) \in \mathcal{B}\). We have
\[
\Re \left( e^{i\gamma} \frac{z(D^m_{\alpha,\beta} f(z))'}{(1-\eta) D^m_{\alpha,\beta} f(z) + \eta z(D^m_{\alpha,\beta} f(z))'} \right) = \Re \left( e^{i\gamma} (1 + (1-\delta) w(z)) \right) = \cos \gamma + (1-\delta) \Re (e^{i\gamma} w(z)) \geq \cos \gamma - (1-\delta) \left| e^{i\gamma} w(z) \right| > \cos \gamma - (1-\delta) \geq \lambda \cos \gamma,
\]
provided that \(|\gamma| \leq \cos^{-1}((1-\delta)/(1-\lambda))\). Thus, the proof is completed. \(\square\)

If in Theorem 2 we take \(\delta = 1 - (1-\lambda) \cos \gamma\), we will obtain the following result.

Corollary 3. Let \(f \in \mathcal{A}\). If
\[
\left| \frac{z(D^m_{\alpha,\beta} f(z))'}{(1-\eta) D^m_{\alpha,\beta} f(z) + \eta z(D^m_{\alpha,\beta} f(z))'} - 1 \right| \leq (1-\lambda) \cos \gamma,
\]
then \(f \in \mathcal{D}^{m}_{\alpha,\beta}(\eta, \gamma, \lambda)\).

A sufficient condition for a function \(f \in \mathcal{A}\) to be in the class \(\mathcal{D}^{m}_{\alpha,\beta}(\eta, \gamma, \lambda)\), in terms of coefficients inequality, is obtained in the next theorem.

Theorem 4. If a function \(f \in \mathcal{A}\) given by (1) satisfies the inequality
\[
\sum_{n=2}^{\infty} \left| (1-\eta)(n-1) \sec \gamma + (1-\lambda)(1+\eta(n-1)) \right| \Phi_n(\alpha, \beta, m) |a_n| \leq 1 - \lambda,
\]
where \(0 \leq \eta < 1, 0 \leq \lambda < 1, |\gamma| < \pi/2\), and \(\Phi_n(\alpha, \beta, m)\) is defined by (10), then it belongs to the class \(\mathcal{D}^{m}_{\alpha,\beta}(\eta, \gamma, \lambda)\).

Proof. In virtue of Corollary 3, it suffices to show that the condition (22) is satisfied. We have
\[
\left| \frac{z(D^m_{\alpha,\beta} f(z))'}{(1-\eta) D^m_{\alpha,\beta} f(z) + \eta z(D^m_{\alpha,\beta} f(z))'} - 1 \right| = (1-\eta) \left| \sum_{n=2}^{\infty} (n-1) \Phi_n(\alpha, \beta, m) a_n z^{n-1} \right| + (1-\eta) \left| \sum_{n=2}^{\infty} (n-1) \Phi_n(\alpha, \beta, m) a_n z^{n-1} \right|
\]
\[
< (1-\eta) \left| \sum_{n=2}^{\infty} (n-1) \Phi_n(\alpha, \beta, m) a_n \right|
\]
\[
= (1-\eta) \left| \sum_{n=2}^{\infty} (n-1) \Phi_n(\alpha, \beta, m) |a_n| \right|
\]
\[
\leq (1-\lambda) \cos \gamma \left( 1 - \sum_{n=2}^{\infty} (1-\eta)(1+\eta(n-1)) \Phi_n(\alpha, \beta, m) |a_n| \right),
\]
which is equivalent to
\[
\sum_{n=2}^{\infty} \left[ (1-\eta)(n-1) \sec \gamma + (1-\lambda)(1+\eta(n-1)) \right] |a_n| \leq 1 - \lambda.
\]
(26)

For special values of \(m, \eta, \gamma, \lambda\), from Theorem 4, we can derive the following sufficient conditions for a function \(f \in \mathcal{A}\) to be in the classes \(\mathcal{D}^{0}_{\alpha,\beta}(\eta, \gamma, \lambda)\), \(\mathcal{D}^{0}_{\alpha,\beta}(0, \gamma, \lambda) = \mathcal{S}^{*}(\gamma, \lambda)\), and \(\mathcal{D}^{0}_{\alpha,\beta}(0, \gamma, \lambda) = \mathcal{D}^{*}(\gamma, \lambda)\), respectively.

Corollary 5. Let \(f \in \mathcal{A}\). If
\[
\sum_{n=2}^{\infty} \left[ (1-\eta)(n-1) \sec \gamma + (1-\lambda)(1+\eta(n-1)) \right] |a_n| \leq 1 - \lambda,
\]
(27)
then \(f \in \mathcal{D}^{0}_{\alpha,\beta}(\eta, \gamma, \lambda)\).

Corollary 6 (see [17]). Let \(f \in \mathcal{A}\). If
\[
\sum_{n=2}^{\infty} \left[ (n-1) \sec \gamma + 1 - \lambda \right] |a_n| \leq 1 - \lambda,
\]
(28)
where \(0 \leq \lambda < 1, |\gamma| < \pi/2\), then \(f \in \mathcal{D}^{0}_{\alpha,\beta}(\eta, \gamma, \lambda)\).

Corollary 7 (see [18]). Let \(f \in \mathcal{A}\). If
\[
\sum_{n=2}^{\infty} \left[ 1 + (n-1) \sec \gamma \right] |a_n| \leq 1,
\]
(29)
where \(|\gamma| < \pi/2\), then \(f \in \mathcal{D}^{*}(\gamma)\).
A necessary and sufficient condition for a function to be in the class $S^m_{\alpha, \beta}(\eta, \gamma, \lambda)$ can be given in terms of integral representation.

**Theorem 8.** A function $f \in \mathcal{A}$ is in the class $S^m_{\alpha, \beta}(\eta, \gamma, \lambda)$ if and only if there exists $w \in \mathcal{B}$ such that

$$f(z) = g_{\alpha, \beta}^{(-1)}(z) * z \exp \left( \int_0^z \left[ \frac{p_{\lambda, y}(w(\xi)) - 1}{1 - \eta p_{\lambda, y}(w(\xi))} \right] d\xi \right),$$

where $p_{\lambda, y}(z)$ and $g_{\alpha, \beta}^{(-1)}(z)$ are defined by (3) and (13), respectively.

**Proof.** In virtue of (15), $f \in S^m_{\alpha, \beta}(\eta, \gamma, \lambda)$ if and only if there exists $w \in \mathcal{B}$ such that

$$z(D^m_{\alpha, \beta} f(z))' = p_{\lambda, y}(w(z)).$$

From this last equality, we obtain

$$D^m_{\alpha, \beta} f(z) = z \exp \left( \int_0^z \left[ p_{\lambda, y}(w(\xi)) - 1 \right] \frac{d\xi}{\zeta} \right),$$

and thus, the proof is completed. \qed

For $0 \leq \theta \leq 2\pi, 0 \leq \tau \leq 1$, define the function $\Psi(z, \theta, \tau)$

$$\Psi(z, \theta, \tau) = g_{\alpha, \beta}^{(-1)}(z) * z \exp \left( \int_0^z \left[ \frac{p_{\lambda, y}(e^{\theta \zeta} (\zeta + \tau)/(1 + \tau \zeta)) - 1}{1 - \eta p_{\lambda, y}(e^{\theta \zeta} (\zeta + \tau)/(1 + \tau \zeta))} \right] d\xi \right),$$

where $p_{\lambda, y}(z)$ and $g_{\alpha, \beta}^{(-1)}(z)$ are defined by (3) and (13), respectively.

In virtue of Theorem 8, the function $\Psi(z, \theta, \tau)$ belongs to the class $S^m_{\alpha, \beta}(\eta, \gamma, \lambda)$. Note that $\Psi(z, 0, 0)$ is an odd function.

**3. The Fekete-Szegö Problem**

In order to obtain sharp upper bounds for the Fekete-Szegö functional for the class $S^m_{\alpha, \beta}(\eta, \gamma, \lambda)$, the following lemma is required (see, e.g., [19, page 108]).

**Lemma 9.** Let the function $w \in \mathcal{B}$ be given by

$$w(z) = \sum_{n=1}^{\infty} w_n z^n, \quad z \in \mathcal{U}.$$  (35)

Then

$$|w_1| \leq 1, \quad |w_2| \leq 1 - |w_1|^2, \quad |w_2 - sw|^2 \leq \max\{1, |s|\}, \quad \text{for any complex number } s.$$  (36) (37)

The functions $w(z) = z$ and $w(z) = z^2$, or one of their rotations, show that both inequalities (36) and (37) are sharp.

First we obtain sharp upper bounds for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ with $\mu$ real parameter.

**Theorem 10.** Let $f \in S^m_{\alpha, \beta}(\eta, \gamma, \lambda)$ be given by (1), and let $\mu$ be a real number. Then

$$|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{(1 - \lambda) \cos \gamma}{(1 - \eta)^2 \Phi_3(\alpha, \beta, m)} \times \left[ \eta + 3 - 2\lambda(1 + \eta) - \mu \frac{4(1 - \lambda) \Phi_3(\alpha, \beta, m)}{\Phi_2^2(\alpha, \beta, m)} \right], & \text{if } \mu \leq \sigma_1, \\
\frac{(1 - \lambda) \cos \gamma}{(1 - \eta)^2 \Phi_3(\alpha, \beta, m)} \times \left[ \mu \frac{4(1 - \lambda) \Phi_3(\alpha, \beta, m)}{\Phi_2^2(\alpha, \beta, m)} + 2\lambda(1 + \eta) - \eta - 3 \right], & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\
\frac{(1 - \lambda) \cos \gamma}{(1 - \eta)^2 \Phi_3(\alpha, \beta, m)} \times \left[ \frac{4(1 - \lambda) \Phi_3(\alpha, \beta, m)}{\Phi_2^2(\alpha, \beta, m)} + \lambda(1 + \eta) - \eta - 3 \right], & \text{if } \mu \geq \sigma_2, 
\end{array} \right.$$  (38)

where

$$\sigma_1 = \frac{(1 + \eta) \Phi_2^2(\alpha, \beta, m)}{2\Phi_3(\alpha, \beta, m)},$$  (39)

$$\sigma_2 = \frac{2 - \lambda(1 + \eta)}{1 - \lambda} \frac{\Phi_2^2(\alpha, \beta, m)}{2\Phi_3(\alpha, \beta, m)}.$$  (40)

and $\Phi_2(\alpha, \beta, m)$, $\Phi_3(\alpha, \beta, m)$ are defined by (10) with $n = 2$ and $n = 3$, respectively.

All estimates are sharp.
Proof. Suppose that \( f \in S_{\alpha, \beta}(\eta, \gamma, \lambda) \) is given by (1). Then, from the definition of the class \( S_{\alpha, \beta}(\eta, \gamma, \lambda) \), there exist \( w \in \mathcal{R} \), \( w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \ldots \) such that
\[
\frac{z(D^m_{\alpha, \beta} f(z))'}{(1-\eta) D^m_{\alpha, \beta} f(z) + \eta z(D^m_{\alpha, \beta} f(z))'} = p_{\lambda, \gamma}(w(z)),
\]
where \( z \in \mathcal{H} \).

(41)

Set \( p_{\lambda, \gamma}(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \). Equating the coefficients of \( z \) and \( z^2 \) on both sides of (41), we obtain
\[
a_2 = \frac{p_1 w_1}{(1-\eta) \Phi_3(\alpha, \beta, m)},
\]
\[
a_3 = \frac{1}{2(1-\eta) \Phi_3(\alpha, \beta, m)} \left( \frac{1 + \eta}{1 - \eta} p_1^2 + p_2 \right) w_1^2 + p_1 w_2.
\]

(42)

From (5), we have \( p_1 = p_2 = 2e^{-i\gamma}(1-\lambda) \cos \gamma \), and thus we obtain
\[
a_2 = \frac{2e^{-i\gamma}(1-\lambda) \cos \gamma}{(1-\eta) \Phi_3(\alpha, \beta, m)} w_1,
\]
\[
a_3 = \frac{e^{-i\gamma}(1-\lambda) \cos \gamma}{(1-\eta) \Phi_3(\alpha, \beta, m)} \left( 2e^{-i\gamma}(1-\lambda) \cos \gamma \frac{1 + \eta}{1 - \eta} + 1 \right) w_1^2 + p_1 w_2.
\]

(43)

It follows that
\[
|a_3 - \mu a_2^2| \leq \frac{(1-\lambda) \cos \gamma}{(1-\eta) \Phi_3(\alpha, \beta, m)} \times \left\{ \frac{2e^{-i\gamma}(1-\lambda) \cos \gamma}{1 - \eta} \left( 1 + \frac{2\Phi_3(\alpha, \beta, m)}{\Phi_2^2(\alpha, \beta, m)} \right) + 1 \right\} \times |w_1|^2 + |w_2|.
\]

(44)

Making use of Lemma 9 (36), we have
\[
|a_3 - \mu a_2^2| \leq \frac{(1-\lambda) \cos \gamma}{(1-\eta) \Phi_3(\alpha, \beta, m)} \times \left\{ 1 + \left| \frac{2e^{-i\gamma}(1-\lambda) \cos \gamma}{1 - \eta} \left( 1 + \frac{2\Phi_3(\alpha, \beta, m)}{\Phi_2^2(\alpha, \beta, m)} \right) + 1 \right| \right\} \times |w_1|^2.
\]

(45)
In view of Lemma 9, the results are sharp for \( w(z) = z \) and \( w(z) = z^2 \) or one of their rotations. From (41), we obtain that the extremal functions are \( \Psi(z, \theta, 1) \) and \( \Psi(z, \theta, 0) \) defined by (34) with \( \tau = 1 \) and \( \tau = 0 \).

Next, we consider the Fekete-Szegő problem for the class \( \delta^m_{\alpha, \beta}(\eta, \gamma, \lambda) \) with \( \mu \) complex parameter.

**Theorem 11.** Let \( f \in \delta^m_{\alpha, \beta}(\eta, \gamma, \lambda) \) be given by (1), and let \( \mu \) be a complex number. Then,

\[
\left| a_3 - \mu a_2^2 \right| \leq \frac{(1 - \lambda) \cos \gamma}{(1 - \eta) \Phi_3(\alpha, \beta, m)} \times \max \left\{ 1, \frac{2(1 - \lambda) \cos \gamma}{1 - \eta} \times \left( \mu \frac{2\Phi_3(\alpha, \beta, m)}{\Phi_2(\alpha, \beta, m)} - 1 - \eta \right) e^{i\eta} \right\}.
\]

The result is sharp.

**Proof.** Assume that \( f \in \delta^m_{\alpha, \beta}(\eta, \gamma, \lambda) \). Making use of (43), we obtain

\[
\left| a_3 - \mu a_2^2 \right| \leq \frac{(1 - \lambda) \cos \gamma}{(1 - \eta) \Phi_3(\alpha, \beta, m)} \times \frac{2e^{-i\gamma}(1 - \lambda) \cos \gamma}{1 - \eta} \times \left( \mu \frac{2\Phi_3(\alpha, \beta, m)}{\Phi_2(\alpha, \beta, m)} - 1 - \eta \right) e^{i\eta} \left| w_2 \right|.
\]

The inequality (53) follows as an application of Lemma 9 (37) with

\[
s = \frac{2e^{-i\gamma}(1 - \lambda) \cos \gamma}{1 - \eta} \left( \mu \frac{2\Phi_3(\alpha, \beta, m)}{\Phi_2(\alpha, \beta, m)} - 1 - \eta \right) - 1.
\]

The functions \( \Psi(z, \theta, 1) \) and \( \Psi(z, \theta, 0) \) defined by (34) with \( \tau = 1 \) and \( \tau = 0 \) show that the inequality (53) is sharp.

Our Theorems 10 and 11 include several various results for special values of \( m, \eta, \gamma, \) and \( \lambda \). For example, taking \( m = \eta = \gamma = \lambda = 0 \), in Theorem 10, we obtain the Fekete-Szegő inequalities for the class \( \delta^* \) (see [2, 11]). The special case \( m = \eta = \lambda = 0 \) leads to the Fekete-Szegő inequalities for the class \( \delta^*(\gamma) \) (see [2]). The Fekete-Szegő inequalities for the class \( \delta^*(\gamma, \lambda) \) (see [2]) are also included in Theorems 10 and 11.

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