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Retraction

Retracted: Numerical Solution of Nonlinear Fredholm Integrodifferential Equations by Hybrid of Block-Pulse Functions and Normalized Bernstein Polynomials

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The paper titled "Numerical Solution of Nonlinear Fredholm Integrodifferential Equations by Hybrid of Block-Pulse Functions and Normalized Bernstein Polynomials" [1], published in Abstract and Applied Analysis, has been retracted as an almost identical paper by the same author has been simultaneously submitted to and published in Journal of Computational and Applied Mathematics. The other publication is "Solution of nonlinear Fredholm integro-differential equations using a hybrid of block pulse functions and normalized Bernstein polynomials," Volume 260, April 2014, Pages 258–265, DOI: 10.1016/j.cam.2013.09.036.

References

[1] S. H. Behiry, "Numerical solution of nonlinear fredholm integrodifferential equations by hybrid of block-pulse functions and normalized bernstein polynomials," *Abstract and Applied Analysis*, vol. 2013, Article ID 416757, 8 pages, 2013.

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Research Article

Numerical Solution of Nonlinear Fredholm Integrodifferential Equations by Hybrid of Block-Pulse Functions and Normalized Bernstein Polynomials

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A numerical method for solving nonlinear Fredholm integrodifferential equations is proposed. The method is based on hybrid functions approximate. The properties of hybrid of block pulse functions and orthonormal Bernstein polynomials are presented and utilized to reduce the problem to the solution of nonlinear algebraic equations. Numerical examples are introduced to illustrate the effectiveness and simplicity of the present method.

1. Introduction

Integrodifferential equations are often involved in mathematical formulation of physical phenomena. Fredholm integrodifferential equations play an important role in many fields such as economics, biomechanics, control, elasticity, fluid dynamics, heat and mass transfer, oscillation theory, and airfoil theory; for examples see [1–3] and references cited therein. Finding numerical solutions for Fredholm integrodifferential equations is one of the oldest problems in applied mathematics. Numerous works have been focusing on the development of more advanced and efficient methods for solving integrodifferential equations such as wavelets method [4, 5], Walsh functions method [6], sinc-collocation method [7], homotopy analysis method [8], differential transform method [9], the hybrid Legendre polynomials and block-pulse functions [10], Chebyshev polynomials method [11], and Bernoulli matrix method [12].

Block-pulse functions have been studied and applied extensively as a basic set of functions for signals and functions approximations. All these studies and applications show that block-pulse functions have definite advantages for solving problems involving integrals and derivatives due to their clearness in expressions and their simplicity in formulations; see [13]. Also, Bernstein polynomials play a prominent role in various areas of mathematics. Many authors have used these polynomials in the solution of integral equations, differential equations, and approximation theory; see for instance [14–17].

The purpose of this work is to utilize the hybrid functions consisting of combination of block-pulse functions with normalized Bernstein polynomials for obtaining numerical solution of nonlinear Fredholm integrodifferential equation:

$$\sum_{i=0}^{s} p_i(x)\, y^{(i)}(x) = g(x) + \lambda \int_0^1 k(x,t)\,[y(t)]^q dt, \qquad 0 \le x,\ t < 1, \tag{1}$$

with the conditions

$$y^{(i)}(0) = \alpha_i, \quad 0 \le i \le s-1, \tag{2}$$

where $y^{(i)}(x)$ is the ith derivative of the unknown function that will be determined, $k(x,t)$ is the kernel of the integral, $g(x)$ and $p_i(x)$ are known analytic functions, q is a positive integer, and λ and α_i are suitable constants. The proposed approach for solving this problem uses few numbers of basis and benefits of the orthogonality of block-pulse functions and the advantages of orthonormal Bernstein polynomials properties to reduce the nonlinear integrodifferential equation to easily solvable nonlinear algebraic equations.

This paper is organized as follows. In the next section, we present Bernstein polynomials and hybrid of block-pulse functions. Also, their useful properties such as functions approximation, convergence analysis, operational matrix of product, and operational matrix of differentiation are given. In Section 3, the numerical scheme for the solution of (1) and (2) is described. In Section 4, the proposed method is applied to some nonlinear Fredholm integrodifferential equations, and comparisons are mad with the existing analytic or numerical solutions that were reported in other published works in the literature. Finally conclusions are given in Section 5.

2. Properties of Hybrid Functions

2.1. Hybrid of Block-Pulse Functions and Orthonormal Bernstein Polynomials. The Bernstein polynomials of nth degree are defined on the interval $[0, 1]$ as [16]

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad \text{for } i = 0, 1, 2, \dots, n, \tag{3}$$

where

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}. \tag{4}$$

There are $(n+1)$ nth degree Bernstein polynomials. Using Gram-Schmidt orthonormalization process on $B_{i,n}(x)$, we obtain a class of orthonormal polynomials from the Bernstein polynomials. We call them orthonormal Bernstein polynomials of degree n and denote them by $b_{i,n}(x)$, $0 \le i \le n$. For $n = 3$, the four orthonormal Bernstein polynomials are given by

$$\begin{aligned} b_{0,3}(x) &= -\sqrt{7}\left[x^3 - 3x^2 + 3x - 1\right], \\ b_{1,3}(x) &= \sqrt{5}\left[7x^3 - 15x^2 + 9x - 1\right], \\ b_{2,3}(x) &= -\sqrt{3}\left[21x^3 - 33x^2 + 13x - 1\right], \\ b_{3,3}(x) &= 35x^3 - 45x^2 + 15x - 1. \end{aligned} \tag{5}$$

Hybrid functions $h_{ji}(x)$, $j = 1, 2, \dots, m$ and $i = 0, 1, \dots, n$ are defined on the interval $[0, 1)$ as

$$h_{ji}(x) = \begin{cases} \sqrt{m}\, b_{i,n}(mx - j + 1), & \dfrac{j-1}{m} \le x < \dfrac{j}{m}, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

where j and n are the order of block-pulse functions and degree of orthonormal Bernstein polynomials, respectively.

It is clear that these sets of hybrid functions in (6) are orthonormal and disjoint.

2.2. Functions Approximation. A function $y(x) \in L^2[0, 1)$ may be approximated as

$$y(x) \approx \sum_{j=1}^{m}\sum_{i=0}^{n} c_{ji} h_{ji}(x) = \mathbf{C}^T \mathbf{H}(x), \tag{7}$$

where

$$\begin{aligned} \mathbf{C} &= \left[\mathbf{C}_1^T, \mathbf{C}_2^T, \dots, \mathbf{C}_j^T, \dots, \mathbf{C}_m^T\right]^T, \\ \mathbf{C}_j &= \left[c_{j0}, c_{j1}, c_{j2}, \dots, c_{jn}\right]^T, \quad j = 1, 2, \dots, m, \end{aligned} \tag{8}$$

$$\mathbf{H}(x) = \left[\mathbf{H}_1^T(x), \mathbf{H}_2^T(x), \dots, \mathbf{H}_j^T(x), \dots, \mathbf{H}_m^T(x)\right]^T, \tag{9}$$

and $\mathbf{H}_j(x) = [h_{j0}(x), h_{j1}(x), \dots, h_{jn}(x)]^T$, $j = 1, 2, \dots, m$. The constant coefficients c_{ji} are $(y(x), h_{ji}(x))$, $i = 0, 1, 2, \dots, n$, $j = 1, 2, \dots, m$, and $(\cdot, \cdot)$ is the standard inner product on $L^2[0, 1)$.

We can also approximate the function $k(x, t) \in L^2([0, 1) \times [0, 1))$ by

$$k(x,t) \approx \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{l=0}^{n}\sum_{r=0}^{n} k_{lr}^{ij} h_{il}(x) h_{jr}(t) = \mathbf{H}^T(x)\mathbf{K}\mathbf{H}(t), \tag{10}$$

where $\mathbf{K} = [\mathbf{K}^{ij}]$ is an $m(n+1) \times m(n+1)$ matrix, such that the elements of the sub matrix $\mathbf{k}^{ij}$ are

$$\begin{gathered} k_{lr}^{ij} = \int_{i-1/m}^{i/m}\int_{j-1/m}^{j/m} k(x,t)\, h_{i(l-1)}(x)\, h_{j(r-1)}(t)\, dx dt, \\ l, r = 1, 2, \dots, n+1, \; i, j = 1, 2, \dots, m, \end{gathered} \tag{11}$$

utilizing properties of block-pulse function and orthonormal Bernstein polynomials.

2.3. Convergence Analysis. In this section, the error bound and convergence is established by the following lemma.

Lemma 1. *Suppose that $f \in C^{(n+1)}[0, 1)$ is $n+1$ times continuously differentiable function such that $f = \sum_{j=1}^{m} f_j$, and let $Y_j = \operatorname{Span}\{h_{j0}(x), h_{j1}(x), \dots, h_{jn}(x)\}$, $j = 1, 2, \dots, m$. If $\mathbf{C}_j^T \mathbf{H}_j(x)$ is the best approximation to f_j from Y_j, then $\mathbf{C}^T\mathbf{H}(x)$ approximates f with the following error bound:*

$$\begin{gathered} \left\| f - \mathbf{C}^T \mathbf{H}(x) \right\|_2 \le \frac{\gamma}{m^{n+1}(n+1)!\sqrt{2n+3}}, \\ \gamma = \max_{x \in [0.1)} \left| f^{(n+1)}(x) \right|. \end{gathered} \tag{12}$$

Proof. The Taylor expansion for the function $f_j(x)$ is

$$\begin{aligned} \tilde{f}_j(x) = {} & f_j\left(\frac{j-1}{m}\right) + f_j'\left(\frac{j-1}{m}\right)\left(x - \frac{j-1}{m}\right) \\ & + \cdots + f_j^{(n)}\left(\frac{j-1}{m}\right)\frac{(x - (j-1/m))^n}{n!}, \\ & \frac{j-1}{m} \le x < \frac{j}{m}, \end{aligned} \tag{13}$$

for which it is known that

$$\begin{gathered} \left| f_j(x) - \tilde{f}_j(x) \right| \le \left| f^{(n+1)}(\eta) \right| \frac{(x - (j-1/m))^{n+1}}{(n+1)!}, \\ \eta \in \left[\frac{j-1}{m}, \frac{j}{m}\right), \; j = 1, 2, \dots, m. \end{gathered} \tag{14}$$

Since $\mathbf{C}_j^T\mathbf{H}_j(x)$ is the best approximation to f_j form Y_j and $\tilde{f}_j \in Y_j$, using (14) we have

$$\begin{aligned}\left\|f_j - \mathbf{C}_j^T\mathbf{H}_j(x)\right\|_2^2 &\le \left\|f_j - \tilde{f}_j\right\|^2 \\ &= \int_{j-1/m}^{j/m} \left|f_j(x) - \tilde{f}_j(x)\right|^2 dx \\ &\le \int_{j-1/m}^{j/m} \left[\frac{f^{(n+1)}(\eta)\,(x-(j-1/m))^{n+1}}{(n+1)!}\right]^2 dx \\ &\le \left[\frac{\gamma}{(n+1)!}\right]^2 \int_{j-1/m}^{j/m} \left(x - \frac{j-1}{m}\right)^{2n+2} dx \\ &= \left[\frac{\gamma}{(n+1)!}\right]^2 \frac{1}{m^{2n+3}(2n+3)}.\end{aligned} \tag{15}$$

Now,

$$\begin{aligned}\left\|f - \mathbf{C}^T\mathbf{H}(x)\right\|_2^2 &\le \sum_{j=1}^{m} \left\|f_j - \mathbf{C}_j^T\mathbf{H}_j(x)\right\|_2^2 \\ &\le \frac{\gamma^2}{m^{2n+2}[(n+1)!]^2(2n+3)}.\end{aligned} \tag{16}$$

By taking the square roots we have the above bound. □

2.4. The Operational Matrix of Product. In this section, we present a general formula for finding the $m(n+1)\times m(n+1)$ operational matrix of product $\widetilde{\mathbf{C}}$ whenever

$$\mathbf{C}^T\mathbf{H}(x)\,\mathbf{H}^T(x) \approx \mathbf{H}^T(x)\,\widetilde{\mathbf{C}}, \tag{17}$$

where

$$\widetilde{\mathbf{C}} = \operatorname{diag}\left[\widetilde{\mathbf{C}}_1, \widetilde{\mathbf{C}}_2, \dots, \widetilde{\mathbf{C}}_j, \dots, \widetilde{\mathbf{C}}_m\right]. \tag{18}$$

In (18), $\widetilde{\mathbf{C}}_j = [c_{lr}^j]$ are $(n+1)\times(n+1)$ symmetric matrices depending on n, where

$$c_{lr}^j = \int_{j-1/m}^{j/m} \left(h_{j(l-1)}(x)\,h_{j(r-1)}(x)\sum_{i=0}^{n} c_{ji}h_{ji}(x)\right)dx, \qquad l,r = 1,2,\dots,n+1. \tag{19}$$

Furthermore, the integration of cross-product of two hybrid functions vectors is

$$\int_0^1 \mathbf{H}(x)\,\mathbf{H}^T(x)\,dx = \mathbf{I}, \tag{20}$$

where $\mathbf{I}$ is the $m(n+1)$ identity matrix.

2.5. The Operational Matrix of Differentiation. The operational matrix of derivative of the hybrid functions vector $\mathbf{H}(x)$ is defined by

$$\frac{d}{dx}\mathbf{H}(x) = \mathbf{D}\mathbf{H}(x), \tag{21}$$

where $\mathbf{D}$ is the $m(n+1)\times m(n+1)$ operational matrix of derivative given as

$$\begin{aligned}\mathbf{H}(x) &= \left[\mathbf{H}_1^T(x), \mathbf{H}_2^T(x), \dots, \mathbf{H}_j^T(x), \dots, \mathbf{H}_m^T(x)\right]^T \\ &= \widetilde{\mathbf{A}}\widetilde{\mathbf{T}}(x),\end{aligned} \tag{22}$$

where $\widetilde{\mathbf{A}} = \operatorname{diag}[\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_j, \dots, \mathbf{A}_m]$ is the $m(n+1)\times m(n+1)$ coefficient matrix of the $(n+1)\times(n+1)$ coefficient submatrix $\mathbf{A}_j$, and $\widetilde{\mathbf{T}}(x) = [\mathbf{t}_1(x), \mathbf{t}_2(x), \dots, \mathbf{t}_j(x), \dots, \mathbf{t}_m(x)]^T$ is the $m(n+1)$ vector with $\mathbf{t}_j(x) = [1, x, x^2, \dots, x^n]^T$, such that $\mathbf{H}_j(x) = \mathbf{A}_j\mathbf{t}_j(x)$. Now

$$\frac{d}{dx}\mathbf{H}(x) = \widetilde{\mathbf{A}}\widetilde{\mathbf{Q}}\widetilde{\mathbf{T}}(x) = \widetilde{\mathbf{A}}\widetilde{\mathbf{Q}}\widetilde{\mathbf{A}}^{-1}\mathbf{H}(x), \tag{23}$$

where $\widetilde{\mathbf{Q}} = \operatorname{diag}[\mathbf{Q}, \dots, \mathbf{Q}]$ is the $m(n+1)\times m(n+1)$ matrix of the $(n+1)\times(n+1)$ sub-matrix $\mathbf{Q}$, such that

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n & 0 \end{bmatrix}. \tag{24}$$

Hence,

$$\mathbf{D} = \widetilde{\mathbf{A}}\widetilde{\mathbf{Q}}\widetilde{\mathbf{A}}^{-1}. \tag{25}$$

In general, we can have

$$\frac{d^k}{dx^k}\mathbf{H}(x) = \mathbf{D}^k\mathbf{H}(x), \quad k = 1,2,3,\dots. \tag{26}$$

3. Outline of the Solution Method

This section presents the derivation of the method for solving sth-order nonlinear Fredholm integrodifferential equation (1) with the initial conditions (2).

Step 1. The functions $y^{(i)}(x)$, $i = 0,1,2,\dots,s$ are being approximated by

$$y^{(i)}(x) = \mathbf{C}^T(\mathbf{H}(x))^{(i)} = \mathbf{C}^T\mathbf{D}^i\mathbf{H}(x), \quad i = 0,1,2,\dots,s, \tag{27}$$

where $\mathbf{D}$ is given by (25).

Step 2. The function $k(x,t)$ is being approximated by (10).

Step 3. In this step, we present a general formula for approximate $y^q(x)$. By using (7) and (17), we can have

$$y^2(x) = \left[\mathbf{C}^T\mathbf{H}(x)\right]^2 = \mathbf{C}^T\mathbf{H}(x)\,\mathbf{H}^T(x)\,\mathbf{C} = \mathbf{H}^T(x)\,\widetilde{\mathbf{C}}\mathbf{C}, \tag{28}$$

$$\begin{aligned}y^3(x) &= \mathbf{C}^T\mathbf{H}(x)\left[\mathbf{C}^T\mathbf{H}(x)\right]^2 = \mathbf{C}^T\mathbf{H}(x)\,\mathbf{H}^T(x)\,\widetilde{\mathbf{C}}\mathbf{C} \\ &= \mathbf{H}^T(x)\,\widetilde{\mathbf{C}}\widetilde{\mathbf{C}}\mathbf{C} = \mathbf{H}^T(x)\left(\widetilde{\mathbf{C}}\right)^2\mathbf{C},\end{aligned} \tag{29}$$

and so by use of induction, $y^q(x)$ will be approximated as

$$y^q(x) = \mathbf{H}^T(x)\left(\widetilde{\mathbf{C}}\right)^{q-1}\mathbf{C}. \tag{30}$$

Table 1: Numerical comparison of absolute difference errors for Example 3.

x	Method of [17]	The proposed method	
	$n = 7$	$n = 2$, $m = 30$	$n = 3$, $m = 30$
0.0	$3.2038E - 009$	$3.1309E - 007$	$4.0173E - 010$
0.2	$7.1841E - 010$	$3.8241E - 007$	$4.9068E - 010$
0.4	$1.4151E - 010$	$4.6707E - 007$	$5.9932E - 010$
0.6	$4.0671E - 011$	$5.7048E - 007$	$7.3201E - 010$
0.8	$9.1044E - 010$	$6.9679E - 007$	$8.9407E - 010$
1.0	$3.7002E - 009$	$8.2709E - 007$	$1.4907E - 010$

Step 4. Approximate the functions $g(x)$ and $p_i(x)$ by

$$g(x) \approx \mathbf{G}^T \mathbf{H}(x), \tag{31}$$

$$p_i(x) \approx \mathbf{P}_i^T H(x), \quad i = 0, 1, 2, \ldots, s, \tag{32}$$

where $\mathbf{G}$ and $\mathbf{P}_i$ are constant coefficient vectors which are defined similarly to (7).

Now, using (27)–(32) and (10) to substitute into (1), we can obtain

$$\sum_{i=0}^{s} \mathbf{P}_i^T \mathbf{H}(x) \mathbf{H}^T(x) \left(\mathbf{D}^i\right)^T \mathbf{C} = \mathbf{H}^T(x) \mathbf{G} + \lambda \int_0^1 \mathbf{H}^T(x) \mathbf{K} \mathbf{H}(t) \mathbf{H}^T(t) \left(\widetilde{\mathbf{C}}\right)^{q-1} \mathbf{C} dt. \tag{33}$$

Utilizing (17) and (20), we may have

$$\sum_{i=0}^{s} \mathbf{H}^T(x) \widetilde{\mathbf{P}}_i \left(\mathbf{D}^i\right)^T \mathbf{C} = \mathbf{H}^T(x) \mathbf{G} + \lambda \mathbf{H}^T(x) \mathbf{K} \left(\widetilde{\mathbf{C}}\right)^{q-1} \mathbf{C}, \tag{34}$$

and hence we get

$$\sum_{i=0}^{s} \widetilde{\mathbf{P}}_i \left(\mathbf{D}^i\right)^T \mathbf{C} - \lambda \mathbf{K} \left(\widetilde{\mathbf{C}}\right)^{q-1} \mathbf{C} = \mathbf{G}. \tag{35}$$

The matrix (35) gives a system of $m(n+1)$ nonlinear algebraic equations which can be solved utilizing the initial condition for the elements of $\mathbf{C}$. Once $\mathbf{C}$ is known, $y(x)$ can be constructed by using (7).

4. Applications and Numerical Results

In this section, numerical results of some examples are presented to validate accuracy, applicability, and convergence of the proposed method. Absolute difference errors of this method is compared with the existing methods reported in the literature [5, 6, 17, 18]. The computations associated with these examples were performed using MATLAB 9.0.

Example 1. Consider the first-order nonlinear Fredholm integrodifferential equation [17, 18] as follows:

$$y'(x) = 1 - \frac{1}{3}x + \int_0^1 x\, y^2(t)\, dt, \quad 0 \le x < 1, \tag{36}$$

with the initial condition

$$y(0) = 0. \tag{37}$$

In this example, we have $p_0 = 0$, $p_1 = 1$, $g(x) = 1 - (1/3)x$, $\lambda = 1$, $k(x,t) = x$, and $q = 2$.

The matrix (35) for this example is

$$\widetilde{\mathbf{P}}_1 \mathbf{D}^T \mathbf{C} - \mathbf{K} \left(\widetilde{\mathbf{C}}\right) \mathbf{C} = \mathbf{G}, \tag{38}$$

where for $n = 1$ and $m = 2$ we have

$$\widetilde{\mathbf{P}}_1 = \mathbf{I}, \qquad \mathbf{D}^T = \begin{bmatrix} -3 & 3\sqrt{3} & 0 & 0 \\ -\sqrt{3} & 3 & 0 & 0 \\ 0 & 0 & -3 & 3\sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} c_{10} \\ c_{11} \\ c_{20} \\ c_{21} \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} \frac{1}{16} & \frac{\sqrt{3}}{48} & \frac{1}{16} & \frac{\sqrt{3}}{48} \\ \frac{\sqrt{3}}{16} & \frac{1}{16} & \frac{\sqrt{3}}{16} & \frac{1}{16} \\ \frac{1}{4} & \frac{\sqrt{3}}{12} & \frac{1}{4} & \frac{\sqrt{3}}{12} \\ \frac{\sqrt{3}}{8} & \frac{1}{8} & \frac{\sqrt{3}}{8} & \frac{1}{8} \end{bmatrix},$$

$$\widetilde{\mathbf{C}} = \frac{1}{4} \begin{bmatrix} 3\sqrt{6}c_{10} - \sqrt{2}c_{11} & -\sqrt{2}c_{10} + \sqrt{6}c_{11} & 0 & 0 \\ -\sqrt{2}c_{10} + \sqrt{6}c_{11} & \sqrt{6}c_{10} + 5\sqrt{2}c_{11} & 0 & 0 \\ 0 & 0 & 3\sqrt{6}c_{20} - \sqrt{2}c_{21} & -\sqrt{2}c_{20} + \sqrt{6}c_{21} \\ 0 & 0 & -\sqrt{2}c_{20} + \sqrt{6}c_{21} & \sqrt{6}c_{20} + 5\sqrt{2}c_{21} \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} \dfrac{17\sqrt{6}}{72} \\ \dfrac{5\sqrt{2}}{24} \\ \dfrac{7\sqrt{6}}{36} \\ \dfrac{\sqrt{2}}{6} \end{bmatrix}. \tag{39}$$

Equation (38) gives a system of nonlinear algebraic equations that can be solved utilizing the initial condition (37); that is, $\sqrt{6}c_{10} - \sqrt{2}c_{11} = 0$, we obtain

$$c_{10} = \frac{\sqrt{6}}{24}, \quad c_{11} = \frac{\sqrt{2}}{8}, \qquad c_{20} = \frac{\sqrt{6}}{6}, \quad c_{21} = \frac{\sqrt{2}}{4}. \tag{40}$$

Substituting these values into (7), the result will be $y(x) = x$, that is, the exact solution. It is noted that the result gives the exact solution as in [17], while in [18] using the sinc method the maximum absolute error is 1.52165×10^{-3}.

Example 2. Consider the first-order nonlinear Fredholm integrodifferential equation [6, 17] as follows:

$$xy'(x) - y(x) = -\frac{1}{6} + \frac{4}{5}x^2 + \int_0^1 \left(x^2 + t\right) y^2(t)\, dt, \quad 0 \le x < 1. \tag{41}$$

with the initial condition

$$y(0) = 0. \tag{42}$$

In this example, we have $p_0 = -1$, $p_1 = x$, $g(x) = -(1/6) + (4/5)x^2$, $\lambda = 1$, $k(x,t) = x^2 + t$, and $q = 2$.

The matrix (35) for this example is

$$\left(\widetilde{\mathbf{P}}_0 + \widetilde{\mathbf{P}}_1 \mathbf{D}^T\right)\mathbf{C} - \mathbf{K}\left(\widetilde{\mathbf{C}}\right)\mathbf{C} = \mathbf{G}, \tag{43}$$

where for $n = 2$ and $m = 2$ we have

$$\widetilde{\mathbf{P}}_0 = -\mathbf{I}, \qquad \widetilde{\mathbf{P}}_1 = \begin{bmatrix} \dfrac{1}{12} & \dfrac{\sqrt{15}}{60} & \dfrac{-\sqrt{5}}{120} & 0 & 0 & 0 \\ \dfrac{\sqrt{15}}{60} & \dfrac{1}{4} & \dfrac{\sqrt{3}}{24} & 0 & 0 & 0 \\ \dfrac{-\sqrt{5}}{120} & \dfrac{\sqrt{3}}{24} & \dfrac{5}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{7}{12} & \dfrac{\sqrt{15}}{60} & \dfrac{-\sqrt{5}}{120} \\ 0 & 0 & 0 & \dfrac{\sqrt{15}}{60} & \dfrac{3}{4} & \dfrac{\sqrt{3}}{24} \\ 0 & 0 & 0 & \dfrac{-\sqrt{5}}{120} & \dfrac{\sqrt{3}}{24} & \dfrac{11}{12} \end{bmatrix},$$

$$\mathbf{D}^T = \begin{bmatrix} -5 & \dfrac{7\sqrt{15}}{3} & -2\sqrt{5} & 0 & 0 & 0 \\ \dfrac{-\sqrt{15}}{3} & -3 & \dfrac{14\sqrt{3}}{3} & 0 & 0 & 0 \\ 0 & \dfrac{-8\sqrt{3}}{3} & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & \dfrac{7\sqrt{15}}{3} & -2\sqrt{5} \\ 0 & 0 & 0 & \dfrac{-\sqrt{15}}{3} & -3 & \dfrac{14\sqrt{3}}{3} \\ 0 & 0 & 0 & 0 & \dfrac{-8\sqrt{3}}{3} & 8 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} c_{10} \\ c_{11} \\ c_{12} \\ c_{20} \\ c_{21} \\ c_{22} \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} -\dfrac{11\sqrt{10}}{450} \\ \dfrac{-\sqrt{6}}{90} \\ \dfrac{\sqrt{2}}{180} \\ \dfrac{23\sqrt{10}}{900} \\ \dfrac{13\sqrt{6}}{180} \\ \dfrac{19\sqrt{2}}{180} \end{bmatrix},$$

$$\mathbf{K}=\begin{bmatrix}\frac{1}{24} & \frac{\sqrt{15}}{45} & \frac{7\sqrt{5}}{240} & \frac{13}{72} & \frac{\sqrt{15}}{20} & \frac{41\sqrt{5}}{720}\\ \frac{\sqrt{15}}{72} & \frac{1}{12} & \frac{5\sqrt{3}}{144} & \frac{\sqrt{15}}{24} & \frac{1}{16} & \frac{\sqrt{3}}{16}\\ \frac{\sqrt{5}}{48} & \frac{5\sqrt{3}}{144} & \frac{1}{24} & \frac{7\sqrt{5}}{144} & \frac{\sqrt{3}}{16} & \frac{5}{72}\\ \frac{7}{48} & \frac{31\sqrt{15}}{720} & \frac{\sqrt{15}}{20} & \frac{41}{144} & \frac{17\sqrt{15}}{240} & \frac{7\sqrt{5}}{90}\\ \frac{7\sqrt{15}}{144} & \frac{3}{16} & \frac{5\sqrt{3}}{72} & \frac{11\sqrt{15}}{144} & \frac{13}{48} & \frac{7\sqrt{3}}{72}\\ \frac{\sqrt{5}}{16} & \frac{11\sqrt{3}}{144} & \frac{1}{12} & \frac{13\sqrt{5}}{144} & \frac{5\sqrt{3}}{48} & \frac{1}{9}\end{bmatrix},\qquad \widetilde{\mathbf{C}}=\begin{bmatrix}\widetilde{\mathbf{c}}_1 & \mathbf{0}\\ \mathbf{0} & \widetilde{\mathbf{c}}_2\end{bmatrix},$$

$$\widetilde{\mathbf{c}}_j=\begin{bmatrix}\frac{5\sqrt{10}}{7}c_{j0}-\frac{5\sqrt{6}}{21}c_{j1}+\frac{\sqrt{2}}{7}c_{j2} & -\frac{5\sqrt{6}}{21}c_{j0}+\frac{11\sqrt{10}}{35}c_{j1}-\frac{8\sqrt{30}}{105}c_{j2} & \frac{\sqrt{2}}{7}c_{j0}-\frac{8\sqrt{30}}{105}c_{j1}+\frac{3\sqrt{10}}{35}c_{j2}\\ -\frac{5\sqrt{6}}{21}c_{j0}+\frac{11\sqrt{10}}{35}c_{j1}-\frac{8\sqrt{30}}{105}c_{j2} & \frac{11\sqrt{10}}{35}c_{j0}+\frac{3\sqrt{6}}{7}c_{j1}+\frac{\sqrt{2}}{7}c_{j2} & -\frac{8\sqrt{30}}{105}c_{j0}+\frac{\sqrt{2}}{7}c_{j1}+\frac{5\sqrt{6}}{21}c_{j2}\\ \frac{\sqrt{2}}{7}c_{j0}-\frac{8\sqrt{30}}{105}c_{j1}+\frac{3\sqrt{10}}{35}c_{j2} & -\frac{8\sqrt{30}}{105}c_{j0}+\frac{\sqrt{2}}{7}c_{j1}+\frac{5\sqrt{6}}{21}c_{j2} & \frac{3\sqrt{10}}{35}c_{j0}+\frac{5\sqrt{6}}{21}c_{j1}+\frac{13\sqrt{2}}{7}c_{j2}\end{bmatrix},$$

$$j=1,2. \tag{44}$$

Equation (43) gives a system of nonlinear algebraic equations that can be solved utilizing the initial condition (42); that is, $\sqrt{10}c_{10}-\sqrt{6}c_{11}+\sqrt{2}c_{12}=0$, we obtain

$$c_{10}=\frac{\sqrt{10}}{240},\quad c_{11}=\frac{\sqrt{6}}{48},$$
$$c_{12}=\frac{\sqrt{2}}{24},\quad c_{20}=\frac{\sqrt{10}}{15}, \tag{45}$$
$$c_{21}=\frac{\sqrt{6}}{8},\quad c_{22}=\frac{\sqrt{2}}{6}.$$

Substituting these values into (7), the result will be $y(x)=x^2$, that is, the exact solution. It is noted that the result gives the exact solution as in [17], while in [6] approximate solution is obtained with maximum absolute error 1.0000×10^{-5}.

Example 3. Consider the second-order nonlinear Fredholm integrodifferential equation [17] as follows:

$$y''(x)+xy'(x)-xy(x)=e^{x}\sin x+\int_0^1\sin x\cdot e^{-2t}y^2(t)\,dt,\qquad 0\le x<1, \tag{46}$$

with the initial conditions

$$y(0)=y'(0)=1. \tag{47}$$

The exact solution is $y(x)=e^{x}$. We solve this example by using the proposed method with $n=2$, $m=30$ and $n=3$, $m=30$. Comparison among the proposed method and methods in [17] is shown in Table 1. It is clear from this table that the results obtained by the proposed method, using few numbers of basis, are very promising and superior to that of [17].

Example 4. Consider the following nonlinear Fredholm integrodifferential equation [5, 17]:

$$y'(x)+y(x)=\frac{1}{2}\left(e^{-2}-1\right)+\int_0^1 y^2(t)\,dt,\qquad 0\le x<1, \tag{48}$$

with the initial conditions

$$y(0)=1. \tag{49}$$

The exact solution of this problem is $y(x)=e^{-x}$. In Table 2 we have compared the absolute difference errors of the proposed method with the collocation method based on Haar wavelets in [5] and method in [17].

Maximum absolute errors of Example 4 for some different values of n and m are shown in Table 3. As it is seen from Table 3, for a certain value of n as m increases the accuracy increases, and for a certain value of m as n increases the accuracy increases as well. In case of $m=1$, the numerical solution obtained is based on orthonormal Bernstein polynomials only, while in case of $n=0$, the numerical solution obtained is based on block-pulse functions only.

Table 2: Numerical comparison of absolute difference errors for Example 4.

x	Method of [5] Number of collocation points $N = 128$	Method of [17] $n = 7$	The proposed method $n = 3$, $m = 35$	$n = 4$, $m = 15$
0.125	$3.7591E-007$	$2.4509E-010$	$5.5200E-011$	$1.6710E-011$
0.250	$6.6413E-007$	$1.0202E-010$	$8.9982E-011$	$3.9705E-012$
0.375	$8.6917E-007$	$1.6139E-010$	$9.4606E-011$	$1.2126E-011$
0.500	$1.0020E-006$	$3.2362E-010$	$9.2457E-011$	$1.8312E-012$
0.625	$1.0757E-006$	$1.9197E-010$	$7.4991E-011$	$8.1299E-012$
0.750	$1.1029E-006$	$6.6120E-011$	$4.9442E-011$	$7.7237E-012$
0.875	$1.0944E-006$	$2.2417E-010$	$2.6083E-011$	$2.5547E-012$

Table 3: Maximum absolute errors for different values of n and m for Example 4.

n	m = 1	5	10	15	20	25	30	35
0	$5.7735E-01$	$1.1547E-01$	$5.7735E-02$	$3.8490E-02$	$2.8868E-02$	$2.3094E-02$	$1.9245E-02$	$1.6496E-02$
1	$2.2361E-01$	$8.9443E-03$	$2.2361E-03$	$9.9381E-04$	$5.5902E-04$	$3.5777E-04$	$2.4845E-04$	$1.8254E-04$
2	$6.2994E-02$	$5.0395E-04$	$6.2994E-05$	$1.8665E-05$	$7.8743E-06$	$4.0316E-06$	$2.3331E-06$	$1.4693E-06$
3	$1.3889E-02$	$2.2222E-05$	$1.3889E-06$	$2.7435E-07$	$8.6806E-08$	$3.5556E-08$	$1.7147E-08$	$9.2554E-09$
4	$2.5126E-03$	$8.0403E-07$	$2.5126E-08$	$3.3088E-09$	$7.8519E-10$	$2.5729E-10$	$1.0340E-10$	$4.7839E-11$
5	$3.8521E-04$	$2.4653E-08$	$3.8521E-10$	$3.3818E-11$	$6.0189E-12$	$1.5778E-12$	$5.2841E-13$	$2.0955E-13$
6	$5.1230E-05$	$6.5574E-10$	$5.1230E-12$	$2.9984E-13$	$4.0023E-14$	$8.3935E-15$	$2.3425E-15$	$7.9625E-16$

5. Conclusion

In this work, we present a numerical method for solving nonlinear Fredholm integrodifferential equations based on hybrid of block-pulse functions and normalized Bernstein polynomials. One of the most important properties of this method is obtaining the analytical solutions if the equation has an exact solution, that is, a polynomial function. Another considerable advantage is this method has high relative accuracy for small numbers of basis n. The matrices $\mathbf{K}$, $\widetilde{\mathbf{C}}$, and $\mathbf{D}$ in (10), (17), and (25), respectively, have large numbers of zero elements, and they are sparse; hence, the present method is very attractive and reduces the CPU time and computer memory. Moreover, satisfactory results of illustrative examples with respect to several other methods (e.g., Haar wavelets method, Walsh functions method, Bernstein polynomials method, and sinc collocation method) are included to demonstrate the validity and applicability of the proposed method.

References

[1] M. Dehghan and A. Saadatmandi, "Chebyshev finite difference method for Fredholm integro-differential equation," *International Journal of Computer Mathematics*, vol. 85, no. 1, pp. 123–130, 2008.

[2] Z. Lackiewicz, M. Rahman, and B. D. Welfert, "Numerical solution of a Fredholm integro-differential equation modelling neural networks," *Applied Numerical Mathematics*, vol. 56, no. 3-4, pp. 423–432, 2006.

[3] M. Lakestani, M. Razzaghi, and M. Dehghan, "Semiorthogonal spline wavelets approximation for Fredholm integro-differential equations," *Mathematical Problems in Engineering*, vol. 2006, Article ID 96184, 12 pages, 2006.

[4] S. H. Behiry and H. Hashish, "Wavelet methods for the numerical solution of Fredholm integro-differential equations," *International Journal of Applied Mathematics*, vol. 11, no. 1, pp. 27–35, 2002.

[5] S. Islam, I. Aziz, and M. Fayyaz, "A new approach for numerical solution of integro-differential equations via Haar wavelets," *International Journal of Computer Mathematics*, 2013.

[6] Y. Ordokhani, "An application of Walsh functions for Fredholm-Hammerstein integro-differential equations," *International Journal of Contemporary Mathematical Sciences*, vol. 5, no. 22, pp. 1055–1063, 2010.

[7] S. Yeganeh, Y. Ordokhani, and A. Saadatmandi, "A sinc-collocation method for second-order boundary value problems of nonlinear integro-differential equation," *Journal of Information and Computing Science*, vol. 7, no. 2, pp. 151–160, 2012.

[8] Sh. S. Behzadi, S. Abbasbandy, T. Allahviranloo, and A. Yildirim, "Application of homotopy analysis method for solving a class of nonlinear Volterra-Fredholm integro-differential equations," *The Journal of Applied Analysis and Computation*, vol. 2, no. 2, pp. 127–136, 2012.

[9] S. H. Behiry and S. I. Mohamed, "Solving high-order nonlinear Volterra-Fredholm integro-differential equations by differential transform method," *Natural Science*, vol. 4, no. 8, pp. 581–587, 2012.

[10] K. Maleknejad, B. Basirat, and E. Hashemizadeh, "Hybrid Legendre polynomials and block-pulse functions approach for nonlinear Volterra-Fredholm integro-differential equations," *Computers & Mathematics with Applications*, vol. 61, no. 9, pp. 2821–2828, 2011.

[11] R. Ezzati and S. Najafalizadeh, "Application of Chebyshev polynomials for solving nonlinear Volterra-Fredholm integral equations system and convergence analysis," *Indian Journal of Science and Technology*, vol. 5, no. 2, pp. 2060–2064, 2012.

[12] A. H. Bhrawy, E. Tohidi, and F. Soleymani, "A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals," *Applied Mathematics and Computation*, vol. 219, no. 2, pp. 482–497, 2012.

[13] Z. H. Jiang and W. Schaufelberger, *Block Pulse Functions and Their Applications in Control Systems*, Springer, Berlin, Germany, 1992.

[14] S. A. Yousefi, Z. Barikbin, and M. Dehghan, "Ritz-Galerkin method with Bernstein polynomial basis for finding the product solution form of heat equation with non-classic boundary conditions," *International Journal of Numerical Methods for Heat & Fluid Flow*, vol. 22, no. 1, pp. 39–48, 2012.

[15] K. Maleknejad, E. Hashemizadeh, and B. Basirat, "Computational method based on Bernstein operational matrices for nonlinear Volterra-Fredholm-Hammerstein integral equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 1, pp. 52–61, 2012.

[16] G. Tachev, "Pointwise approximation by Bernstein polynomials," *Bulletin of the Australian Mathematical Society*, vol. 85, no. 3, pp. 353–358, 2012.

[17] Y. Ordokhani and S. Davaei far, "Application of the Bernstein polynomials for solving the nonlinear Fredholm integro-differential equations," *Journal of Applied Mathematics and Bioinformatics*, vol. 1, no. 2, pp. 13–31, 2011.

[18] K. Jalaei, M. Zarebnia, and M. M. Chalaki, "Development of the sinc method for nonlinear integro-differential eequations," *Australian Journal of Basic and Applied Sciences*, vol. 4, no. 11, pp. 5508–5515, 2010.