Retraction

Retracted: Numerical Solution of Nonlinear Fredholm Integrodifferential Equations by Hybrid of Block-Pulse Functions and Normalized Bernstein Polynomials

Abstract and Applied Analysis

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The paper titled “Numerical Solution of Nonlinear Fredholm Integrodifferential Equations by Hybrid of Block-Pulse Functions and Normalized Bernstein Polynomials” [1], published in Abstract and Applied Analysis, has been retracted as an almost identical paper by the same author has been simultaneously submitted to and published in Journal of Computational and Applied Mathematics. The other publication is “Solution of nonlinear Fredholm integrodifferential equations using a hybrid of block pulse functions and normalized Bernstein polynomials,” Volume 260, April 2014, Pages 258–265, DOI: 10.1016/j.cam.2013.09.036.

References

Research Article

Numerical Solution of Nonlinear Fredholm Integrodifferential Equations by Hybrid of Block-Pulse Functions and Normalized Bernstein Polynomials

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A numerical method for solving nonlinear Fredholm integrodifferential equations is proposed. The method is based on hybrid functions approximate. The properties of hybrid of block pulse functions and orthonormal Bernstein polynomials are presented and utilized to reduce the problem to the solution of nonlinear algebraic equations. Numerical examples are introduced to illustrate the effectiveness and simplicity of the present method.

1. Introduction

Integrodifferential equations are often involved in mathematical formulation of physical phenomena. Fredholm integrodifferential equations play an important role in many fields such as economics, biomechanics, control, elasticity, fluid dynamics, heat and mass transfer, oscillation theory, and airfoil theory; for examples see [1–3] and references cited therein. Finding numerical solutions for Fredholm integrodifferential equations is one of the oldest problems in applied mathematics. Numerous works have been focusing on the development of more advanced and efficient methods for solving integrodifferential equations such as wavelets method [4, 5], Walsh functions method [6], sinc-collocation method [7], homotopy analysis method [8], differential transform method [9], the hybrid Legendre polynomials and block-pulse functions [10], Chebyshev polynomials method [11], and Bernoulli matrix method [12].

Block-pulse functions have been studied and applied extensively as a basic set of functions for signals and functions approximations. All these studies and applications show that block-pulse functions have definite advantages for solving problems involving integrals and derivatives due to their clearness in expressions and their simplicity in formulations; see [13]. Also, Bernstein polynomials play a prominent role in various areas of mathematics. Many authors have used these polynomials in the solution of integral equations, differential equations, and approximation theory; see for instance [14–17].

The purpose of this work is to utilize the hybrid functions consisting of combination of block-pulse functions with normalized Bernstein polynomials for obtaining numerical solution of nonlinear Fredholm integrodifferential equation:

$$\sum_{i=0}^{s} p_i(x) y^{(i)}(x) = g(x) + \lambda \int_{0}^{1} k(x, t) \left[ y(t) \right]^q dt, \quad 0 \leq x, \ t < 1,$$

with the conditions

$$y^{(i)}(0) = \alpha_i, \quad 0 \leq i \leq s - 1,$$

where $y^{(i)}(x)$ is the $i$th derivative of the unknown function that will be determined, $k(x, t)$ is the kernel of the integral, $g(x)$ and $p_i(x)$ are known analytic functions, $q$ is a positive integer, and $\lambda$ and $\alpha_i$ are suitable constants. The proposed approach for solving this problem uses few numbers of basis and benefits of the orthogonality of block-pulse functions and the advantages of orthonormal Bernstein polynomials properties to reduce the nonlinear integrodifferential equation to easily solvable nonlinear algebraic equations.
This paper is organized as follows. In the next section, we present Bernstein polynomials and hybrid of block-pulse functions. Also, their useful properties such as functions approximation, convergence analysis, operational matrix of product, and operational matrix of differentiation are given. In Section 3, the numerical scheme for the solution of (1) and (2) is described. In Section 4, the proposed method is applied to some nonlinear Fredholm integrodifferential equations, and comparisons are mad with the existing analytic or numerical solutions that were reported in other published works in the literature. Finally conclusions are given in Section 5.

2. Properties of Hybrid Functions

2.1. Hybrid of Block-Pulse Functions and Orthonormal Bernstein Polynomials. The Bernstein polynomials of nth degree are defined on the interval \([0,1)\) as \(B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}\), for \(i = 0, 1, 2, \ldots, n\), (3) where

\[
\binom{n}{i} = \frac{n!}{i! (n-i)!}.
\]

There are \((n+1)\) \(n\)th degree Bernstein polynomials. Using Gram-Schmidt orthonormalization process on \(B_{i,n}(x)\), we obtain a class of orthonormal polynomials from the Bernstein polynomials. We call them orthonormal Bernstein polynomials of degree \(n\) and denote them by \(b_{i,n}(x)\), \(0 \leq i \leq n\). For \(n = 3\), the four orthonormal Bernstein polynomials are given by

\[
\begin{align*}
B_{0,3}(x) &= -\sqrt{7} \left[ x^3 - 3x^2 + 3x - 1 \right], \\
B_{1,3}(x) &= \sqrt{5} \left[ 7x^3 - 15x^2 + 9x - 1 \right], \\
B_{2,3}(x) &= -\sqrt{3} \left[ 21x^3 - 33x^2 + 13x - 1 \right], \\
B_{3,3}(x) &= 35x^3 - 45x^2 + 15x - 1.
\end{align*}
\]

(5)

Hybrid functions \(h_{ji}(x), j = 1, 2, \ldots, m\) and \(i = 0, 1, \ldots, n\) are defined on the interval \([0,1)\) as

\[
h_{ji}(x) = \begin{cases} 
\sqrt{m} \, b_{i,n}(mx - j + 1), & j \leq x < j + \frac{1}{m}, \\
0, & \text{otherwise},
\end{cases}
\]

(6)

where \(j\) and \(n\) are the order of block-pulse functions and degree of orthonormal Bernstein polynomials, respectively.

It is clear that these sets of hybrid functions in (6) are orthonormal and disjoint.

2.2. Functions Approximation. A function \(y(x) \in L^2[0,1)\) may be approximated as

\[
y(x) = \sum_{j=0}^{m} \sum_{i=0}^{n} c_{ji} h_{ji}(x) = C^T H(x),
\]

(7)

where

\[
C = [C_1^T, C_2^T, \ldots, C_m^T]^T,
\]

\[
C_j = [c_{j,0}, c_{j,1}, c_{j,2}, \ldots, c_{j,n}]^T,
\]

and \(H(x) = [H_1^T(x), H_2^T(x), \ldots, H_m^T(x)]^T\).

2.2. Functions Approximation. A function \(y(x) \in L^2[0,1)\) may be approximated as

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\]

\[
C_j = [c_{j,0}, c_{j,1}, c_{j,2}, \ldots, c_{j,n}]^T,
\]

and \(H(x) = [H_1^T(x), H_2^T(x), \ldots, H_m^T(x)]^T\).

(9)

and \(H_j(x) = [h_{j0}(x), h_{j1}(x), \ldots, h_{jm}(x)]^T, j = 1, 2, \ldots, m\). The constant coefficients \(c_{ji}\) are \(c_{ji} = (y(x), h_{ji}(x)), i = 0, 1, 2, \ldots, n, j = 1, 2, \ldots, m\), and \((\cdot, \cdot)\) is the standard inner product on \(L^2[0,1)\).

We can also approximate the function \(k(x, t) \in L^2((0,1)\times (0,1))\) by

\[
k(x, t) = \sum_{j=1}^{m} \sum_{i=0}^{n} \sum_{r=0}^{n} k_{ijr} h_{ij}(x) h_{ir}(t) = H^T(x) K H(t),
\]

(10)

where \(K = [K_{ijr}]\) is an \((m+1) \times (n+1)\) matrix, such that the elements of the sub matrix \(k_{ijr}\) are

\[
k_{ijr} = \int_{j-1/m}^{j/m} \int_{i-1/m}^{i/m} k(x,t) h_{ij}(x) h_{ir}(t) \, dx \, dt,
\]

(11)

utilizing properties of block-pulse function and orthonormal Bernstein polynomials.

2.3. Convergence Analysis. In this section, the error bound and convergence is established by the following lemma.

Lemma 1. Suppose that \(f \in C^{n+1}((0,1))\) is \(n + 1\) times continuously differentiable function such that \(f = \sum_{i=0}^{m} f_i\) and let \(Y_f = \text{Span}\{h_{j0}(x), h_{j1}(x), \ldots, h_{jm}(x)\}, j = 1, 2, \ldots, m\). If \(C_{j}^T H_j(x)\) is the best approximation to \(f_j\) from \(Y_f\), then \(C_{j}^T H_j(x)\) approximates \(f\) with the following error bound:

\[
\|f - C^T H(x)\|_2 \leq \frac{\gamma}{m^{n+1} (n+1)! \sqrt{2n+3}},
\]

(12)

\[
\gamma = \max_{x \in (0,1)} \left| f^{(n+1)}(x) \right|.
\]

Proof. The Taylor expansion for the function \(f_j(x)\) is

\[
f_j(x) = f_j \left( \frac{j - 1}{m} \right) + f_j' \left( \frac{j - 1}{m} \right) \left( x - \frac{j - 1}{m} \right) + \cdots + f_j^{(n)} \left( \frac{j - 1}{m} \right) \left( x - \frac{j - 1}{m} \right)^n,
\]

(13)

for which it is known that

\[
\left| f_j(x) - f_j(x) \right| \leq \left| f^{(n+1)}(x) \right| \frac{(x - (j - 1/m))^{n+1}}{(n+1)!},
\]

(14)

\[
\eta \in \left[ \frac{j - 1}{m}, \frac{j}{m} \right], j = 1, 2, \ldots, m.
\]
Since $C_j^T H_j(x)$ is the best approximation to $f_j$ form $Y_j$ and $\tilde{f}_j \in Y_j$, using (14) we have
\[
\left\| f_j - C_j^T H_j(x) \right\|_2^2 \leq \left\| f_j - \tilde{f}_j \right\|_2^2 = \int_{j-1/m}^{j/m} \left[ f_j(x) - \tilde{f}_j(x) \right]^2 dx \\
\leq \int_{j-1/m}^{j/m} \left[ f^{(m)}(x) \right]^2 dx \leq \left[ \frac{y}{(n+1)!} \right]^2 \int_{j-1/m}^{j/m} \left( x - j - 1 \right)^{2n+2} dx \\
= \left[ \frac{y}{(n+1)!} \right]^2 \frac{1}{m^{2n+3} (2n+3)}. 
\]

Now,
\[
\left\| f - C^T H(x) \right\|_2^2 \leq \sum_{j=1}^m \left\| f_j - C_j^T H_j(x) \right\|_2^2 \\
\leq m^{2n+2} \frac{(n+1)!^2}{(2n+3)}. \tag{15}
\]
By taking the square roots we have the above bound. $\square$

2.4. The Operational Matrix of Product. In this section, we present a general formula for finding the $m(n+1) \times m(n+1)$ operational matrix of product $C$ whenever
\[
C^T H(x) H^T(x) = H^T(x) C, \tag{17}
\]
where
\[
C = \text{diag} \left[ C_1, C_2, \ldots, C_m \right]. \tag{18}
\]
In (18), $C_j = [c_{ij}]$ are $(n+1) \times (n+1)$ symmetric matrices depending on $n$, where
\[
c_{ir} = \int_{j-1/m}^{j/m} \left( \sum_{i=0}^n c_i h_{ij}(x) \right) dx, \quad l, r = 1, 2, \ldots, n+1. \tag{19}
\]
Furthermore, the integration of cross-product of two hybrid functions vectors is
\[
\int_0^1 H(x) H^T(x) dx = I, \tag{20}
\]
where $I$ is the $m(n+1)$ identity matrix.

2.5. The Operational Matrix of Differentiation. The operational matrix of derivative of the hybrid functions vector $H(x)$ is defined by
\[
\frac{d}{dx} H(x) = D H(x), \tag{21}
\]
where $D$ is the $m(n+1) \times m(n+1)$ operational matrix of derivative given as
\[
H(x) = \left[ H_1^T(x), H_2^T(x), \ldots, H_j^T(x), \ldots, H_m^T(x) \right]^T
\]
where $\tilde{A} = \text{diag} [A_1, A_2, \ldots, A_{n+1}]$ is the $m(n+1) \times m(n+1)$ coefficient matrix of the $(n+1) \times (n+1)$ coefficient submatrix $A_j$, and $\tilde{T}(x) = [t_1(x), t_2(x), \ldots, t_m(x)]^T$ is the $m(n+1)$ vector with $t_j(x) = [1, x, x^2, \ldots, x^m]^T$, such that $H_j(x) = A_j t_j(x)$. Now
\[
\frac{d}{dx} H(x) = \tilde{A} Q \tilde{T}(x) = \tilde{A} Q \tilde{A}^{-1} H(x), \tag{23}
\]
where $\tilde{Q} = \text{diag} \{ Q_1, Q_2, \ldots, Q_m \}$ is the $m(n+1) \times m(n+1)$ matrix of the $(n+1) \times (n+1)$ sub-matrix $Q$, such that
\[
Q = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
& & \ddots & \vdots \\
& & & 0 & 0 & \cdots & 0
\end{bmatrix}. \tag{24}
\]
Hence,
\[
D = \tilde{A} Q \tilde{A}^{-1}. \tag{25}
\]
In general, we can have
\[
\frac{d^k}{dx^k} H(x) = D^k H(x), \quad k = 1, 2, 3, \ldots. \tag{26}
\]

3. Outline of the Solution Method

This section presents the derivation of the method for solving $s^{th}$-order nonlinear Fredholm integro-differential equation (1) with the initial conditions (2).

Step 1. The functions $y^{(i)}(x), i = 0, 1, 2, \ldots, s$ are being approximated by
\[
y^{(i)}(x) = C^T H(x) y^{(i)} = C^T D^i H(x), \quad i = 0, 1, 2, \ldots, s, \tag{27}
\]
where $D$ is given by (25).

Step 2. The function $k(x, t)$ is being approximated by (10).

Step 3. In this step, we present a general formula for approximate $y^{(i)}(x)$. By using (7) and (17), we can have
\[
y^{(2)}(x) = \left[ C^T H(x) \right]^2 = C^T H(x) H^T(x) C = H^T(x) \tilde{C} C, \tag{28}
\]
\[
y^{(3)}(x) = C^T H(x) \left[ C^T H(x) \right]^2 = C^T H(x) H^T(x) \tilde{C} C
\]
\[
= H^T(x) \tilde{C} C \tilde{C} C = H^T(x) \left( \tilde{C}^2 \right) C, \tag{29}
\]
and so by use of induction, $y^{(i)}(x)$ will be approximated as
\[
y^{(i)}(x) = H^T(x) \left( \tilde{C}^i \right) C. \tag{30}
\]
Table 1: Numerical comparison of absolute difference errors for Example 3.

<table>
<thead>
<tr>
<th>x</th>
<th>Method of [17]</th>
<th>The proposed method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n = 7$</td>
<td>$n = 2, m = 30$</td>
</tr>
<tr>
<td>0.0</td>
<td>$3.2038E - 009$</td>
<td>$3.1309E - 007$</td>
</tr>
<tr>
<td>0.2</td>
<td>$7.1841E - 010$</td>
<td>$3.8241E - 007$</td>
</tr>
<tr>
<td>0.4</td>
<td>$1.4151E - 010$</td>
<td>$4.6707E - 007$</td>
</tr>
<tr>
<td>0.6</td>
<td>$4.0671E - 011$</td>
<td>$5.7048E - 007$</td>
</tr>
<tr>
<td>0.8</td>
<td>$9.1044E - 010$</td>
<td>$6.9679E - 007$</td>
</tr>
<tr>
<td>1.0</td>
<td>$3.7002E - 009$</td>
<td>$8.2709E - 007$</td>
</tr>
</tbody>
</table>

Step 4. Approximate the functions $g(x)$ and $p_i(x)$ by

- $g(x) \approx G^T H(x)$,  \( i = 0, 1, 2, \ldots, s \) \hspace{1cm} (31)
- $p_i(x) \approx P_i^T H(x)$,  \( i = 0, 1, 2, \ldots, s \) \hspace{1cm} (32)

where $G$ and $P_i$ are constant coefficient vectors which are defined similarly to (7).

Now, using (27)–(32) and (10) to substitute into (1), we can obtain

$$
\sum_{i=0}^{s} P_i^T H(x) H^T(x) (D^T)^T C = H^T(x) G + \lambda \int_{0}^{1} H^T(x) K H(t) H^T(t) (C)_{q-1} C dt.
$$

(33)

Utilizing (17) and (20), we may have

$$
\sum_{i=0}^{s} H^T(x) P_i (D^T)^T C = H^T(x) G + \lambda H^T(x) K (C)_{q-1} C,
$$

(34)

and hence we get

$$
\sum_{i=0}^{s} P_i (D^T)^T C - \lambda K (C)_{q-1} C = G.
$$

(35)

The matrix (35) gives a system of $m(n+1)$ nonlinear algebraic equations which can be solved utilizing the initial condition for the elements of $C$. Once $C$ is known, $y(x)$ can be constructed by using (7).

4. Applications and Numerical Results

In this section, numerical results of some examples are presented to validate accuracy, applicability, and convergence of the proposed method. Absolute difference errors of this method is compared with the existing methods reported in the literature [5, 6, 17, 18]. The computations associated with these examples were performed using MATLAB 9.0.

Example 1. Consider the first-order nonlinear Fredholm integrodifferential equation [17, 18] as follows:

$$
y'(x) = 1 - \frac{1}{3} x + \int_{0}^{1} x y^2(t) dt, \quad 0 \leq x < 1,
$$

(36)

with the initial condition

$$
y(0) = 0.
$$

(37)

In this example, we have $p_0 = 0$, $p_1 = 1$, $g(x) = 1 - (1/3)x$, $\lambda = 1$, $k(x,t) = x$, and $q = 2$.

The matrix (35) for this example is

$$
\tilde{P}_1 D^T C - K (\tilde{C}) C = G,
$$

(38)

where for $n = 1$ and $m = 2$ we have

$$
\tilde{P}_1 = I, \quad D^T = \begin{bmatrix} -3 & -3 \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 \end{bmatrix}, \quad C = \begin{bmatrix} c_10 \\ c_11 \\ c_20 \\ c_21 \end{bmatrix},
$$

$$
K = \begin{bmatrix} 1 & \sqrt{3} & 1 & \sqrt{3} \\ 16 & 48 & 16 & 48 \\ \sqrt{3} & 1 & \sqrt{3} & 1 \\ 16 & 16 & 16 & 16 \\ 1 & \sqrt{3} & 1 & \sqrt{3} \\ 4 & 12 & 4 & 12 \\ \sqrt{3} & 1 & \sqrt{3} & 1 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix},
$$

$$
\tilde{C} = \frac{1}{4} \begin{bmatrix} 3 \sqrt{6} c_{10} - \sqrt{2} c_{11} & -\sqrt{2} c_{10} + \sqrt{6} c_{11} & 0 & 0 \\ -\sqrt{2} c_{10} + \sqrt{6} c_{11} & \sqrt{6} c_{10} + 5 \sqrt{2} c_{11} & 0 & 0 \\ 0 & 0 & 3 \sqrt{6} c_{20} - \sqrt{2} c_{21} & -\sqrt{2} c_{20} + \sqrt{6} c_{21} \\ 0 & 0 & -\sqrt{2} c_{20} + \sqrt{6} c_{21} & \sqrt{6} c_{20} + 5 \sqrt{2} c_{21} \end{bmatrix}.
$$


Equation (38) gives a system of nonlinear algebraic equations that can be solved utilizing the initial condition (37); that is, \( \sqrt{6}c_{10} - \sqrt{2}c_{11} = 0 \), we obtain

\[
\begin{align*}
\frac{c_{10}}{24} &= \frac{\sqrt{6}}{8}, \\
\frac{c_{11}}{24} &= \frac{\sqrt{2}}{8},
\end{align*}
\]

(40)

Substituting these values into (7), the result will be \( y(x) = x \), that is, the exact solution. It is noted that the result gives the exact solution as in [17], while in [18] using the sinc method the maximum absolute error is \( 1.52165 \times 10^{-3} \).

Example 2. Consider the first-order nonlinear Fredholm integrodifferential equation [6, 17] as follows:

\[
xy'(x) - y(x) = -\frac{1}{6} + \frac{4}{5}x^2 + \int_0^1 \left( x^2 + t \right) y^2(t) \, dt, \tag{41}
\]

with the initial condition \( y(0) = 0 \). (42)

In this example, we have \( p_0 = -1 \), \( p_1 = x \), \( g(x) = -(1/6) + (4/5)x^2 \), \( \lambda = 1 \), \( k(x, t) = x^2 + t \), and \( q = 2 \).

The matrix (35) for this example is

\[
\left( \tilde{P}_0 + \tilde{P}_1 D^T \right) C - K(\tilde{C}) C = G, \tag{43}
\]

where for \( n = 2 \) and \( m = 2 \) we have

\[
\tilde{P}_0 = -I, \quad \tilde{P}_1 = \begin{bmatrix}
1 & \frac{\sqrt{15}}{60} & \frac{\sqrt{2}}{120} & 0 & 0 & 0 \\
\frac{\sqrt{15}}{60} & \frac{1}{120} & \frac{\sqrt{3}}{3} & 0 & 0 & 0 \\
\frac{1}{120} & \frac{\sqrt{3}}{3} & \frac{5}{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 7 & \frac{\sqrt{15}}{120} & -\sqrt{5} \\
0 & 0 & 0 & \frac{12}{60} & \frac{1}{120} & \frac{\sqrt{3}}{3} \\
0 & 0 & 0 & \frac{60}{\sqrt{15}} & \frac{4}{\sqrt{3}} & \frac{24}{12} \\
0 & 0 & 0 & \frac{120}{\sqrt{5}} & \frac{\sqrt{3}}{11} & \frac{12}{24} \\
0 & 0 & 0 & \frac{120}{\sqrt{5}} & \frac{\sqrt{3}}{11} & \frac{12}{24}
\end{bmatrix},
\]

\[
D^T = \begin{bmatrix}
-5 & \frac{7\sqrt{15}}{3} & -2\sqrt{5} & 0 & 0 & 0 \\
-\frac{\sqrt{15}}{3} & -3 & 14\sqrt{3} & 0 & 0 & 0 \\
0 & -8\sqrt{3} & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & -5 & \frac{7\sqrt{15}}{3} & -2\sqrt{5} \\
0 & 0 & 0 & -\frac{\sqrt{15}}{3} & -3 & 14\sqrt{3} \\
0 & 0 & 0 & 0 & -8\sqrt{3} & 8
\end{bmatrix}, \quad C = \begin{bmatrix}
c_{10} \\
c_{11} \\
c_{12} \\
c_{20} \\
c_{21} \\
c_{22}
\end{bmatrix}, \quad G = \begin{bmatrix}
-\frac{11\sqrt{10}}{450} \\
-\frac{\sqrt{6}}{90} \\
\frac{\sqrt{2}}{180} \\
\frac{23\sqrt{10}}{900} \\
13\sqrt{6} \\
180 \end{bmatrix}.
\]
Substituting these values into (7), the result will be
\[
\begin{pmatrix}
1 & \sqrt{15} & 7\sqrt{5} & 13 & \sqrt{15} & 41\sqrt{5} \\
24 & 45 & 240 & 72 & 720 & \\
\sqrt{15} & 1 & 5\sqrt{3} & \sqrt{15} & 1 & \sqrt{3} \\
72 & 12 & 144 & 24 & 16 & 16 \\
v & 5\sqrt{3} & 1 & 7\sqrt{5} & \sqrt{3} & 5 \\
48 & 144 & 24 & 144 & 16 & 72 \\
7 & 31\sqrt{15} & \sqrt{15} & 41 & 17\sqrt{15} & 7\sqrt{5} \\
48 & 720 & 20 & 144 & 240 & 90 \\
7\sqrt{15} & 3 & 5\sqrt{3} & 11\sqrt{15} & 13 & 7\sqrt{3} \\
144 & 16 & 72 & 144 & 48 & 72 \\
v & 5\sqrt{3} & 1 & 13\sqrt{5} & 5\sqrt{3} & 1 \\
16 & 144 & 12 & 144 & 48 & \frac{3}{9} \\
v & 11\sqrt{3} & 1 & 13\sqrt{5} & 5\sqrt{3} & 1 \\
16 & 144 & 12 & 144 & 48 & \frac{3}{9} \\
\end{pmatrix}
\]
\[K = \begin{pmatrix}
\begin{align*}
5\sqrt{10} & 5\sqrt{6} & \frac{\sqrt{2}}{7} & \frac{11\sqrt{10}}{35} & -8\sqrt{30} & \frac{\sqrt{2}}{7} \\
21 & 21 & 35 & 35 & 105 & 105 \\
\frac{v}{21} & \frac{11\sqrt{10}}{35} & 8\sqrt{30} & 11\sqrt{10} & 3\sqrt{30} & \frac{\sqrt{2}}{7} \\
21 & 35 & 35 & 105 & 105 & 105 \\
\frac{\sqrt{2}}{7} & \frac{8\sqrt{30}}{105} & \frac{3\sqrt{10}}{35} & \frac{8\sqrt{30}}{105} & \frac{5\sqrt{6}}{21} & \frac{13\sqrt{9}}{7} \\
21 & 35 & 35 & 105 & 105 & 105 \\
\frac{\sqrt{2}}{7} & \frac{8\sqrt{30}}{105} & \frac{3\sqrt{10}}{35} & \frac{8\sqrt{30}}{105} & \frac{5\sqrt{6}}{21} & \frac{13\sqrt{9}}{7} \\
\end{align*}
\end{pmatrix}
\]
\[\tilde{c}_j = \begin{pmatrix}
\frac{\sqrt{10}}{21} & \frac{\sqrt{6}}{48} \\
\frac{\sqrt{2}}{24} & \frac{\sqrt{10}}{15} \\
\frac{\sqrt{6}}{8} & \frac{\sqrt{2}}{6} \\
\end{pmatrix}
\]

Equation (43) gives a system of nonlinear algebraic equations that can be solved utilizing the initial condition (42); that is, \(v\sqrt{10}c_{10} - \sqrt{6}c_{11} + \sqrt{2}c_{12} = 0\), we obtain
\[c_{10} = \frac{\sqrt{10}}{240}, \quad c_{11} = \frac{\sqrt{6}}{48}, \quad c_{12} = \frac{\sqrt{2}}{24}, \quad c_{20} = \frac{\sqrt{10}}{15}, \quad c_{21} = \frac{\sqrt{6}}{8}, \quad c_{22} = \frac{\sqrt{2}}{6}. \quad (45)\]

Substituting these values into (7), the result will be \(y(x) = x^2\), that is, the exact solution. It is noted that the result gives the exact solution as in [17], while in [6] approximate solution is obtained with maximum absolute error \(1,0000 \times 10^{-5}\).

**Example 3.** Consider the second-order nonlinear Fredholm integro-differential equation [17] as follows:
\[
y''(x) + xy'(x) - xy(x) = e^x \sin x + \int_0^1 \sin x \cdot e^{-2t} y^2(t) \, dt, \quad 0 \leq x < 1,
\]
with the initial conditions
\[y(0) = y'(0) = 1. \quad (47)\]
The exact solution is \(y(x) = e^x\). We solve this example by using the proposed method with \(n = 2, m = 30\) and \(n = 3, m = 30\}. Comparison among the proposed method and methods in [17] is shown in Table 1. It is clear from this table that the results obtained by the proposed method, using few numbers of basis, are very promising and superior to that of [17].

**Example 4.** Consider the following nonlinear Fredholm integro-differential equation [5, 17]:
\[
y'(x) + y(x) = \frac{1}{2} (e^{-x} - 1) + \int_0^1 y^2(t) \, dt, \quad 0 \leq x < 1,
\]
with the initial conditions
\[y(0) = 1. \quad (49)\]
The exact solution of this problem is \(y(x) = e^{-x}\). In Table 2 we have compared the absolute difference errors of the proposed method with the collocation method based on Haar wavelets in [5] and method in [17].

Maximum absolute errors of Example 4 for some different values of \(n\) and \(m\) are shown in Table 3. As it is seen from Table 3, for a certain value of \(n\) as \(m\) increases the accuracy increases, and for a certain value of \(m\) as \(n\) increases the accuracy increases as well. In case of \(m = 1\), the numerical solution obtained is based on orthonormal Bernstein polynomials only, while in case of \(n = 0\), the numerical solution obtained is based on block-pulse functions only.
**Table 2: Numerical comparison of absolute difference errors for Example 4.**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of collocation points</td>
<td>n = 7</td>
<td>n = 3, m = 35</td>
</tr>
<tr>
<td>0.125</td>
<td>N = 128</td>
<td>2.4509E – 010</td>
<td>5.5200E – 011</td>
</tr>
<tr>
<td>0.375</td>
<td>8.6917E – 007</td>
<td>1.6139E – 010</td>
<td>9.4606E – 011</td>
</tr>
<tr>
<td>0.500</td>
<td>1.0208E – 006</td>
<td>2.2417E – 010</td>
<td>5.5200E – 011</td>
</tr>
<tr>
<td>0.625</td>
<td>1.1029E – 006</td>
<td>2.6083E – 011</td>
<td>7.7237E – 012</td>
</tr>
<tr>
<td>0.750</td>
<td>1.1244E – 006</td>
<td>2.3094E – 012</td>
<td>2.5547E – 012</td>
</tr>
</tbody>
</table>

**Table 3: Maximum absolute errors for different values of n and m for Example 4.**

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>Maximum absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>5.7735E – 01</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2.2361E – 01</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>6.2994E – 01</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1.3889E – 01</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>1.0126E – 01</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>5.1230E – 01</td>
</tr>
</tbody>
</table>

5. **Conclusion**

In this work, we present a numerical method for solving nonlinear Fredholm integrodifferential equations based on hybrid of block-pulse functions and normalized Bernstein polynomials. One of the most important properties of this method is obtaining the analytical solutions if the equation has an exact solution, that is, a polynomial function. Another considerable advantage is this method has high relative accuracy for small numbers of basis n. The matrices K, C, and D in (10), (17), and (25), respectively, have large numbers of zero elements, and they are sparse; hence, the present method is very attractive and reduces the CPU time and computer memory. Moreover, satisfactory results of illustrative examples with respect to several other methods (e.g., Haar wavelets method, Walsh functions method, Bernstein polynomials method, and sinc collocation method) are included to demonstrate the validity and applicability of the proposed method.

**References**


